Reflexivity and Eigenform: The Shape of Process

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Purpose – The paper discusses the concept of a reflexive domain, an arena where the apparent objects as entities of the domain are actually processes and transformations of the domain as a whole. Human actions in the world partake of the patterns of reflexivity, and the productions of human beings, including science and mathematics, can be seen in this light.

Methodology – Simple mathematical models are used to make conceptual points.

Context – The paper begins with a review of the author's previous work on eigenforms - objects as tokens for eigenbehaviors, the study of recursions and fixed points of recursions. The paper also studies eigenforms in the Boolean reflexive models of Vladimir Lefebvre.

Findings –The paper gives a mathematical definition of a reflexive domain and proves that every transformation of such a domain has a fixed point. (This point of view has been taken by W. Lawvere in the context of logic and category theory.) Thus eigenforms exist in reflexive domains. We discuss a related concept called a "magma." A magma is composed entirely of its own structure-preserving transformations. Thus a magma can be regarded as a model of reflexivity and we call a magma "reflexive" if it encompasses all of its structure-preserving transformations (plus a side condition explained in the paper). We prove a fixed point theorem for reflexive magmas. We then show how magmas are related to knot theory and to an extension of set theory using knot diagrammatic topology. This work brings formalisms for self-reference into a wider arena of process algebra, combinatorics, non-standard set theory and topology. The paper then discusses how these findings are related to lambda calculus, set theory and models for self-reference. The last section of the paper is an account of a computer experiment with a variant of the Life cellular automaton of John H. Conway. In this variant , 7-Life, the recursions lead to self-sustaining processes with very long evolutionary patterns. We show how examples of novel phenomena arise in these patterns over the course of large time scales.

Value – The paper provides a wider context and mathematical conceptual tools for the cybernetic study of reflexivity and circularity in systems.

Keywords – Reflexive, eigenform, cybernetics, Boolean algebra, knots, magma, Russell paradox, cellular automata

1. Introduction

"Reflexive" is a term that refers to the presence of a relationship between an entity and itself. One can be aware of one's own thoughts. An organism produces itself through its own action and its own productions. A market or a system of finance is composed of actions and individuals, and the actions of those individuals influence the market just as the global information from the market influences the actions of the individuals. Here it is the self-relations of the market through its own structure and the structure of its individuals that moves its evolution forward. Nowhere is there a way to cut an individual participant from the market effectively and make him into an objective observer. His action in the market is concomitant to his being reflexively linked with that market. It is just so for theorists of the market, for their theories, if communicated, become part of the action and decision-making of the market. Social systems partake of this same reflexivity, and so does apparently objective science and mathematics. In order to see the reflexivity of the practice of physical science or mathematics, one must leave the idea of an objective domain of investigation in brackets and see the enterprise as a wide-ranging conversation among a group of investigators. Then, at once, the process is seen to be a reflexive interaction among the members of this group. Mathematical results, like all technical inventions, have a certain stability over time that gives them an air of permanence, but the process that produces these novelties is every bit as fraught with circularity and mutual influence as any other conversation or social interaction.

How then, shall we describe a reflexive domain? It is the purpose of this paper to give a very abstract definition that nevertheless captures what I believe to be the main conceptual feature of reflexivity. We then immediately prove that eigenforms, fixed points of transformations, are present for all transformations of the reflexive domain. This will encourage us and will give us pause to think further about the relationship of reflexivity and eigenform.

The existence of eigenforms will encourage us, for we have previously studied them with the notion that "objects are tokens for eigenbehavior." Eigenforms are the natural emergence of those tokens by way of recursion. So to find the eigenforms dictated by a larger concept is pleasing. The existence of fixed points for arbitrary transformations shows us that the domain we have postulated is indeed very wide. It is not an objectively existing domain. It is a clearing in which structures can arise and new structures can arise. A reflexive domain is not an already-existing structure. To be what it claims to be, a reflexive domain must be a combination of an existing structure and an invitation to create new structures and new concepts. The new will become platforms from which further flights of creativity can be made. Thus in the course of examining the concept of reflexivity we will find that the essence of the matter is an opening into creativity; and that will become the actual theme of this paper.

This essay begins with a discussion of the notion of "eigenform" as pioneered by Heinz von Foerster in his papers (Foerster 1981a–c) and explored in papers by the author (Kauffman 1987, 2003, 2005). We include some of the material from (Kauffman 2005) in this paper for the sake of completeness. In (Foerster 1981a) the familiar objects of our existence can be seen to be nothing more than tokens for the behaviors of the organism, creating apparently stable forms.

In this view, the object is *both* an element of a world *and* a token or symbol for the process of its production/observation.

An object, in itself, is a symbolic entity, participating in a network of interactions, taking on its apparent solidity and stability from these interactions. We ourselves are such objects: we, as human beings, are "signs for ourselves," a concept originally developed by the American philosopher, Charles S. Peirce (Kauffman 2001). Eigenforms are mathematical companions to Peirce's work.

In an observing system, what is observed is not distinct from the system itself, nor can one make a complete separation between the observer and the observed. The observer and the observed stand together in a coalescence of perception. From the stance of the observing system, all objects are non-local, depending upon the presence of the system as a whole. It is within that paradigm that these models begin to live, act and enter into conversation with us.

After this journey into objects and eigenforms, we take a wider stance and consider the structure of spaces and domains that partake of the reflexivity of object and process. In Section 6 we give a definition of a *reflexive domain*. Our definition populates a space (domain) with entities that could be construed as objects, and we assume that each object acts as a transformation on the space. Essentially this means that given entities A and B, there is a new entity C that is the result of A and B acting together in the order AB (so that one can say that "A acts on B" for AB and "B acts on A" for BA). This means that the reflexive space is endowed with a non-commutative and non-associative algebraic structure. The reflexive space is expandable in the sense that whenever we define a process, using entities that have already been constructed or defined, then that process can take a name, becoming a new entity/transformation of a space that is expanded to include itself. Reflexive spaces are open to evolution over time as new processes are invented and new forms emerge from their interaction.

Remarkably, reflexive spaces always have eigenforms for every element/transformation/entity in the space! The proof is simple but requires discussion.

Given F in a reflexive domain, define G by Gx = F(xx).

Then GG = F(GG) and so GG is an eigenform for F.

Just as promised, in a reflexive domain, every entity has an eigenform. From this standpoint, one should start with the concept of reflexivity and see that from it emerge eigenforms. Are we satisfied with this approach? We are not. In order to start with reflexivity, we need to posit objects and processes. As we have already argued in this essay, objects are tokens for eigenbehaviors. And a correct or natural beginning is a process where objects are seen as tokens of processes.

By now the reader begins to see that the story we have to tell is a circular one. We give a way to understand this circularity in our last section, where we discuss creativity in recursive processes and the emergence of novelty.

The paper continues in Section 6 by studying an allied concept that we call a *magma*. A magma is a domain with a binary operation ***** that allows one to combine elements **a** and **b** of the domain to form a new element **a * b** of that domain. In the magma each element **a** is also a mapping of the domain to itself via left combination: **x → a * x**. We assume that each such transformation preserves the structure of the combinatory operation. Magmas are very close in concept to reflexive domains. We define the notion of a reflexive magma and show that such magmas satisfy a fixed point theorem and so contain eigenforms. In Section 7 we show how magmas arise naturally in the context of knot theory and a theory of knot sets. Sections 8 and 9 discuss the relationships of reflexivity with the lambda calculus of Church and Curry and with Cantor's diagonal argument and the Russell paradox. Section 10 is a minimalist discussion of self-reference and reflexivity in relation to the conceptualization of a universe that comes to observe itself. Section 11 is an account of a computer experiment with a variant of the Life cellular automaton of John H. Conway. In this variant that we have discovered, 7-Life, the recursions lead to self-sustaining processes with very long evolutionary patterns. We show how examples of novel phenomena arise over the course of large time scales. This example will be a later springboard for the discussion of the emergence of novelty from deterministic processes. Here, it is an example showing how the course of a process is just as important as its eigenform or infinite concatenation.

The paper ends with a discussion of the wider context of reflexivity. We are acutely aware that this paper about reflexivity only gives certain conceptual tools and does not yet address the actuality of the reflexive condition of persons and observers who are inextricably part of the universes that they hope to study. In so doing they will adopt points of view and these very points of view will create patterns, new forms, objects of study and will act as a veil over the original intent. It is only through working with many points of view and many investigations that the particularities of single lenses will begin to fall away and a wider understanding will emerge.

2. Objects as tokens for eigenbehaviors

In his paper *Objects as Tokens for Eigenbehaviors*, von Foerster (1981a) suggests that we think seriously about the mathematical structure behind the constructivist doctrine that *perceived worlds are worlds created by the observer.* At first glance such a statement appears to be nothing more than solipsism. At second glance, the statement appears to be a tautology, for who else can create the rich subjectivity of the immediate impression of the senses? At third glance, something more is needed. In that paper he suggests that the familiar objects of our experience are the fixed points of operators. These operators *are* the structure of our perception. To the extent that the operators are shared, there is no solipsism in this point of view. It is the beginning of a mathematics of second order cybernetics.

Consider the relationship between an observer **O** and an "object" **A**. The key point about the observer and the object is that "the object remains in constant form with respect to the observer." This constancy of form does not preclude motion or change of shape. Form is more malleable than the geometry of Euclid. In fact, ultimately, the form of an "object" is the form of the distinction that "it" makes in the space of our perception. In any attempt to speak absolutely about the nature of form we take the form of distinction for the form (paraphrasing Spencer-Brown 1969). It is the form of distinction that remains constant and produces an apparent object for the observer. How can you write an equation for this? The simplest route is to write

$$\mathbf{O(A) = A}.$$

The object **A** is a fixed point for the observer **O**. The object is an eigenform. We must emphasize that this is the most schematically possible description of the condition of the observer in relation to an object **A**. We only record that the observer as an actor (operator) manages through his acting to leave the (form of) the object unchanged. This can be a recognition of the symmetry of the object but it also can be a description of how the observer, searching for an object, makes that object up (like a good fairy tale) from the very ingredients that are the observer herself. This is the situation that Heinz von Foerster has been most interested in studying. As he puts it, if you give a person an undecideable problem, then the answer that he gives you is a description of himself. And so, by working on hard and undecideable problems we go deeply into the discovery of who we really are. All this is symbolized in the little equation **O(A) = A**.

And what about this matter of the object as a token for eigenbehavior? This is the crucial step. We forget about the object and focus on the observer. We attempt to "solve" the equation **O(A) = A** with **A** as the unknown. Not only do we admit that the "inner" structure of the object is unknown, we adhere to whatever knowledge we have of the observer and attempt to find what such an observer could observe based upon that structure.

We can start anew from the dictum that the perceiver and the perceived arise together in the condition of observation. This is a stance that insists on mutuality (neither perceiver nor the perceived causes the other). A distinction has emerged and with it a world with an observer and an observed. The distinction is itself an eigenform.

3. Compresence and coalescence

We identify the world in terms of how we shape it. We shape the world in response to how it changes us. We change the world and the world changes us. Objects arise as tokens of a behavior that leads to seemingly unchanging forms. Forms are seen to be unchanging through their invariance under our attempts to change, to shape them.

For an observer there are two primary modes of perception – *compresence* and *coalescence*. Compresence connotes the coexistence of separate entities together in one including space. Coalescence connotes the one space holding, in perception, the observer and the observed, inseparable in an unbroken wholeness. Coalescence is the constant condition of our awareness. Coalescence is the world taken in simplicity; compresence is the world taken in apparent multiplicity.

This distinction between compresence and coalescence, drawn by Henri Bortoft (1971), can act as a compass in traversing the domains of object and reference. *Eigenform is a first step towards a mathematical description of coalescence*. In the world of eigenform, the observer and the observed are one in a process that recursively gives rise to each.

4. The eigenform model

We have seen how the concept of an object has evolved to make what we call objects (and the objective world), processes that are interdependent with the actions of observers.

