# Anticipatory Kaldor-Kalecki Model of Business Cycle

## Daniel M. Dubois

U.E.R. Systèmes d'Information, HEC•Liège
rue Louvrex 14, B-4000 Liège, Belgium
Daniel.Dubois@hec.be - http://www.hec.be/~ddubois

Centre for Hyperincursion and Anticipation in Ordered Systems,
CHAOS asbl, Institute of Mathematics, B37, University of Liège
Grande Traverse 12, B–4000 LIEGE 1, Belgium
Fax: +32(0)4 366 94 89 - Daniel.Dubois@ulg.ac.be - http://www.ulg.ac.be/mathgen/CHAOS

## Abstract

This paper extends the Kaldor-Kalecki model of business cycle with retardation to include anticipatory capabilities. There is a time shift between a decision of investment and its actual installation. The results, presented in this paper, deal with the inclusion of an anticipated capital stock in the Kaldor-Kalecki model. A third equation is added to the classical two equations of Kaldor-Kalecki. This anticipated capital stock is directly related to the future potential value of the current capital stock, that is $KA(t) = K(t + \tau)$. Numerical simulations on computer confirm the feasibility of such a method that can be applied to many other systems with delay in economy ecology, biology, physics and engineering.

## 1 Introduction

This paper addresses the problem of the mathematical implementation of anticipatory capabilities in models of business cycle. Most models published in the scientific literature deal mainly with the implementation of retardation with no anticipation (e.g. Hale and Verduyn Lunel, 1993). Cybernetics with the feed-back process in control systems deals with an explicit goal or purpose given to a system. The anticipatory systems discussed in this paper deal with a behaviour for which the future state of the system is built by the system itself, without explicit goal. A system with weak anticipation is based on a predictive model of the system, while a system with strong anticipation builds its own future by itself (Dubois, 2000).

In this paper, the Kaldor-Kalecki model, described by functional differential equations with retardation, is extended by the inclusion of anticipatory capabilities, described by an anticipation time, $t + \tau$, added to the current time, t, and the retardation time, $t - \tau$.

Initially, this model was created to include decisions of investments. There is a time lag, $\tau$, between the investment decision and installation of investment goods. So the Kaldor-Kalecki model is represented by two differential difference equations of the gross product $Y(t)$, $Y(t - \tau)$, at current time and with retardation, and the capital stock $K(t)$, at the current time. This paper extends this model to include a new equation giving a computational anticipated capital stock $KA(t)$, at the current time. This anticipated capital stock is directly related to the future potential value of the current capital stock, that is $KA(t) = K(t + \tau)$.

This extended model is simulated on computer and shows how well it works.

Methods of resolution of such functional differential equations with both retardation and anticipation were proposed recently (Dubois, 2001, 2002).

These methods are very general and can be applied to a lot of problems dealing with delayed systems, in economy, ecology, biology, physics and engineering.

## 2 The Linear Kalecki Model with Retardation

The following formalization of the Kalecki model of capital stock in continuous time, due to R.G.D. Allen (1963), can be found, for example, in G. Gabisch and H.W. Lorenz (1987). For Kalecki (1935), the evolution of the capital $K(t)$ is given by the following differential difference retarded equation

$$dK(t)/dt = (\alpha/\tau)K(t) - (\delta + \alpha/\tau).K(t - \tau) \qquad (1)$$

where $\alpha$ is the adjustment coefficient in the goods market, and $\delta$ the depreciation rate of the capital, and $\tau$ is the retardation of the decision to invest before the equipment is installed. It is assumed that the retardation time $\tau$ is constant. When the retardation is small, $K(t - \tau)$ can be developed as

$$K(t - \tau) = K(t) - \tau.dK(t)/dt$$

so the equation (1) is then written as

$$dK(t)/dt = (\alpha/\tau)K(t) - (\delta + \alpha/\tau).(K(t) - \tau.dK(t)/dt)$$

or

$$dK(t)/dt = (- \delta/(1 - \alpha - \delta\tau)).K(t) \qquad (2)$$

which gives a solution as a simple growth or decay process.