The notion of a fixed object has become a notion of a process that produces the apparent stability of the object. This process can be simplified in a model to become a recursive process where a rule or rules are applied time and time again. The resulting object of such a process is the *eigenform* of the process, and the process itself is the *eigenbehavior.*

In this way we have a model for thinking about object as token for eigenbehavior. This model examines the result of a simple recursive process carried to its limit. For example, suppose that

$$F(X) = \boxed{\quad X \quad}$$

That is, each step in the process encloses the results of the previous step within a box. Here is an illustration of the first few steps of the process applied to an empty box, X:

$$\begin{array}{cccc} \boxed{} & \boxed{\boxed{}} & \boxed{\boxed{\boxed{}}} & \boxed{\boxed{\boxed{\boxed{}}}} \\ X & F(X) & F(F(X)) & F(F(F(X))) \end{array}$$

If we continue this process, then successive nests of boxes resemble one another, and in the limit of infinitely many boxes, we find that

$$X=F(F(F(...)))= \boxed{\boxed{\boxed{\boxed{\boxed{\cdots}}}}}$$

$$F(X) = \boxed{\boxed{\boxed{\boxed{\boxed{\boxed{\cdots}}}}}} = X$$

The infinite nest of boxes is invariant under the addition of one more surrounding box. Hence this infinite nest of boxes is a fixed point for the recursion. In other words, if X denotes the infinite nest of boxes, then

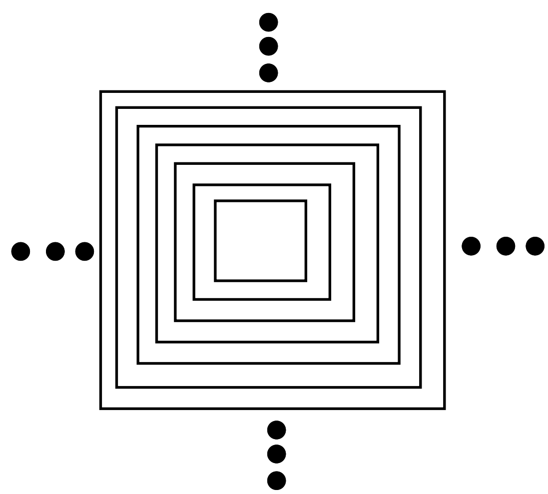
$$\mathbf{X = F(X)}.$$

This equation is a description of a state of affairs. The form of an infinite nest of boxes is invariant under the operation of adding one more surrounding box. The infinite nest of boxes is one of the simplest eigenforms.

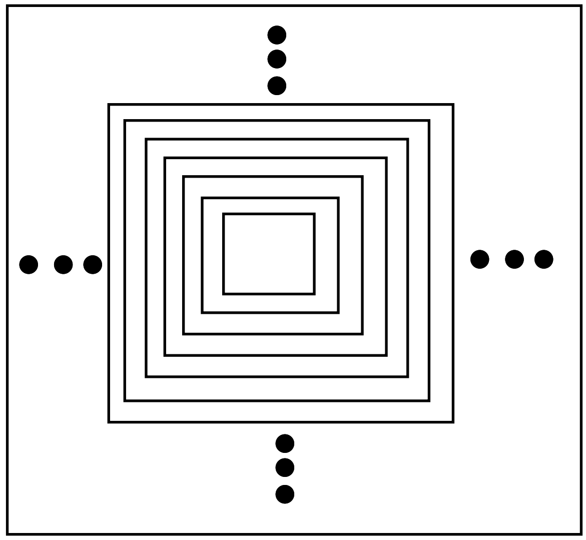
Remark. On reading the above description of the limiting process

$$X \dashrightarrow F(X) \dashrightarrow F(F(X)) \dashrightarrow \ldots$$

the reader may find herself thinking along the following lines: “Doesn’t he mean to put those three dots in the nested boxes on the *outside* of the boxes rather than on the inside? After all, the operation F surrounds X with a square, so at each stage, a square is added from the outside. Shouldn’t the picture then be like this one below?”



“I have illustrated the new picture with the three dots on the left, the right, the top and the bottom to show how in this way of thinking the nest of boxes grows outward and consequently it grows in all these directions. If we take this construction to infinity, then it will either fill the plane with boxes, or the widths between successive boxes will have to grow smaller and smaller, just as, with the three dots inside, you had to make the boxes smaller and smaller. But really, this second picture is quite different from the first picture. In fact if we *do* make the second picture, and imagine that it is a solution to the equation F(X) = X, it does not seem to be a solution! Look at the picture below.”



"Now I have put a box around the outwardly growing infinite nest of boxes, but this means that I have allowed an infinite number of boxes to grow there (going out but staying in a finite amount of space by crowding one next to another) and then I put *one more* box around all of them. The result is *not* the same! This is a *new* form of boxes.

So with the outward growth, I make new infinities, but I do not solve the equation X = F(X). Now I see what you were doing with the inward nest of boxes. You let it grow inwardly and obtained a limit form that did not see the one box more that you put around the outside. I had to try this other method in order to see what you were doing. And I am sure that other readers will have to experiment in this way and in new ways to really understand this construction of eigenforms."

Comment on the remark. Indeed the patient reader was right that there is more than one way to go to infinity. A simpler example can be seen in the equation x = ax where we solve it by letting x = aaa..., an infinite repetition of a's going off to the right.

Then

$$ax = a(aaa...) = aaaa... = aaa... = x.$$

But if we do it in the other order and take xa, we find that

$$xa = aaaa...a$$

which means an infinite row of a's followed by one more a.

And we see that in this way of thinking xa is not equal to ax.

Similarly, y = ...aaa is not a solution to ay = y but it is a solution to ya = a. This may seem a bit strange and abstract, so it is better to think with the boxes (I think). But in ordinary mathematics we use this same sort of infinite construction. For example, we write

$$x = 1 + a + aa + aaa + aaaa + ...$$

and rewrite it in the form

$$x = 1 + a(1 + a + aa + aaa + ...) = 1 + ax$$

and conclude that

$$x - ax = 1,$$

$$x(1 - a) = 1,$$

$$x = 1 / (1 - a).$$

Hence

$$1 / (1 - a) = 1 + a + aa + aaa + aaaa + ...$$

Here we are using aa for the product of a with itself, so these can be numbers. And one can verify that indeed if a is a number and the absolute value of a is less than one, then this formula is true. For example, if a = 1 / 2, then 1 / (1 – (1 / 2)) = 2 and the formula asserts that

$$2 = 1 + 1/2 + 1/4 + 1/8 + 1/16 + ...$$

This is true, and the reader should ask herself how she *knows* that it is true! The reader will also be interested in seeing what happens when a is bigger than or equal to 1 in absolute value. For example, if a = 2, then our formula would seem to say that

$$-1 = 1 + 2 + 4 + 8 + 16 + 32 + ...$$

Is there some truth in this absurdity?

We can see what has actually happened by making a closer analysis.

Let $X = 1 + 2 + 4 + ... + 2^N$ where 2^N means 2 multiplied by itself N times. Then we have

$$X = 1 + 2(1 + 2 +... + 2^{N-1}),$$

$$X = 1 + 2(1 + 2 +... + 2^{N-1} + 2^N) - 2(2^N),$$

$$X = 1 + 2X - 2^{N+1},$$

So we have

$$X - 2X = 1 - 2^{N+1},$$

which is the same as saying

$$X = -1 + 2^{N+1}.$$

Do you see what has happened? We are interested in finding out what happens when N goes to infinity. But here if we ignore the term 2^{N+1} we will get the wildly wrong answer of –1. You have to take infinity with a grain of salt as well as looking at it as the vastness of all the grains of sand on the beach. *End of Comment.*

A further comment: Perhaps you thought that we showed that the equation: -1 = 1 + 2 + 4 + … is wrong. There is a point of view in which it is right! Consider that in binary arithmetic we represent 1 by 1, 2 by 10, 4 by 100, 8 by 1000 and so on. Then 1+ 2+ 4 + 8 is represented in binary by 1111, and when you add 1 to 1111 you find a series of carrys taking you to the answer 10000. Suppose you had a computer that could only handle binary numbers up to four bits. Then when you added 1 to 1111 you would get 0000, since the computer would throw away the last bit. In this sense 1111 represents -1 in such a limited computer, and in the same way the infinite sum 1+2+4+8+… represents -1 in an infinite computer that is not prepared to have bits beyond the first infinity!

In the process of observation, we interact with ourselves and with the world to produce stabilities that become the objects of our perception. These objects, like the infinite nest of boxes, may go beyond the specific properties of the world in which we operate. They attain their stability through the limiting process that goes outside the immediate world of individual actions. We make an imaginative leap to complete such objects to become tokens for eigenbehaviors. It is impossible to make an infinite nest of boxes. We do not make it. We *imagine* it. And in imagining that infinite nest of boxes, we arrive at the eigenform.

The leap of imagination to the infinite eigenform is a model of the human ability to create signs and symbols. In the case of the eigenform **X** with **X = F(X)**, **X** can be regarded as the name of the process itself or as the name of the limiting process. Note that if you are told that

$$\mathbf{X = F(X),}$$

then, substituting **F(X)** for **X**, you can write

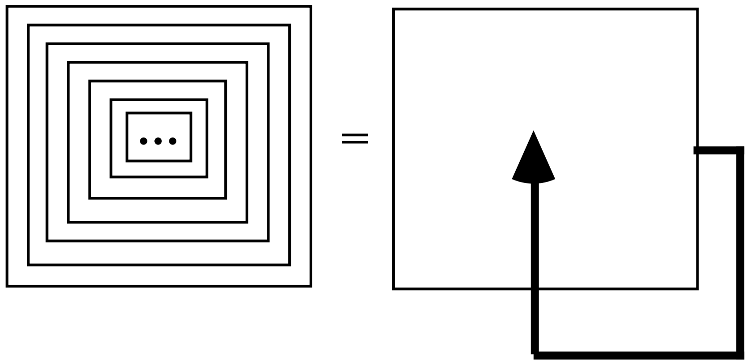
$$\mathbf{X = F(F(X))}.$$

Substituting again and again, you have

$$\mathbf{X = F(F(F(X))) = F(F(F(F(X)))) = F(F(F(F(F(X))))) = ...}$$

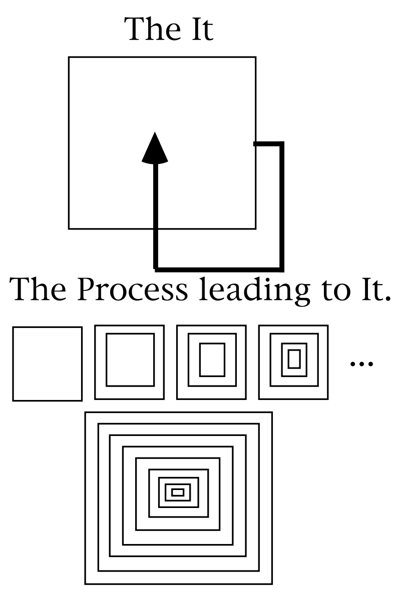
The process arises from the symbolic expression of its eigenform. In this view, *the eigenform is an implicate order for the process that generates it.* (Here we refer to implicate order in the sense of David Bohm (1980).)

Sometimes one stylizes the structure by indicating where the eigenform **X** reenters its own indicational space with an arrow or other graphical device. See the picture below for the case of the nested boxes.



Does the infinite nest of boxes exist? Certainly it does not exist on this page or anywhere in the physical world with which we are familiar. The infinite nest of boxes exists in the imagination. It is a symbolic entity.

The eigenform is the imagined boundary in the reciprocal relationship of the object (the “It”) and the process leading to the object (the process leading to “It”). In the diagram below we have indicated these relationships with respect to the eigenform of nested boxes. Note that the “It” is illustrated as a finite approximation (to the infinite limit) that is sufficient to allow an observer to infer/perceive the generating process that underlies it.



Just so, an object in the world (cognitive, physical, ideal, etc.) provides a conceptual center for the exploration of a skein of relationships related to its context and to the processes that generate it. An object can have varying degrees of reality, just as an eigenform does. If we take the suggestion to heart that objects are tokens for eigenbehaviors, then an object in itself is an entity, participating in a network of interactions, taking on its apparent solidity and stability from these interactions.

An object is an amphibian between the symbolic and imaginary world of the mind and the complex world of personal experience. The object, when viewed as a process, is a dialogue between these worlds. The object, when seen as a sign for itself, or in and of itself, is imaginary.

Why are objects apparently solid? Of course you cannot walk through a brick wall even if you think about it differently. I do not mean apparent in the sense of thought alone. I mean apparent in the sense of appearance. The wall appears solid to me because of the actions that I can perform. The wall is quite transparent to a neutrino, and will not even be an eigenform for that neutrino.

This example shows quite sharply how the nature of an object is entailed in the properties of its observer.

The eigenform model can be expressed in quite abstract and general terms. Suppose that we are given a recursion (not necessarily numerical) with the equation

$$\mathbf{X(t + 1) = F(X(t))}.$$

Here $\mathbf{X(t)}$ denotes the condition of observation at time $\mathbf{t}$. $\mathbf{X(t)}$ could be as simple as a set of nested boxes, or as complex as the entire configuration of your body in relation to the known universe at time $\mathbf{t}$. Then $\mathbf{F(X(t))}$ denotes the result of applying the operations symbolized by $\mathbf{F}$ to the condition at time $\mathbf{t}$. You could, for simplicity, assume that $\mathbf{F}$ is independent of time. Time independence of the recursion $\mathbf{F}$ will give us simple answers and we can later discuss what will happen if the actions depend upon the time. In the time-independent case we can write

$$\mathbf{J = F(F(F(...)))}$$

- the infinite concatenation of F upon itself. Then

$$\mathbf{F(J) = J}$$

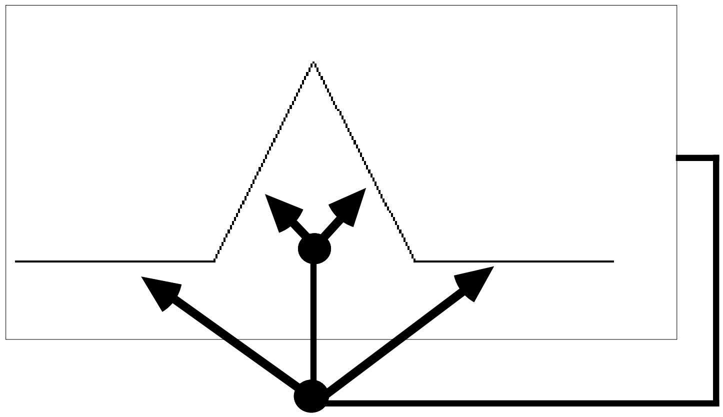
since adding one more $\mathbf{F}$ to the concatenation changes nothing.

Thus **J,** the infinite concatenation of the operation upon itself leads to a fixed point for **F**. **J** is said to be the eigenform for the recursion **F**. We see that every recursion has an eigenform. Every recursion has an (imaginary) fixed point.

We end this section with one more example. This is the eigenform of the Koch fractal (Mandelbrot 1982). In this case one can write symbolically the eigenform equation

$$\mathbf{K = K \{ K\ K \}\ K}$$

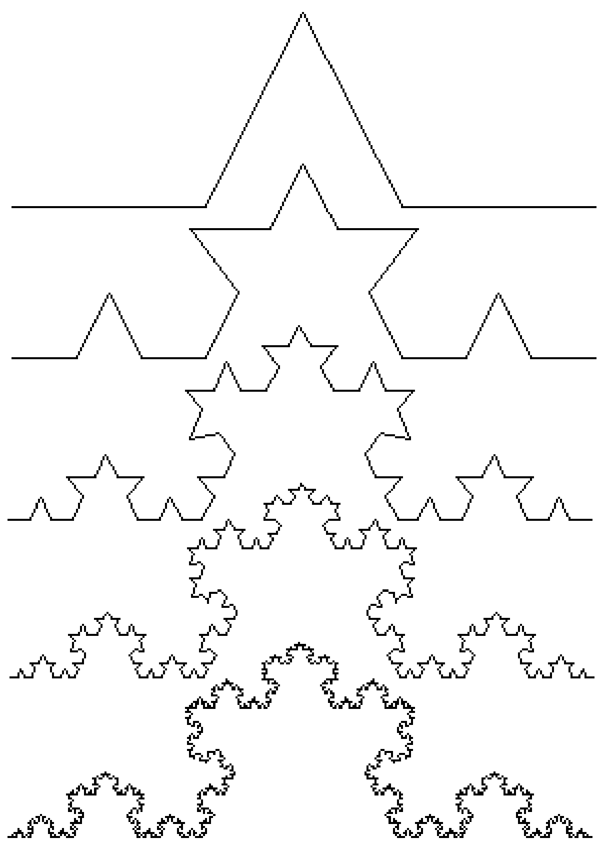
to indicate that the Koch Fractal reenters its own indicational space four times (that is, it is made up of four copies of itself, each one-third the size of the original. The curly brackets in the center of this equation refer to the fact that the two middle copies within the fractal are inclined with respect to one another and with respect to the two outer copies. In the figure below we show the geometric configuration of the reentry.



K = K { K K } K

In the geometric recursion, each line segment at a given stage is replaced by four line segments of one third of its length, arranged according to the pattern of reentry as shown in the figure above.

The recursion corresponding to the Koch eigenform is illustrated in the next figure. Here we see the sequence of approximations leading to the infinite self-reflecting eigenform that is known as the Koch snowflake fractal.



Five stages of recursion are shown. To the eye, the last stage vividly illustrates how the ideal fractal form contains four copies of itself, each one-third the size of the whole. The abstract schema

K = K { K K } K

for this fractal can itself be iterated to produce a "skeleton" of the geometric recursion:

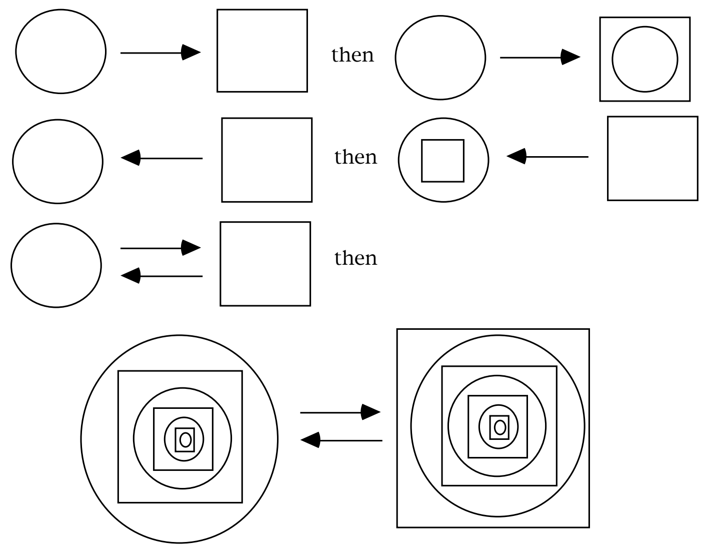
K = K { K K } K
= K { K K } K { K { K K } K K { K K } K } K { K K } K
= ...

We have only performed one line of this skeletal recursion. There are sixteen **K**'s in this second expression, just as there are sixteen line segments in the second stage of the geometric recursion. Comparison with this symbolic recursion shows how geometry aids the intuition. The interaction of eigenforms with the geometry of physical, mental, symbolic and spiritual landscapes is an entire subject that is in need of deep exploration.

It is usually thought that the miracle of recognition of an object arises in some simple way from the assumed existence of the object and the action of our perceiving systems. This is fine tuning to the point where the action of the perceiver and the perception of the object are indistinguishable. Such tuning requires an intermixing of the perceiver and the perceived that goes beyond description. Yet at the mathematical levels, such as number or fractal pattern, part of the process is slowed down to the point where we can begin to apprehend the process. There is a stability in the comparison, in the correspondence that is a process happening at once in the present time. The closed loop of perception occurs in the eternity of present individual time. Each such process depends upon linked and ongoing eigenbehaviors and yet is seen as simple by the perceiving mind. The perceiving mind is itself an eigenform.

Mirror-mirror

In the next figure we illustrate how an eigenform can arise from a process of mutual reflection. The figure shows a circle with an arrow pointing to a rectangle and a rectangle with an arrow pointing toward a circle. For this example, we take the rule that an arrow between two entities ($P \rightarrow Q$) means that the second entity will create an internal image of the first entity (Q will make an image of P). If $P \rightarrow Q$ and $Q \rightarrow P$, then each entity makes an image of the other. A recursion will ensue. Each of P and Q generates eigenforms in this mutuality.



In this example we can denote the initial forms by C (for circle) and B (for box). We have C → B and B → C. The rule of imaging is (symbolically):

If P → Q then P → QP.

If P ← Q, then PQ ← Q.

We start with the mutual reference C ↔B.

This condition of mutual mirroring can be described by two operators C and B:

C(P) = CP corresponds to C → P.

B(Q) = BQ corresponds to Q ← B.

Solving the eigenform equations

C(Y) = X,

B(X) = Y,

we have the mirror-mirror solution

X = BCBCBCBC...,

Y = CBCBCBCB...,

just as in the figure.

We are quite familiar with this form of mutual mirroring in the physical realm where one can have two facing mirrors, and in the realm of human relations where the complexity of exchange (mutual mirroring) between two individuals leads to the eigenform of their relationship.

5. Boolean self-reference and the work of Vladimir Lefebvre

Vladimir Lefebvre (1982) models ethical situations involving multiple reflections (I think about your thoughts about me, while you think about my thoughts about you.) using Boolean algebra and a graphical formalism.

The crux of this endeavor begins with examining self-referential equations in the Boolean context. In this context we have the arithmetic of **0** and **1** with $\mathbf{1 + 1 = 1}$ and otherwise **0** and **1** behaving as in ordinary arithmetic, with complementation ($\mathbf{a \rightarrow a'}$) interchanging them: $\mathbf{0' = 1}$ and $\mathbf{1' = 0}$.

In interpreting Boolean algebra for logic, we take **a+b** to mean **"a or b"** and **ab** to mean "**a and b**."

Thus we take **1** as **T** (True) and **0** as **F** (False).

Note that $\mathbf{a > b}$ (a implies b) is represented by $\mathbf{a' + b}$ in this system.

A very simple form of Boolean self-reference is the equation

$$\mathbf{x = x}.$$

This just says that x is equal to x. It is like the biblical "I am that I am."

A diabolical form of Boolean self reference is the equation

$$\mathbf{x = x'}.$$

This says that x is equal to not x, and can be interpreted as the statement of the liar who asserts that he is lying, "I am a liar." I like to think of the solution to this equation as an *oscillation* between 0 and 1.

After all, if $x = 0$, then $x = x' = 0' = 1$ and if $x = 1$, then $x = x' = 1' = 0$.

So x oscillates just like a buzzer of a doorbell.

The simplest general form of a Boolean self-referential equation is

$$\mathbf{x = ax + bx'}.$$

What are the possibilities?

We can have

$$\mathbf{x = x + x' = 1 \text{ (constantly true)}},$$
$$\mathbf{x = 1x + 0x' = x \text{ (self-affirming)}},$$

$$x = 0x + 1x' = x' \text{ (self-denying)},$$
$$x = 0x + 0x' = 0 \text{ (just false)}.$$

Thus there does not seem to be a lot of structure in this simplest version of self-reference. However, we should think a bit further and realize that $\mathbf{a}$ and $\mathbf{b}$ can be *propositions* that have relative truth values and we may not need to know the actual truth values of $\mathbf{a}$ and $\mathbf{b}$. Consider the equation

$$x = (b' + c)x + cx'.$$

If $x = 0$, then we have $0 = c$.

So we conclude that for $x = 0$ to be a solution, this equation reduces to $x = x$ or $x = 0$. However, if $x = 1$ is a solution, then we have $1 = b' + c$, and the equation will have a solution just so long as $\mathbf{b}$ implies $\mathbf{c}$ is true. In either case the equation has a non-oscillatory solution. This is the form of the self-referential equation at the base of Vladimir Lefebvre's analysis of ethics and reflectivity.

The next thing to notice is that

$(a > b) > c = (a' + b)' + c = ab' + c = (b' + c)a + ca'$.

Thus we have

$$a = (a > b) > c$$

as an allowable self referential Boolean equation.

Lefebrve interprets the right hand side of this equation as **"c thinks of b thinking of a."** Thus the self-reference is **"a is thinking of c thinking of b thinking of a."**

Lefebvre takes b^a as notation for **"a implies b."** Thus

$$b^a = b + a'$$

and b^a stands for **$a > b$,** which is interpreted as **"b is thinking about a"** or **"b has an internal image of a."**

Thus our self-referential equation becomes $a = c^x$ where $x = b^a$.