If taking the second order of the development,

$$K(t - \tau) = K(t) - \tau.dK(t)/dt + (\tau^2/2).d^2K/dt^2$$

the equation (1) becomes

$$d^2K(t)/dt^2 + [2.(1 - \delta\tau - \alpha)/(\delta\tau^2 + \alpha\tau)].dK(t)/dt =$$

$$[- 2.\delta/(\delta\tau^2 + \alpha\tau)].K(t) \qquad (3)$$

the solution of which is given by an oscillatory behaviour. Sustained oscillations occur for $1 - \delta\tau - \alpha = 0$, with a frequency equal to $\omega^2 = 2.\delta^2/(1 - \alpha)$, and the period of oscillations is given by $T = 2\pi\tau/\sqrt{[2.(1 - \alpha)]}$. Figures 1-abc give the simulation of equation 1 with the following values of the parameters: $\alpha = 0.25$, $\delta = 0.0115$, and $\tau = 90$, 100, and 110. From the second order approximation the period would be $T \approx 500$.
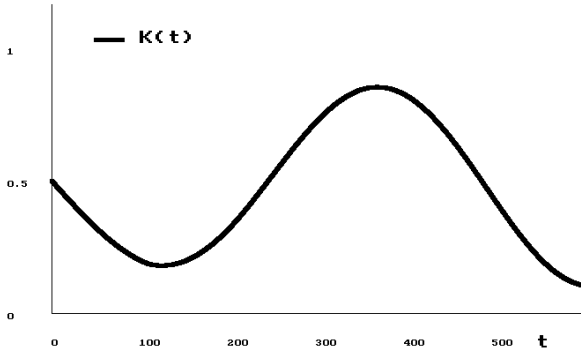


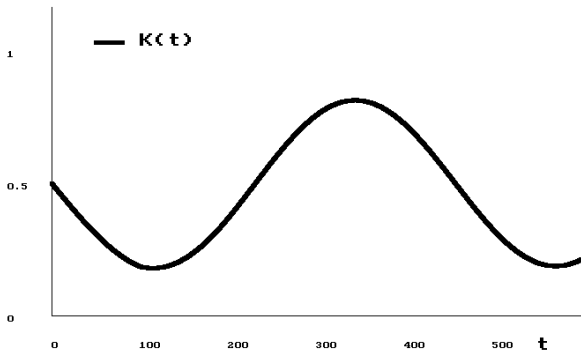Figure 1a: Simulation of the retarded system given by eq. 1 with $\tau = 110$.



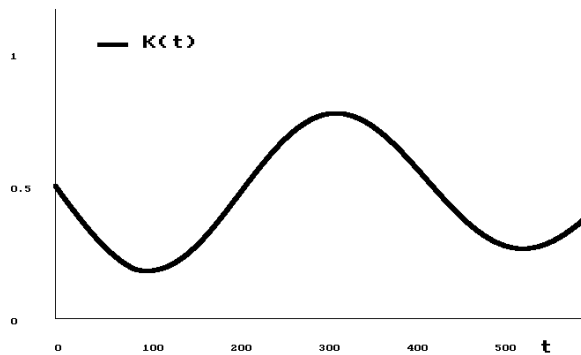Figure 1b: Simulation of the retarded system given by eq. 1 with $\tau = 100$.



Figure 1c: Simulation of the retarded system given by eq. 1 with $\tau = 90$.

In looking at Figure 1b, the sustained oscillations are well seen, and the period is in agreement with the second order approximation of the retardation. Figure 1a is the same simulation with $\tau = 90$, and it is seen that the amplitude increases. Figure 1c is the same simulation with $\tau = 110$, and it is seen that the amplitude decreases. Thus these simulations confirm the results given for the second order approximation.

Now, let us consider the more practical application of a model of business cycle.

## 3 Anticipatory Modelling and Computing of the Kaldor-Kalecki Model of Business Cycle

Let us consider the Kaldor-Kalecki model of business cycle.