Using Laws of Form

Here is a second take on this theme, using Laws of Form (Spencer 1969) bracket notation. In the Laws of Form notation, we take $a' = <a>$ and $\mathbf{ab}$ stands for $a + b$ while the conjunction $\mathbf{ab}$ in Boolean algebra becomes $<<a><b>>$ in accordance with DeMorgan's Law. We also have $\mathbf{0}$ as the void state in Laws of Form and $1 = < >$, the marked state, a single crossing from the void. Then the Boolean arithmetic of $\mathbf{0}$ and $\mathbf{1}$ corresponds to the Laws of Calling $< > < > = < >$ and Crossing $<< >> =$ **"void."**

In Laws of Form notation, **“a implies b”** is written as

<a>b = b <a>.

In Lefebvre’s notation this is the same as

b<a> = $\mathbf{b}^{\mathbf{a}}$.

Thus Laws of Form is a useful alternate formalism for this theory.

We can interpret **b<a>** as **“b thinks of a**.”

Consider the self-referential equation

a = <<a>b>c.

“a is thinking of c thinking of b who thinks of a.”

This is a self-reference that can be made inside two-valued primary arithmetic, since it never oscillates like **a = <a>**. You can think of this fixed point in the form of the infinite reentry:

a = <<<<<<<<<<<<<<...>b>c>b>c>b>c>b>c>b>c>b>c>b>c.

It is amusing to write this in ordinary Boolean form as

a = c + da where **d = <b>** and **xy = <<x><y>>** and **x + y replaces x y** (LOF juxtaposition). Then we get

a = c + da

a = c + d(c + da) = c + dc + d^2 a

and so on,

a = c + d^2 c + d^3 c + d^4 c + ...

a = “c/(1-d).”

The infinite reentry expressions in **LOF** become an infinite power series in Boolean algebra. This brings us closer to classical mathematics and its role in producing imaginary values.

Vladimir Lefebvre (1982) in his “Algebra of Conscience” models structures such as

a = <b>a

b = <a>b

as “a thinking about himself with an image of b” and “b thinking about himself with an image of a.”

We can use the **LOF** notation to represent the self-referential algebra of Lefebvre, and it is useful to do this.

It is important to see how fixed point equations and reflexivity are intertwined in the Boolean structure. One might think that these concepts would not live in the Boolean context, but of course we do manage to discuss them in the Boolean context of our own thought. So Lefebvre’s model is a microcosm of our condition, and of course this is exactly the point!

This section is just a small introduction to Lefebrvre’s theory of reflexivity. It is worth pointing out that he uses the Boolean background skillfully when it is required, but uses the symbolism of reflection on the surface in a way that corresponds to nested linguistic statements. For example, **a<a<a>>** represents “a thinking about a, who has a self-image that corresponds to the true (external) a.” When we evaluate this expression we find

a<a<a>> = a < < > > = a.

Thus the non-self-doubting **a** is simply himself.

On the other hand, **a<a< <a> >>** represents **a** with an image of himself whose image of himself is false (**<a**>). Evaluating this expression, we find

a<a< <a> >> = a<a a> = a < a > = < >.

Thus the individual with a doubting self image receives a marked value for his skepticism. What about an individual who directly doubts

himself? Then we have **a < <a> > = a a = a**. He is in the same boat as the individual with a self-image who doubts. From these examples, we see that the Lefebvre system needs to be examined carefully for its internal meanings. This will be the subject of another paper.

6. Reflexive domains and the magma

A reflexive domain **D** is an arena where actions and processes that transform the domain can also be seen as the elements that compose the domain. Every element of the domain can be seen as a transformation of the domain to itself.

In actual practice, an element of a domain may be a person or company (collective of persons) or a physical object or mechanism that is seen to be in action. In actual practice we must note that what are regarded as objects or entities depends upon the way in which observers inside or outside the domain divide their worlds.

It is very difficult to make a detailed mathematical model of such situations. Each actor is an actor in more than one play. His actions undergo separate but related interpretations, depending upon the others with whom he interacts. Mutual feedback of a multiplicity of ongoing processes is not easily described in the Platonic terms of pure mathematics.

Nevertheless, we take as a general principle for a mathematical model that **D** is a certain set (possibly evolving in time), and we let **[D, D]** denote a selected collection of mappings from **D** to **D**. An element **F** of **[D, D]** is a mapping **F: D → D**.

We shall assume that there is a one-to-one correspondence

$$\boldsymbol{I{:}D \rightarrow [D, D]}.$$

This is the assumption of reflexivity. Every element of the reflexive domain is a transformation of that domain. Each denizen of the reflexive domain has a dual role of actor and actant.

Given an element **g** in **D**, **I(g): D → D** is a mapping from **D** to **D**, and for every mapping **F: D → D**, there is an element **g** in **D** such that **I(g) = F**. The reflexive domain embodies a perfect correspondence between actions and entities that are the recipients of these actions.

See D. Scott (1980) for a specific construction of relflexive domains relevant to computer science and logic. An important precursor to this notion of reflexive domain in mathematics is the notion of Gödel numbering of texts. One chooses a method to encode a text as a specific natural number (a certain product of prime powers). Then texts that speak about numbers can, in principle, speak about other texts and even about themselves. If a text is seen as a transformation on the field of numbers, then that text is itself a number (its Gödelian code) and so can be transforming itself. The precision of this idea enabled Gödel to construct mathematical systems that could talk about their own properties without contradiction and he showed that all sufficiently rich mathematical systems have this property. In this way, these systems become self-limiting due to the possibility of statements whose coded meaning becomes "This statement has no proof in the system of mathematics in which it is written," while the surface meaning of the same statement is a discussion of the properties of certain numerical relations. The domain of numerical relations appears innocuous, and yet it sows the seeds of its own limitations through this ability to reflect itself through the mirror of the Gödel coding.

The Gödelian example is not just a piece of mathematics. It is a reflection with mathematical precision of the condition of our language, thought and action. We are always equipped to comment on our own doings and in so doing to create new language about our old language and new language about our worlds. All our apparent well-thought-out and directed actions in worlds that seem to extend outward from us in an objective way are fraught with the circularity not just of our meta-comments, but also with the circular return of the consequences of those actions and the influence of our very theories of the world on the properties of that world itself.

We now prove a fundamental theorem about reflexive domains.

We show that every mapping **F: D → D** has a fixed point **p**, an element **p** in **D** such that **F(p) = p**. What does this mean? It means that there is another way, in a reflexive domain, to associate a point to a transformation. The point can be seen as the fixed point of a transformation and in that way, the points of the domain disappear into the self-referential nature of the transformations.

Let me tender persuasions. Suppose that **p = F(p)**. Then we can regard this equation as an expression of **p** in terms of **F** and itself and write

$$
\begin{aligned}
\mathbf{p} &= \mathbf{F(p)} \\
&= \mathbf{F(F(p)} \\
&= \mathbf{F(F(F(p)))} \\
&= \mathbf{F(F(F(F(p))))}
\end{aligned}
$$

and continue in this fashion until the appearance of **p** on the right hand side is lost in the depths of the composition of **F** upon itself.

$$\mathbf{p} = F(F(F(F(F(F(F(F(F(F(F(F(F(F(F(F(F(F(\ldots)))))))))))))))))).$$

The infinite composition of **F** upon itself is invariant under one more composition with **F** and so **F(p) = p** is consistent with this process.

To show that an entity **p** is a fixed point for a process **F** is to show that **p** can be formally identified with the infinite concatenation of **F** upon itself. This is an image of the way objects become tokens for eigenbehaviors, in the language of Heinz von Foerster.

Here we show that eigenforms exist in reflexive domains without an infinite limit. The interested reader should compare this argument with the work of William Lawvere (1972). Lawvere proves a more general result in the context of Cartesian closed categories. We have taken his argument and shaped it particularly for this discussion of reflexivity.

Fixed Point Theorem. *Let D be a reflexive domain with 1–1 correspondence $F: D \to [D, D]$. Then every F in [D, D] has a fixed point. That is, there exists a p in D such that $F(p) = p$.*

Proof. Define $G: D \to D$ by the equation $Gx = F(I(x)x)$ for each x in D.

Since $I:D \to [D,D]$ is a $1 - 1$ correspondence, we know that $G = I(g)$ for some g in D.

Hence $Gx = I(g)x = F(I(x)x)$ for all x in D.

Therefore, letting $x = g$, $I(g)g = F(I(g)g)$ and so $p = I(g)g$ is a fixed point for F.

Q.E.D.

We shall discuss this proof and its meaning right now in a series of remarks, and later in the paper in regard to examples that will be constructed.

Remark 1

Suppose that we reduce the notational complexity of our description of the reflexive domain by simply saying that for any two entities g and x in the domain there is a new entity gx that is the result of the interaction of g and x. (We think of gx as I(g)x = I(g) applied to x.)

In mathematical terms, we define

$$gx = I(g)x.$$

Then the proof of the fixed point theorem appears in a simpler form: we define Gx = F(xx) and note that GG = F(GG).

Thus GG is the fixed point for F!

I like to call G "F's Gremlin."[1]

This is an apt description of our G. At first G looks quite harmless. Applying G to any A we just apply A to itself and apply F to the result. GA = F(AA). The dangerous mixture comes when it is possible to apply G to itself! Then GG = F(GG), and GG is sitting right in there surrounded by F and you cannot stop the action. Off goes the recursion

$$\begin{aligned} GG &= F(GG) \\ &= F(F(GG)) \\ &= F(F(F(F(GG)))) \\ &= F(F(F(F(F(F(F(F(GG)))))))) \end{aligned}$$

The diabolical nature of the Gremlin is that he represents a process that once started, is hard to stop.

Gremlins seem innocent. They just duplicate entities that they meet, and set up an operation of the duplicate on the duplicand. But when you let a gremlin meet a gremlin then strange things can happen. It is a bit like the story of the sorcerer's apprentice. A recursion may happen whether you like it or not.

Such are the processes by which we make the world into a field of tokens and symbols and forget the behaviors and processes and reflexive spaces from which they came. Fixed points and self-references are the unavoidable fruits of reflexivity, and reflexivity is the natural condition in a universe where there is no complete separation of part from the whole.

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[1] See Kauffman (2001). According to Webster's New Collegiate Dictionary (1956) a gremlin is "one of the impish foot-high gnomes whimsically blamed by airmen for interfering with motors, instruments, machine guns, etc.; hence any like disruptive elf."

Remark 2

A reflexive domain is a place where actions and events coincide. An action is a mapping of the whole space because there is no intrinsic separation of the local and the global. Feedback is an attempt to handle the lack of separation of part and whole by describing their mutual influence.

When we define a new element g of D via $gx = F(x)$ for any mapping $F: D \to D$, and we have a notion of the combination of elements of D: $a,b \to ab$, then we can define $gx = F(xx)$ and so get $gg = F(gg)$. Here we have not made a big separation between the elements of D and the mappings, since each element g of D gives the mapping $I(g)x = gx$. But in fact, we could define $ab = I(a)b$ in a reflexive domain.

Whenever anyone comes up with a transformation, we make that transformation into an element of the domain by the definition $gx = F(x)$. We transmute verbs to nouns. The reflexive domain evolves.

The space is not given a priori. The space evolves in relation to actions and definitions. The road unfolds before us as we travel.

Remark 3

We create languages for evolving concepts. The outer reaches of set theory (and category theory) lead to clear concepts, but these concepts are not themselves sets or categories. A good example is the famous Russellian concept of sets that are not members of themselves. Russell's concept is not a set. Another example is the concept of set itself. There is no set that is the set of all sets.

This very limitation on the notion of a set is its opening. It shows us that set theory can be an evolving language. Language and concepts expand in time.

Here is a transformation on sets: $F(X) = \{X\}$. The transform of a set X is the singleton set whose member is X. If X is not a member of itself, then F(X) is also not a member of itself. But a fixed point of the transformation F is an entity U such that $\{U\} = U$. We have shown that within the domain of sets that are not members of themselves, there is no fixed point for the transformation $X \to \{X\}$. This fragment of set theory (sets that are not members of themselves) is not yet a reflexive domain. We shall allow sets that are members of themselves if we wish to have a set theory with reflexivity.

Remark 4: Transcendence

The leap to infinity via self-reference, the production of the finite base of a new level of infinity, the completion of an incompletion, the emergence of eternity from the world of time – all these metaphors are intimately related to the going back and forth between a process and its eigenform.

How then is observation different from action?

If observation is a form of recursion coupled with the production of the finite base of the limiting form, then observation is a transcendence to a new level. The model of observation as a simple eigen-vector must be shifted to a model of observation as the act of producing an eigenform.

It is not enough to produce an eigenform. The fixed point is itself an active element and can itself engage in transformation.

In the creation of spaces of conversation for human beings, we partake of a reflexivity of action and apparent object, where it is seen that every local manifestation of process, every seemingly fixed entity in a moving world is an indicator of global transformation. The local and the global intertwine in a reflexive and cybernetic unity.

Retuning (returning/tuning/retuning) to thoughts of reflexivity, one creates by going outside oneself, but the creation returns in the form of a conversation with one's self. There is a feedback loop between the person/designer and the world that she makes.

Each one acts in the creation of the other. Priorities may be assigned, but it is the loop that interests us, and the possibility of the stability (or at least temporal persistence) of what is created in that loop.

Remark 5: The magma as reflexive domain

A magma is an algebraic system with a binary operation $\mathbf{a * b}$ that is

left-distributive: $\mathbf{a * (b * c) = (a * b) * (a * c)}$. This means that every element of the magma is a structure preserving mapping of the magma to itself (via left multiplication). A magma is composed of its own symmetries.

It may help the reader to see how elements of a magma become mappings of the magma to itself, preserving the combinational structure. Let $\mathbf{A(x) = a * x}$ for a given element a in a magma $\mathbf{M}$.

Then $\mathbf{A: M \to M}$ and

$\mathbf{A(x * y) = a * (x * y) = (a * x) * (a * y) = A(x) * A(y)}$.

Thus for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{M}$ we have $\mathbf{A(x * y) = A(x) * A(y)}$. Each element of the magma gives rise, by left multiplication, to a structure-preserving mapping of the magma to itself.

Here is an example of a magma. Let $\mathbf{TRI = \{a, b, c\}}$ be a set with three distinct elements $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Define $\mathbf{a * a = a}$, $\mathbf{b * b = b}$ and $\mathbf{c * c = c.}$ And define $\mathbf{a * b = c = b * a, a * c = b = c * a}$ and $\mathbf{b * c = a = c * b}$.

In other words, each element combines with itself to produce itself, and any pair of distinct elements combine to produce the remaining element that is different from either of them. The reader can verify that TRI is indeed a magma. For example,

$$\mathbf{a * (b * c) = a * (a\ ) = a}$$
$$\mathbf{(a * b) * (a * c) = (c) * (b) = a}.$$

Note also that the multiplication in this magma is not associative:

$$\mathbf{a * (a * b) = a * c = b}$$
$$\mathbf{(a * a) * b = a * b = c}.$$

We will return to this magma in the next section and see that **TRI** is intimately related to the simplest knot, the trefoil knot.

Another example to think about is **OM**, the free magma generated by one element **J**. Here we consider all possible expressions and ways that b can combine with itself and with other elements generated from itself. Remarkably, the free magma is an infinitely complex structure. For example, note the following consequences of the distributive law (here using **XY** instead of **X * Y**):

$$\mathbf{J(JJ) = ((JJ)(JJ))}$$
$$\mathbf{= ((JJ)J)((JJ)J)}$$
$$\mathbf{= (((JJ)J)(JJ))(((JJ)J)J))}.$$

In the free magma an infinite structure is generated from one element and all its patterns of self-interaction.

Suppose further that we assume that every structure-preserving mapping of the magma **M** is represented by an element of the magma **M**. This will place us in the position of creating from the magma something like a reflexive domain.

In the next section we shall see that magmas arise very naturally in the topology of knots and links in three-dimensional space. This is an excellent way to think about them, and it provides a way to think about reflexivity in terms of topology. Here we take an abstract point of view and see when the structure-preserving nature of elements of a magma leads to the analog of a reflexive domain.

I shall call a magma **M** *reflexive* if it has the property that every structure-preserving mapping of the algebra is realized by an element of the algebra *and* **(x * x) * z = x * z** for all **x** and **z** in **M**.

A special case of this last property would be where **x * x = x** for all **x** in **M**. We shall see this property come up in the knot theoretic interpretations of the next section.

Suppose that **M** is a reflexive magma. Does **M** satisfy the fixed point theorem? We find that the answer is, yes:

Fixed Point Theorem for Reflexive Magmas. Let **M** be a reflexive magma. Let **F: M → M** be a structure-preserving mapping of **M** to itself. Then there exists an element **b** in **M** such that **F(p) = p**.

Proof. Let **F: M → M** be any structure-preserving mapping of the magma M to itself. This means that we assume that **F(x * y) = F(x) * F(y)** for all **x** and **y** in **M**. Define **G(x) = F(x * x)** and regard **G: M → M**. Is **G** structure preserving? We must compare **G(x * y) = F((x * y) * (x * y)) = F(x * (y * y))** with **G(x) * G(y) = F(x * x) * F(y * y) = F((x * x) * (y * y)).**

Since **(x * x) * z = x * z** for all **x** and **z** in **M**, we conclude that **G(x * y) = G(x) * G(y)** for all **x** and **y** in **M**.

Thus **G** is structure preserving and hence there is an element **g** of **M** such that **G(x) = g * x** for all **x** in **M**. Therefore we have **g * x = F(x * x)**, whence **g * g = F(g * g)**. For **p = g * g,** we have **p = F(p)**. This completes the proof. //

This analysis shows that the concept of a magma is very close to our notion of a reflexive domain. The examples of magmas related to knot theory, given in the previous section, show that magmas are not just abstract structures, but are related directly to the properties of space and topology in the worlds of communication and perception in which we live.

7. Knot sets, topological eigenforms and the left-distributive magma

We shall use knot and link diagrams to represent sets. More about this point of view can be found in the author's paper "Knot Logic" (Kauffman 1995). In this notation the eigenset **Ω** satisfying the equation

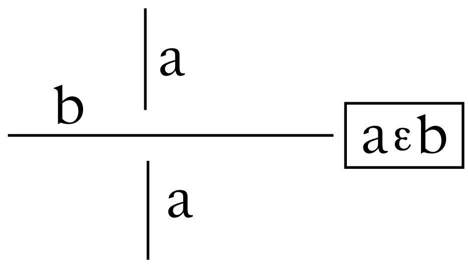
$$\Omega = \{\Omega\}$$

is a topological curl. If you travel along the curl you can start as a member and find that after a while you have become the container.

Further travel takes you back to being a member in an infinite round. In the topological realm, Ω does not have any associated paradox. This section is intended as an introduction to the idea of *topological eigenforms*, a subject that we shall develop more fully elsewhere.

Set theory is about an asymmetric relation called *membership.*

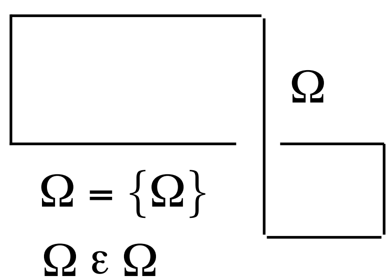
We write **a Ɛ S** to say that **a** is a member of the set **S.** In this section we shall diagram the membership relation as follows:



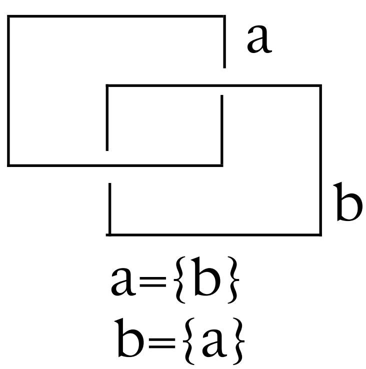
This is *knot-set notation.*

In this notation, if b goes once under a, we write a = {b}. If b goes twice under a, we write a = {b, b}. This means that the “sets” are multi-sets, allowing more than one appearance of a member. For a deeper analysis of the knot-set structure see [KL **REFERENCE?**].

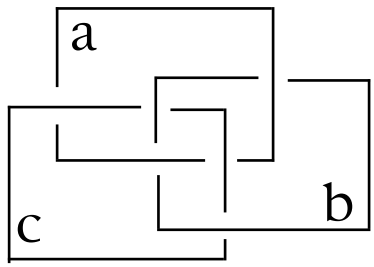
This knot-set notation allows us to have sets that are members of themselves,



and sets can be members of each other.



Here a mutual relationship of **a** and **b** is diagrammed as a topological linking.



a = {b,b}
b = {c,c}
c = {a,a}

Here are the *Borromean Rings.* The Rings have the property that if you remove any one of them, then the other two are topologically unlinked. They form a topological tripartite relation. Their knot-set is described by the three equations in the diagram.

Thus we see that this representative knot-set is a "scissors-paper-stone" pattern. Each component of the Rings lies over one other component, in a cyclic pattern.

Remark. The connection between this formalism and epistemic logic (Hintikka 1962) should be further explored. In epistemic logic the basic expressions are of the form KaKbp ("a knows that b knows that p").

One specific thing to explore is the problem of common knowledge, which can only be reduced to an infinite number of K's as in

E = KaKbKaKbKaKb...

denoting that "a knows that b knows that a knows that ...."

We can write this as

E = KaF
F = KbE

indicating that

E = "a knows F"
F = "b knows E."

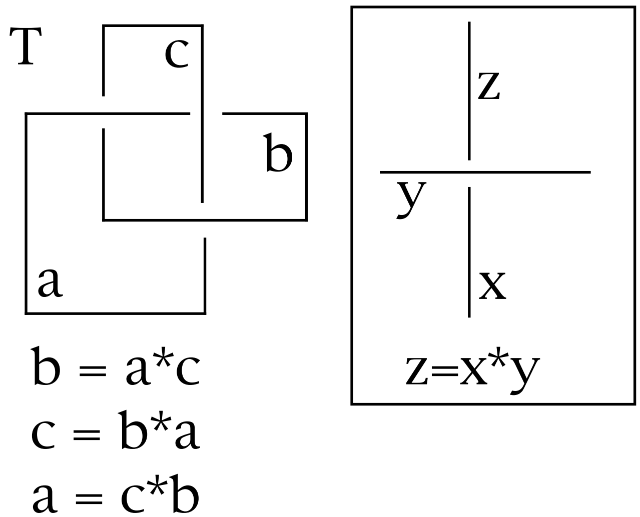
Together these statements indicate common knowledge or mutuality for **a** and **b**. Conversely, we can take the linked sets **A = {B} and B = {A}** as a statement of common knowledge.

Another avenue that should be explored is the relationship between knot set theory and Aczel's theory of self-referential and non-wellfounded sets (Aczel 1988) and the related treatment by Barwise and Moss (1996).

Quandles and colorings of knot diagrams

There is an approach to studying knots and links that is very close to our knot sets, but starts from a rather different premise.

In this approach each arc of the diagram receives a label or "color." An arc of the diagram is a continuous curve in the diagram that starts at one undercrossing and ends at another undercrossing. For example, the trefoil diagram below has three arcs.



Each arc corresponds to an element of a "Trefoil Color Algebra" **IQ(T)**, where **T** denotes the trefoil knot. The algebra is generated by colors **a**, **b** and **c** with the relations

$$\begin{gathered}\mathbf{a} * \mathbf{a} = \mathbf{a},\\ \mathbf{b} * \mathbf{b} = \mathbf{b},\\ \mathbf{c} * \mathbf{c} = \mathbf{c},\\ \mathbf{a} * \mathbf{b} = \mathbf{b} * \mathbf{a} = \mathbf{c},\\ \mathbf{b} * \mathbf{c} = \mathbf{c} * \mathbf{b} = \mathbf{a},\\ \mathbf{a} * \mathbf{c} = \mathbf{c} * \mathbf{a} = \mathbf{b}.\end{gathered}$$

Each of these relations in the diagram above is a description of one of the crossings in **T**. The full set of relations describes the coloring rules for an algebra that contains these relations and allows any two elements to be combined to a third element. This three-element algebra is particularly simple. If two colors are different, they combine to form the remaining third color. If two colors are the same, they combine to form the same color.