### 3.1 The Kaldor-Kalecki Model

The Kalecki model of business cycle (Kalecki, 1935) assumes that the saved part of profit is invested and the capital growth is due to past investment decisions. There is a time shift after which capital equipment is available for production. The Kaldor-Kalecki model of business cycle (Krawiec, Szydlowski, 2001) is represented by the Kaldor model of trade cycle (Kaldor, 1940) in considering the Kalecki time shift of investment. Kaldor assumed that the investment function I(Y) and saving function S(Y) are increasing functions with the gross product Y as a s-shape function. So, the Kaldor-Kalecki model is represented as the following differential difference equation system with retardation:

$$dY(t)/dt = \alpha[I(Y(t), K(t)) - S(Y(t), K (t))] \qquad (4a)$$

$$dK(t)/dt = I(Y(t - \tau), K (t)) - \delta K(t) \qquad (4b)$$

where Y is the gross product and K the capital stock; I is the investment, S the saving function; $\alpha$ is the adjustment coefficient in the goods market, and $\delta$ the depreciation rate of capital stock; $\tau$ is the retardation. It is assumed that the time delay $\tau$ is constant.

It is also assumed that the investment function I(Y(t), K(t)) = I(Y(t)) + I(K(t)), and I(K(t)) are linear such that

$$I(Y(t), K(t)) = I(Y(t)) - \beta K(t), \text{ with } \beta > 0$$

(cfr Krawiec, Szydlowski, 2001). With these assumptions the equation system 4-ab becomes

$$dY(t)/dt = \alpha[I(Y(t)) - \beta K(t) - S(Y(t), K(t))] \qquad (5a)$$

$$dK(t)/dt = I(Y(t - \tau)) - (\delta + \beta)K(t) \qquad (5b)$$

### 3.2 Extension of the Kaldor-Kalecki Model with an Anticipatory Capital Stock

It is of great interest for businessman to know by anticipation the capital stock at the future time. By incursive synchronization (Dubois, 2001, 2002), it is possi-

ble to extend this model for computing the anticipated capital stock. Indeed, in considering the Kaldor-Kalecki equation system 5-ab as a master system, a slave equation for the anticipated capital stock, KA(t), can be added to the master system as follows:

$$dY(t)/dt = \alpha[I(Y(t)) - \beta K(t) - S(Y(t), K(t))] \qquad (6a)$$

$$dK(t)/dt = I(Y(t - \tau)) - (\delta + \beta)K(t) \qquad (6b)$$

$$dKA(t)/dt = I(Y(t)) - (\delta + \beta)KA(t) \qquad (6c)$$

In noting the difference between KA(t) and K(t + τ) as

$$DK(t) = KA(t) - K(t + \tau)$$

the differential equation of DK(t) can be written as

$$dDK(t)/dt = dKA(t)/dt - dK(t + \tau)/dt =$$

$$I(Y(t)) - (\delta + \beta)KA(t) - I(Y(t)) + (\delta + \beta)K(t + \tau) =$$

$$- (\delta + \beta)[KA(t) - K(t + \tau)] = - (\delta + \beta)DK(t) \qquad (6d)$$

and the analytical solution of DK(t) can be obtained as

$$DK(t) = DK(0).\exp(- (\delta + \beta)t)$$

Starting with any initial condition for DK(0), the anticipatory capital stock KA(t) will tend to the future potential value of the capital stock, K(t + τ).

## 3.3 The Linear Extended Kaldor-Kalecki Model

Let us assume (Krawiec, Szydlowski, 2001) that the saving function S only depends on Y and is linear such that

$$S(Y(t), K(t)) = \gamma Y(t), \text{ with } \gamma \in (0,1)$$

and also that the investment function is linear such that

$$I(Y(t)) = \varepsilon Y(t), \text{ with } \varepsilon > 0$$

the system 6-abc is then written as

$$dY(t)/dt = \alpha[(\varepsilon - \gamma)Y(t) - \beta K(t)] \qquad (7a)$$

$$dK(t)/dt = \varepsilon Y(t - \tau) - (\delta + \beta)K(t) \qquad (7b)$$

$$dKA(t)/dt = \varepsilon Y(t) - (\delta + \beta)KA(t) \qquad (7c)$$

Simulations of eqs. 7abc are given in Figure 2, without retardation, and in Figures 3-ab, with retardation.
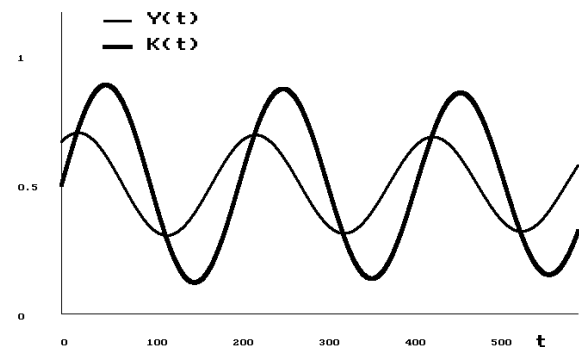