When we take an algebra of this sort, we want its coloring structure to be invariant under the Reidemeister moves (illustrated below).

This means that when you make a new diagram from the old diagram by a topological move, the resulting new diagram inherits a unique coloring from the old diagram. Then one can see from this that the trefoil must be knotted since all diagrams topologically equivalent to it will carry three colors, while an unknotted diagram can carry only one color.

As the next diagram shows, invariance of the coloring rules under the Reidemeister moves implies the following *global relations on the algebra:*

$$\mathbf{x * x = x}$$
$$\mathbf{(x * y) * y = x}$$
$$\mathbf{(x * y) * z = (x * z) * (y * z)}$$

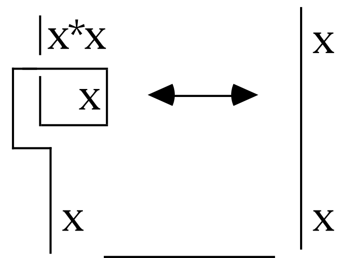
for any **x**, **y** and **z** in the algebra (set of colors) **IQ(T)**.

An algebra that satisfies these rules is called an *Involutory Quandle* (Kauffman 1995), hence the initials **IQ**. Perhaps the most remarkable property of the quandle is its right-distributive law corresponding to the third Reidemeister move, as illustrated below. The reader will be interested to observe that in a multiplicative group **G**, the following operation satisfies all the axioms for the quandle: $\mathbf{g * h = hg^{-1}h}$.

In an additive and commutative version of this axiom we can write $\mathbf{a * b = 2b - a}$. Here the models that are most useful to the knot theorist are to take **a** and **b** to be elements of the integers **Z** or elements of the modular number system $\mathbf{Z/dZ = Z_d}$ for some appropriate modulus **d**. The knot being analyzed restricts the modular possibilities. In the case of the trefoil knot the only possibility is **d = 3**, and in the case of the Figure Eight knot (shown after the Reidemeister moves below) the only possibility is **d = 5**.

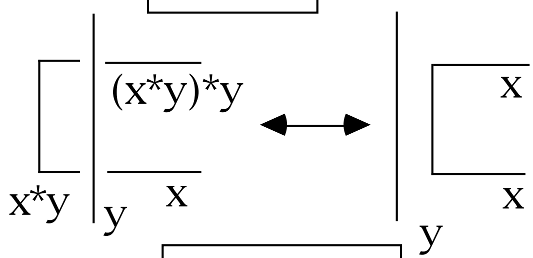
This analysis then shows that there cannot be any sequence of Reidemeister moves connecting the Trefoil and the Figure Eight. They are distinct knot types.

I.



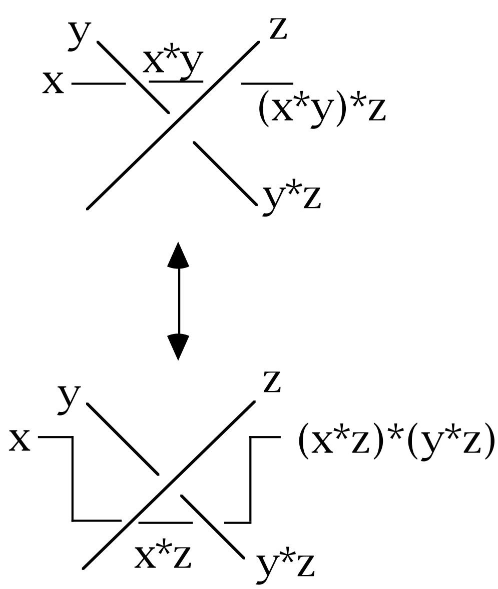
$$\boxed{x*x=x}$$

II.



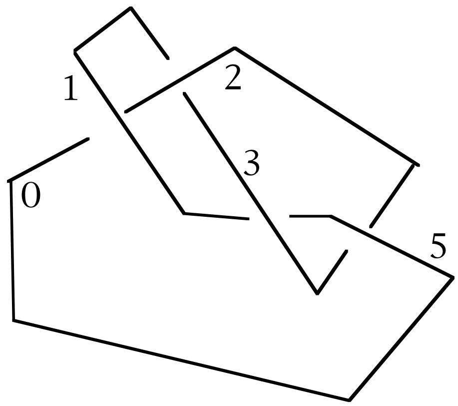
$$\boxed{(x*y)*y = x}$$

III.



$$\boxed{(x*y)*z = (x*z)*(y*z)}$$

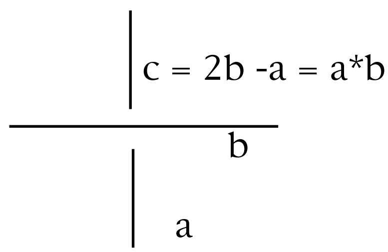
Here is the example for the Figure Eight Knot.



2 x 1 -0 = 1
2 x 2 -1 = 3
2 x 3 - 1 = 5
-> 0 = 5

Z/5Z = {0,1,2,3,4} with 0 = 5.

We have shown how an attempt to label the arcs of the knot according to the quandle rule



a * b = 2b – a leads to a labelling of the Figure Eight knot in **Z/5Z**. In our illustration we have shown that there is a compatible coloring using four out of the five elements of **Z/5Z**. If you apply Reidemeister moves to the diagram for the Figure Eight knot you will see that other versions of the knot require all five colors. It is interesting to prove that there is no diagram of the Figure Eight knot that can be colored in less than four colors.

It should be noted that the knot diagrams give a remarkable picture of non-associative algebra structure and that each arc-label **a** in a diagram is both an element of the algebra and a transformation of the algebra to itself via the mapping $\mathbf{O_a(x) = x * a}$.

Note that the right distributivity of this operation has the equation

$$\mathbf{O_a(x * y) = (x * y) * a = (x * a) * (y * a) = O_a(x) * O_a(y)}$$

That is, we have

$$\mathbf{O_a(x * y) = O_a(x) * O_a(y)}.$$

The right distributive law tells us that e*ach quandle operation is a quandle homomorphism*. That is, each quandle operation is a structure-preserving mapping of the quandle to itself. This is an underlying algebraic meaning of the third Reidemeister move. Since the mappings $\mathbf{O_a}$ are invertible, we see that any quandle **Q** is in 1–1 correspondence with a certain collection of automorphisms of itself. In this sense a quandle is a reflexive domain with a limitation on the allowable collection of self-mappings. In fact we have a very simple fixed point theorem for quandles since

$$\mathbf{O_a(a) = a * a = a}.$$

Every element of the quandle is fixed by its own automorphism.

Since we take **[Q, Q]** to be the set of mappings of **Q** to itself of the form $\mathbf{O_a(x) = x * a}$, we see that any quandle is a reflexive domain of a restricted sort. (Not every set theoretic mapping of **Q** to **Q** is realized in the above manner.)

How far is the quandle from being a reflexive space in the full sense of the word? Let us look at the fixed point construction. We define

G(x) = (x * x) * F for a given element **F** of the quandle.

Is it then the case that **(x * x) * F = x * g** for some g in the quandle?

The answer is, yes, but for a very simple reason:

We have **x * x = x** so that **(x * x) * F = x * F** and consequently **(F * F) * F = F * F**. In fact, **F * F = F**, so **F** is already its own fixed point. We therefore see that in a quandle the fixed point theorem is satisfied automatically due to the axiom **x * x = x** for all **x**.

On the other hand, if **F:Q → Q i**s an arbitrary mapping from **Q** to **Q**, then we cannot expect that **F** will have a fixed point. For example, in the trefoil quandle **TRI** , suppose we define **F(a) = b, F(b) = c** and **F(c) = a**. Then **F** has no fixed point. Note that **F** is a structure-preserving mapping. (In this case the composition of F with itself three times fixes everything. If we make transformations that are permutations of finite sets, then they may be fixed-point free, but some powers of them will certainly have fixed points.)

We have **F(x * y) = F(x) * F(y)** for all **x** and **y** in **TRI**. For example, **F(a * b) = F(c) = a = b * c = F(a) * F(b)**.

In order to extend **TRI** to a reflexive (right-distributive) magma we would have to add an element **f** to the algebra such that **x * f = F(x)** for each **x** in **TRI**, take the consequences of that and continue. We leave the exploration of this extension to the reader.

Left distributivity

We have written the quandle as a right-distributive structure with invertible elements. It is mathematically equivalent to use the formalism of a left distributive operation. In left distributive formalism we have **A * (b * c) = (A * b) * (A * c)**. This corresponds exactly to the interpretation that each element **A** in **Q** is a mapping of **Q** to **Q** where the mapping **A[x] = A * x** is a structure-preserving mapping from **Q** to **Q**.

$$\mathbf{A[b * c] = A[b] * A[c]}.$$

We can ask of a domain that *every element of the domain is itself a structure-preserving mapping of that domain.* This is very similar to the requirement of reflexivity and, as we have seen in the case of quandles, can often be realized for small structures such as the Trefoil quandle.

We call a domain M with an operation * that is left distributive a *magma*. Magmas are more general than the link diagrammatic quandles. We take only the analog of the third Reidemeister move and do not assume any other axioms A magma with no other relations than left-distributivity is called a *free magma.*

The search for structure-preserving mappings can occur in rarefied contexts. See, for example, the work of Laver and Dehornoy (2000; Kauffman 1995), who studied mappings of set theory to itself that would preserve all definable structure in the theory. Dehornoy realized that many of the problems he studied in relation to set theory were accessible in more concrete ways via the use of knots and braids. Thus the knots and braids become a language for understanding the formal properties of self-embedded structures.

Structure-preserving mappings of set theory must begin as the identity mapping since the relations of sets are quite rigid at the beginning. (You would not be able to map an empty set to a set that was not empty for example, and so the empty set would have to go to itself.) The existence of non-trivial structure-preserving mappings of set theory questions the boundaries of definability and involves the postulation of sets of very large size. See Piechocinska (2005) for a good exposition of the philosophical issues about such embeddings and for an approach to wholeness in physics that is based on these ideas.

It is worth making a remark here about sets. Consider the collection Aleph of all sets whose members are themselves sets and such that any investigation into membership will just reveal more sets as members. Typical elements of Aleph are the empty set { }, the set whose member is the empty set { { } } and of course various curious constructs that have infinitely many members such as { { } , {{ }}, {{{ }}}, {{{{ }}}}, ... } and we may even consider sets that are members of themselves (eigen-sets!) such as { { { { { ... } } } } }.

The key thing to understand about Aleph as a class of sets is that any member of Aleph is, by definition, a subset of Aleph. And any subset of Aleph is, by definition, a member of Aleph. This is a beautiful property of the class Aleph, and it is a paradoxical property if we imagine that Aleph is a set! For if Aleph is a set, then we have just shown that Aleph is in 1–1 correspondence with the set of subsets P(Aleph) of Aleph. If X is any set then we denote the set of subsets of X by P(X). Cantor's Theorem (proved here in Section 8 and related in that section to the fixed point theory of reflexive domains) tells us that *for any set X, P(X) is larger than X.*

This means that there cannot be a 1–1 correspondence between Aleph and P(Aleph) if Aleph is a set. We can only conclude that Aleph is not a set. It is a class, to give it a name. It is an unbroken wholeness whose particularities we can always consider, but whose totality will always elude us. The way that the totality of Aleph eludes us is right before our eyes. Any particular element of Aleph is a set and is a collection of sets as well. But we cannot complete Aleph. Any attempt to approximate Aleph as a set will always have some subsets that have not been tallied inside itself and so the set of subsets of the approximation will grow beyond that approximation to a new and larger domain of sets. Philosophically, this observation of the unreachability of Aleph, the set of all sets, as a set itself is very interesting and important. We see here how a perfectly clear mathematical concept may always remain outside the bounds of the formalities to which it refers and yet that concept is indeed composed of these formalities. It is the leading presence of the ultimately huge and unattainable Aleph that leads us to consider exceeding large sets in the pursuit of a flexibility in the self-embeddings of set theory. At the end of Section 8, we take an alternative view of Aleph and consider what would have to change if Aleph were admitted to be a set.

Enough said about the abstract reaches of the magma.

We should not expect that any given structure is a reflexive space. But it is possible to create languages that can expand indefinitely and thus partake of the ideal of reflexivity.

8. Church and Curry

In this section we point out how the notion of a reflexive domain first appeared in the work of Alonzo Church and Haskell Curry (Barendregt 1984) in the 1930s. This method is commonly called the “lambda calculus.” The key to lambda calculus is the construction of a self-reflexive language, a language that can refer and operate upon itself. In this way eigenforms can be woven into the context of languages that are their own metalanguages, hence into the context of natural language and observing systems.

In the Church-Curry language (the lambda calculus), there are two basic rules:

1. Naming. *If you have an expression in the symbols in lambda calculus then there is always a single word in the language that encodes this expression.* The application of this word has the same effect as the application of the expression itself.

2. Reflexivity. *Given any two words, A and B, in the lambda calculus, there is permission to form their concatenation AB, with the interpretation that A operates upon or qualifies B.* In this way, every word in the lambda calculus is both an operator and an operand. The calculus is inherently self-reflexive.

Here is an example. Let **GA** denote the process that creates two copies of **A** and puts them in a box.

$$GA = \boxed{AA}$$

In lambda calculus we are allowed to apply **G** to itself. The result is two copies of **G** next to one another, inside the box.

$$GG = \boxed{GG}$$

This equation about **GG** exhibits **GG** directly as a solution to the eigenform equation

$$X = \boxed{X}$$

thus producing the eigenform without an infinite limiting process.

More generally, we wish to find the eigenform for a process **F**. We want to find a **J** so that **F(J) = J**. We create an operator **G** with the property that

$$\mathbf{GX = F(XX)}$$

for any **X**. When **G** operates on **X**, **G** makes a duplicate of **X** and allows **X** to act on its duplicate. Now comes the kicker.

Let **G** act on itself and look!

$$\mathbf{GG = F(GG)}$$

So **GG** is a fixed point for **F**.

We have solved the eigenform problem without the excursion to infinity. If you reflect on this magic trick of Church and Curry you will see that it has come directly from the postulates of **Naming** and **Reflexivity** that we have discussed above. These notions, *that there should be a name for everything,* and that *words can be applied to the description and production of other words*, allow language to refer to itself and to produce itself from itself. The Church-Curry construction was devised for mathematical logic, but it is fundamental to the logic of logic, the linguistics of linguistics and the cybernetics of cybernetics.

An eigenform must be placed in a context in order for it to have human meaning. The struggle on the mathematical side is to control recursions, bending them to desired ends. The struggle on the human side is to cognize a world sensibly and to communicate well and effectively with others. For each of us, there is a continual manufacture of eigenforms (tokens for eigenbehavior). Such tokens will not pass as the currency of communication unless we achieve mutuality as well. Mutuality itself is a higher eigenform. As with all eigenforms, the abstract version exists. Realization happens over the course of time.

9. Cantor's diagonal argument and Russell's paradox

Let **AB** mean that *B is a member of A*.

Cantor's Theorem. Let S be any set (S can be finite or infinite).

Let P(S) be the set of subsets of S. Then P(S) is bigger than S in the sense that for any mapping $F: S \rightarrow P(S)$ there will be subsets C of S (hence elements of F(S)) that are not of the form F(a) for any a in S. In short, the power set P(S) of any set S is larger than S.

Proof. Suppose that you were given a way to associate to each element x of a set S a subset F(x) of S. Then we can ask whether x is a member of F(x). Either it is or it isn't. So let us form the set of all x such that x is not a member of F(x). Call this new set C. We have the defining equation for C:

$$\mathbf{Cx = \sim F(x)x}.$$

Is C = F(a) for some a in S?

If C = F(a) then for all x we have F(a)x = ~F(x)x.

Take x = a. Then F(a)a = ~F(a)a.

This says that a is a member of F(a) if and only if a is not a member of F(a). This shows that indeed C cannot be of the form F(a), and we have proved Cantor's Theorem that the set of subsets of a set is always larger than the set itself. //

Note the problem that the assumption that C = F(a) gave us.

If C = F(a), then F(a)a = ~F(a)a. We would have a fixed point for negation. But there is no fixed point for negation in classical logic!

If we had enlarged the truth set to

$$\mathbf{\{T, F, I\}}$$

where **~I = I** is an eigenform for negation, then **F(a)a** would have value **I**. What does this mean? It means that the index **a** of the corresponding set **F(a)** would have an oscillating membership value. The element **a** would be like Groucho Marx, who declared that he would not join any club that would have him as a member. We would be propelled into sets that vary in time.

Note that our proof of Cantor's Theorem has exactly the same form as our earlier proof of the existence of fixed points for a reflexive space. The mapping $F: X \rightarrow P(X)$ takes the role of the 1–1 correspondence between D and [D, D]. The reader will enjoy thinking about this analogy. In the Cantor Theorem we have used the non-existence of a fixed point for negation to deduce a difference between set X and its power set P(X). In the study of a reflexive domain we have shown the existence of fixed points, but we have seen that such domains must be open to new elements and new transformations.

Note also how close Cantor's Theorem is to Russell's famous paradox.

Russell devised the set **R** defined by the equation

$$\mathbf{Rx = \sim xx}.$$

An element x is a member of the Russell set if and only if x is not a member of itself.

To see the contradiction, substitute **R** for x and get

$$\mathbf{RR = \sim RR}.$$

This appearance of an eigenform for negation tells us that we either must concede temporality to Russell's construction **R**, or else banish it from the world of sets.

10. The secret

What is the simplest language that is capable of self-reference?

We are all familiar with the abilities of natural language to refer to itself. Why this very sentence is an example of self-referentiality. The American dollar bill declares, "This bill is legal tender." The sentence that you are now reading declares that you, the reader, are complicit in its own act of reference. But what is the simplest language that can refer to itself?

The simplest language would have a simple alphabet. Let us say it has only the letter R. The words in this language will be all strings of Rs. Call the language LS. The words in LS are the following:

R,
RR,
RRR,
RRRR,
and so on.

Two words are equal if they have the same number of letter Rs.

E*ach word makes a meaningful statement of reference via the rule:*

If X is a word in LS, then RX refers to XX.

RX refers to XX, the repetition of X.

Thus RRR refers to RRRR (not to itself), and R refers to the empty word.

There is a word in LS that refers to itself. Can you find it?

Let us see.

RX refers to XX.

So we need XX = RX if RX were to refer to RX.

If XX = RX, then X = R.

So we need X = R.

And RR refers to itself.

The little language LS looks like a pedantic triviality, but it is actually at the root of reflexivity, Gödel's incompleteness Theorem, recursion theory, Russell's paradox and the notion of self-observing and self-referring systems. It seems paradoxical that what looks like a trick of repeating a symbol can be so important. The trick is more than just a trick.

The Russell paradox (see the previous section) continues to act as a mystery at the center of our attempts to relate syntax and semantics. In that center is a little trick of syntactical repetition.

I would like to think that when we eventually discover the true secret of the universe it will turn out to be this simple.

The snake bites its tail. The Universe is constructed in such a way that it can refer to itself. In so doing, the Universe must divide itself into a part that refers and a part to which it refers, a part that sees and a part that is seen.

Let us say that R is the part that refers and U is the referent. The divided universe is RX and RX = U and RX refers to U (itself). Our solution suggests that the Universe divides itself into two identical parts, each of which refers to the universe as a whole. This is

RR.

In other words, the universe can pretend that it is two and then let itself refer to the two, and find that it has in the process referred only to the one, that is, itself.

The Universe plays hide and seek with herself, pretending to divide itself into two when it is really only one. And that is the secret of the Universe and that is the universal source of our trick of self-reference.

11. The World of Recursive Emergence and Creativity

We have repeatedly insisted that a formal fixed point or eigenform is associated with any transformation T in any domain where infinite composition of transformations is possible. Thus we make

$E = T(T(T(T(T(...)))))$ and find that $E = T(E)$. This is the symbolic fixed point that sometimes corresponds to a stability in the original domain of the recursion. We have also seen that one can take a seed z for the recursion and repeatedly form

$$z, T(z), T(T(z)), T(T(T(z))), ...$$

in a temporal sequence or recursive process. Then the finite products of this process can exhibit similarity to the infinite eigenform, and they can also exhibit novelty and emergence structure in ways that are most surprising. It is this appearance of creativity and novelty in recursive process that makes reflexivity more than abstract mathematics and more than a philosophical idea.

The purpose of this last section is to exhibit an example involving cellular automata that illustrates these ideas and gives us a platform for thought. In this example, we are using an algorithm that I call "7-Life." It is a variant of the Life automaton of John H. Conway **(Gardner (1970)**).

Conway's automaton is governed by the rule B3/S23 which means that a white square in the grid is born (B) when it has 3 neighbors and it survives (S) when it has exactly 2 or 3 neighbors. Life has the property that there are many intriguing formations and processes, but statistically most configurations die out to a collection of isolated static patterns (still lifes) and oscillating patterns that do not grow and do not interact outside themselves.

We should mention that there are a vast number of different cellular automata. A good start in learning about these structures is the book by Stephen Wolfram (2002). Wolfram's book concentrates almost entirely on one-dimensional cellular automata and achieves a qualitative classification of the behaviors found in a comprehensive class of the simplest types of line automata. Wolfram finds that a number of these simplest automata are computationally universal in the sense that they can simulate a universal Turing machine on the one dimensional lattice of the automaton. The automata that we are looking at in this section, and 7-Life in particular, are two dimensional and hardly considered by Wolfram in his treatise. Conway's Life is also not analyzed in Wolfram. In fact, it was shown that Conway Life is Turing universal by Conway and his collaborators prior to the onset of Wolfram's work in the 1980s. We mention this background and the difference in dimensionalities to give the reader some perspective so that he will not be surprised and wonder "Why is this phenomenon not discussed in Wolfram?" Indeed the phenomenon of the remarkable emergence of complexity from simple algorithms is the theme of Wolfram's work. This theme plays significantly in all algorithmic mathematics and in all significant studies of cellular automata.

6-Life, defined by the rule B36/S23 is well-known, as a search on the Internet for Life automata will reveal. 6-Life does not have the qualitative self-sustaining properties that are evident in 7-Life, but there are emergent structures there as well. We concentrate here on 7-Life because the long-term self-sustaining interactions of this automaton make it ideal for studies of long term evolution and the emergence of forms.

7-Life has the rule B37/S23 and has many of the properties of Life, plus the phenomenon that many starting configurations grow, self-interact and produce streams of *gliders*. The gliders are five-square formations (occurring in Life as well) that occur spontaneously and regenerate themselves, appearing to move along diagonal directions in the process. The most striking property of 7-Life is the long term persistence of such self-interacting configurations, growing slowly in complexity over time.

In Figures 1, 2, and 3 we indicate the result of applying the 7-Life algorithm to a simple and not-quite symmetrical starting configuration, shown in Figure 1. In Figure 2 we see the result of 33911 iterations of the process. We now have a galaxy of complex interactions. The small entities radiating away from the galaxy are gliders, as described above, and if a reader were to watch the process using a computer program, he or she would see a teeming, seemingly random mass of activity. Then in Figure 3 we see that after 49281 iterations something new has emerged. It seems that a highly patterned *dragon* is emerging from the chaos of the complex process. The tip of this dragon moves forward relentlessly.

The body of the dragon interacts with the glider radiation and begins to roil in the chaotic process. So far, the growing tip of the dragon has not interacted with any gliders.