Figure 2: Simulation of the linear extended Kaldor-Kalecki model 7-abc without retardation (τ = 0): Y(t) is the gross product and K(t) the capital stock.
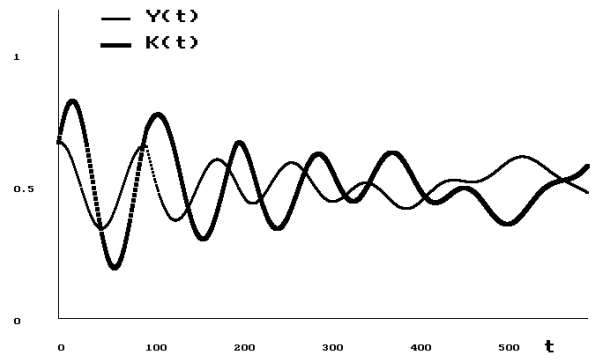


Figure 3a: Simulation of the gross product Y(t) and the capital stock K(t) of the linear extended Kaldor-Kalecki model 7-abc with the retarded gross product Y(t − τ), with τ = 100.
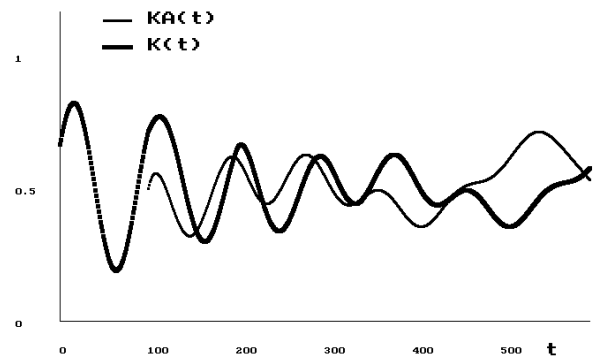


Figure 3b: Simulation of the linear extended Kaldor-Kalecki model 7-abc with retardation:
KA(t) is the anticipated capital stock and K(t) the capital stock: KA(t) = K(t + τ) with τ = 100.

**Remark:** In practice, the functions S and I are, of course, positive as well as Y and K. So, for a practical use of such equation systems, a coordinate transformation should be made such that the system becomes centred at positive steady states S* and I*, and Y* and K*.

### 3.3.1 Linear Kaldor-Kalecki Model Without Retardation

Without retardation, the system 7-abc is given by

$$dY(t)/dt = \alpha[(\varepsilon - \gamma)Y(t) - \beta K(t)] \qquad (8a)$$

$$dK(t)/dt = \varepsilon Y(t) - (\delta + \beta)K(t) \qquad (8b)$$

With the following successive transformations:

$$\varepsilon Y(t) = dK(t)/dt + (\delta + \beta)K(t)$$

$$d^2K(t)/dt^2 + (\delta + \beta)dK(t)/dt = \alpha[(\varepsilon - \gamma)[dK(t)/dt +$$

$$(\delta + \beta)K(t)] - \varepsilon\beta K(t)]$$

a second order equation is obtained:

$$d^2K(t)/dt^2 + [(\delta + \beta) - \alpha(\varepsilon - \gamma)]dK(t)/dt =$$

$$- \alpha[- (\varepsilon - \gamma)(\delta + \beta) + \varepsilon\beta]K(t) \qquad (9)$$

With $\varepsilon\beta > (\varepsilon - \gamma)(\delta + \beta)$, this equation gives oscillatory solutions.

Sustained oscillations occur when the factor of the first derivative vanishes, so when $(\delta + \beta) = \alpha(\varepsilon - \gamma)$.

If $(\delta + \beta) > \alpha(\varepsilon - \gamma)$, the capital stock oscillatory amplitude decreases, and if $(\delta + \beta) < \alpha(\varepsilon - \gamma)$, it increases.