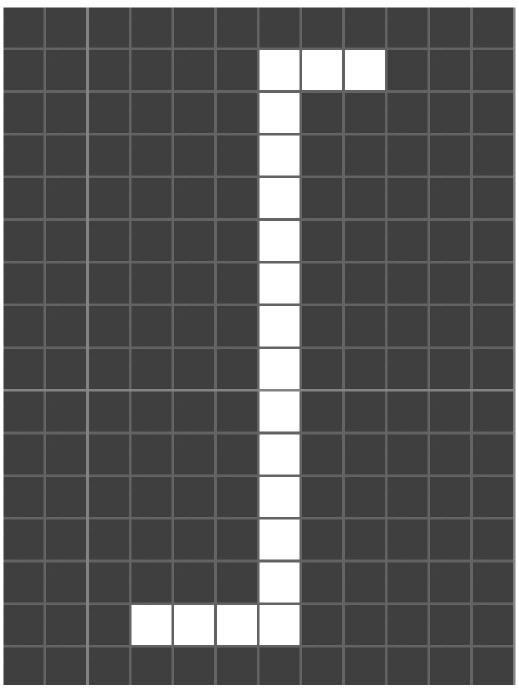


Figure 1. The starting configuration

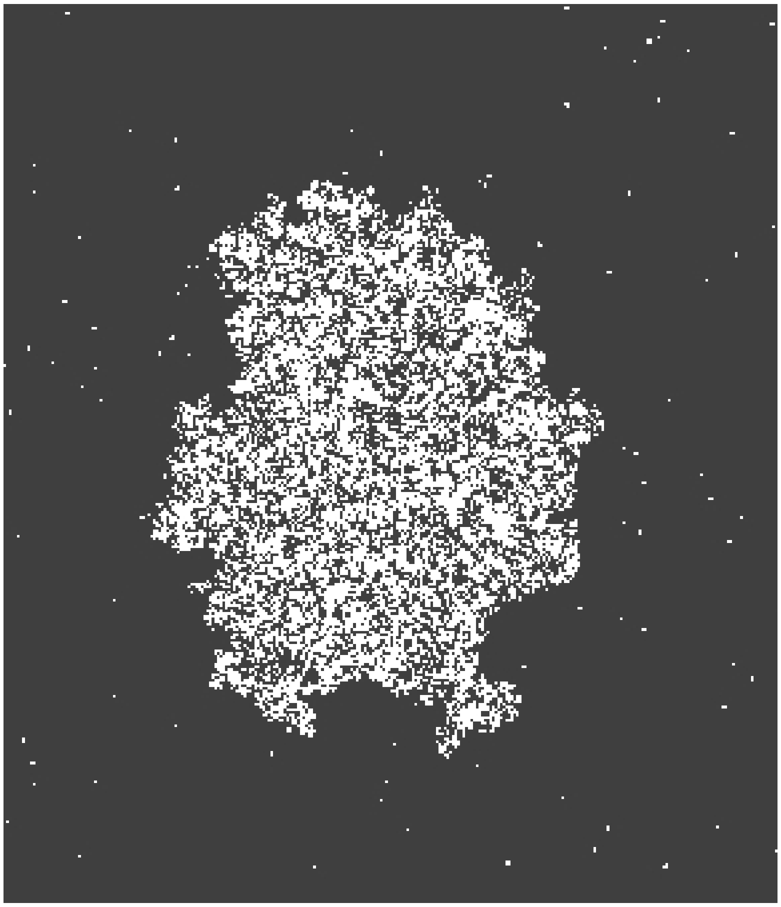


Figure 2. After 33911 iterations

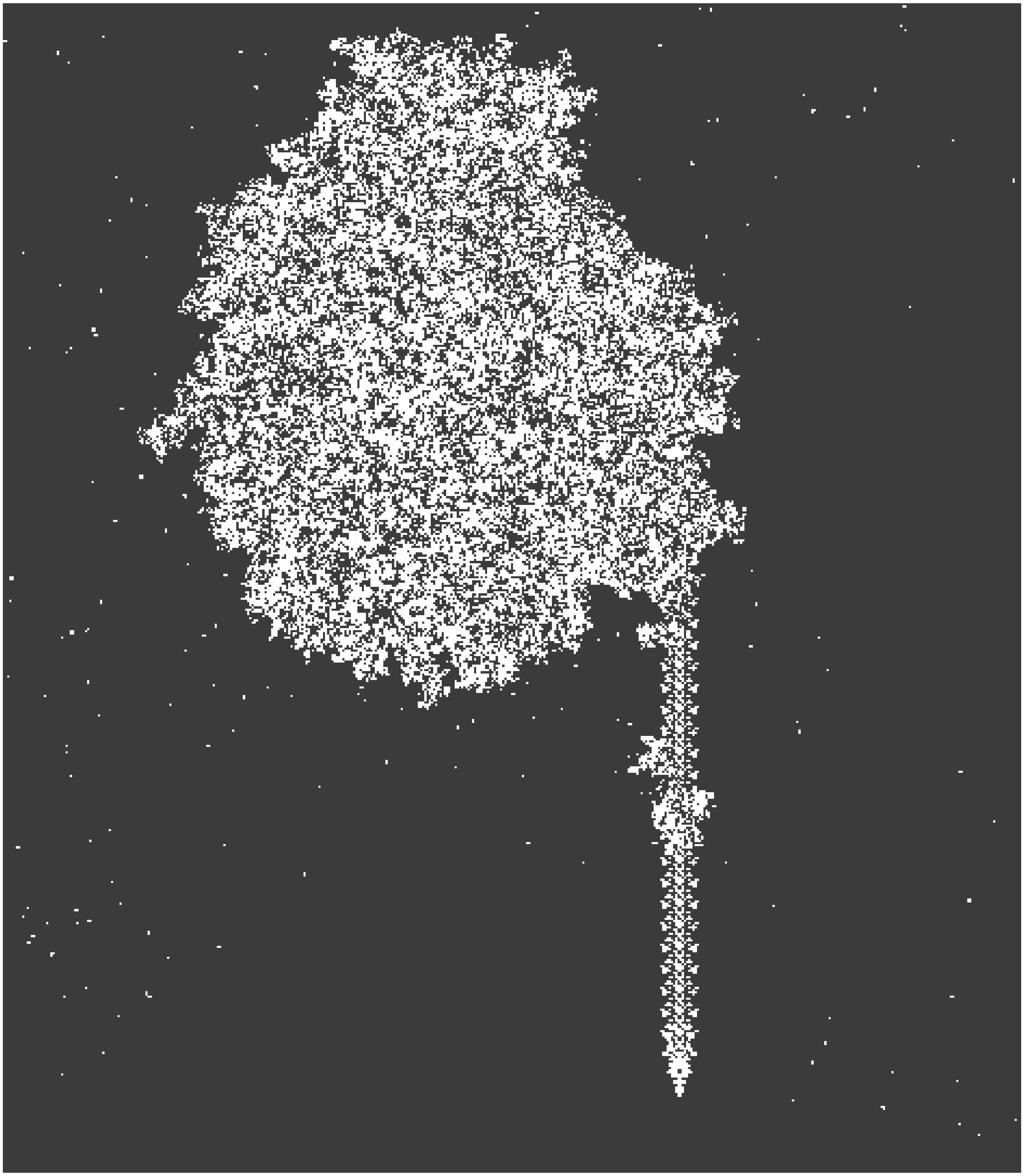


Figure 3. After 49281 iterations

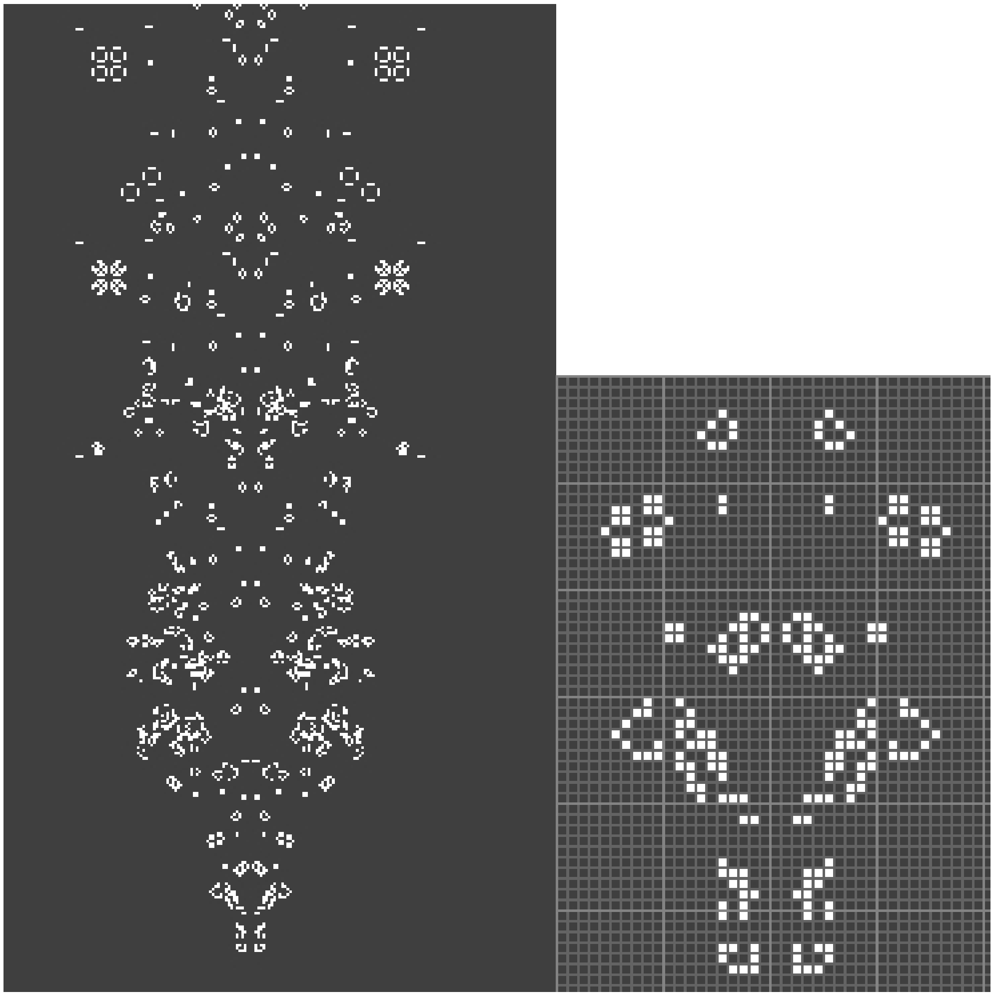


Figure 4. The growing tip

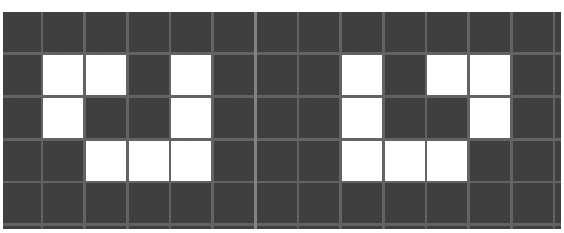


Figure 5. The generating tip GG

Figures 4 gives close-ups of the tip of the dragon and Figure 5 isolates the generator, GG, of the dragon itself. This configuration GG of 16 squares in mirror symmetry, when placed on an otherwise blank lattice, will generate the dragon in the 7-Life algorithm.

What has happened is that this 16-square generator GG has appeared in the course of the complex interactions, and it has had enough room to move forward in its own pattern – forming the dragon behind it and periodically regenerating itself. The generator of the dragon, GG, is not our invention. GG is a natural consequence of the complex process of 7-Life. GG emerges, but with much lower probability than the gliders. The result is an appearance of novelty and creativity in the complex process as it happens over time. We can only speculate what more complex entities would eventually emerge in 7-Life over many more iterations.

In the same way, DNA emerges from the complex process of the world of the earth and sun.

We see from this example that eigenforms that are processes, such as the self-generating GG, c*an and will emerge of their own accord* from complex systems based on recursion. In this sense, such systems begin to generate their own reflexive spaces. The novel and self-reproducing forms that emerge from them can be seen in a similar light.

All these observations are made by an observer. The observer is clever only in the distinctions that he or she makes, and that is enough to found an entire universe.

12. Discussion

In this paper we have covered a number of mathematical structures related to the concept of reflexivity. We have defined the notion of a reflexive domain D as a domain where the elements of that domain and the mappings of the domain to itself are in 1–1 correspondence.

In such a context, every object is inherently a process, and the structure of the domain as a whole comes from the relationships whose exploration constitutes the domain. There is no place to hide in a reflexive domain, no fundamental particle, no irreducible object or building block. Any given entity acquires its properties through its relationships with everything else. The sense of such a domain is not at all like the set theoretic notion of collections or unrelated things, or things related by an identifiable property. It is more like a conversation or an improvisation, held up and moving in its own momentum, creating and lifting sound and meaning in the process of its own exchange. Conversations create spaces and events, and these events create further conversations. The worlds appearing from reflexivity are worlds nevertheless, with those properties of partial longevity, emergence of patterns, and emergence of laws that we have come to associate with seemingly objective reality.

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