### 3.3.2 Linear Kaldor-Kalecki Model with Retardation

The linear system with retardation (7-ab),

$$dY(t)/dt = \alpha[(\varepsilon - \gamma)Y(t) - \beta K(t)] \quad (10a)$$

$$dK(t)/dt = \varepsilon Y(t - \tau) - (\delta + \beta)K(t) \quad (10b)$$

can also be transformed, successively as follows:

$$\varepsilon Y(t) = dK(t + \tau)/dt + (\delta + \beta)K(t + \tau)$$

$$d^2K(t + \tau)/dt^2 + (\delta + \beta)dK(t + \tau)/dt =$$

$$\alpha[(\varepsilon - \gamma)[dK(t + \tau)/dt + (\delta + \beta)K(t + \tau)] - \varepsilon\beta K(t)]$$

to the following single second order equation

$$d^2K(t)/dt^2 + [(\delta + \beta) - \alpha(\varepsilon - \gamma)]dK(t)/dt =$$

$$\alpha[(\varepsilon - \gamma)[(\delta + \beta)K(t)] - \varepsilon\beta K(t - \tau)] \quad (11)$$

For small values of the time shift $\tau$, the retarded function $K(t - \tau)$ can be approximated by a development in Taylor's series as,

$$K(t - \tau) = K(t) - \tau.dK(t)/dt + (\tau^2/2).d^2K(t)/dt^2 - .... \quad (12)$$

In taking into account this development in Taylor's series, eq. 11 becomes

$$d^2K(t)/dt^2 +$$

$$[[(\delta + \beta) - \alpha(\varepsilon - \gamma) - \tau\alpha\varepsilon\beta]/(1 + \tau^2\alpha\varepsilon\beta/2)]dK(t)/dt =$$

$$\alpha[(\varepsilon - \gamma)[(\delta + \beta)K(t)] - \varepsilon\beta K(t)]/(1 + \tau^2\alpha\varepsilon\beta/2) \quad (13)$$

and sustained oscillations occur for

$$(\delta + \beta) - \alpha(\varepsilon - \gamma) = \tau\alpha\varepsilon\beta$$

The factor in $\tau^2$ has the effect to change the period of oscillations, the period increases with the time shift, and the factor of the first derivative. The factor in $\tau$ has the effect to change the stability of oscillations, the instability increases with the time shift.

### 3.4 A Nonlinear Extended Kaldor-Kalecki Model

With a nonlinear investment, the solution of the Kaldor-Kalecki can exhibit a limit cycle solution (see for example, Krawiec, Szydlowski, 2001), that exhibits oscillations with a constant amplitude depending on the parameters of the equation system.

So, let us now consider the extended Kaldor-Kalecki model 6-abc, with the linear saving function $S = \gamma Y(t)$,

$$dY(t)/dt = \alpha[I(Y(t)) - \beta K(t) - \gamma Y(t)] \quad (14a)$$

$$dK(t)/dt = I(Y(t - \tau)) - (\delta + \beta)K(t) \quad (14b)$$

$$dKA(t)/dt = I(Y(t)) - (\delta + \beta)KA(t) \quad (14c)$$

and with a s-shape investment function of the form:

$$I(Y(t)) = \varepsilon.Y(t)/(1 + \varepsilon_1.|Y(t)|) \quad (15)$$

where $|Y(t)|$ is the absolute value of $Y(t)$, and $\varepsilon_1$ is a new parameter.

The numerical simulations of eqs. 14-ab, with the nonlinear function 15, without retardation, are given in the Figures 4-abc. The solution is then given now by a limit cycle: this is an oscillatory behaviour with an amplitude given by the parameters.
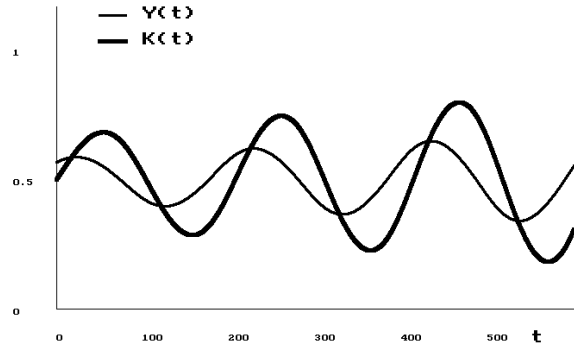


Figure 4a: Simulation of eqs. 14-ab
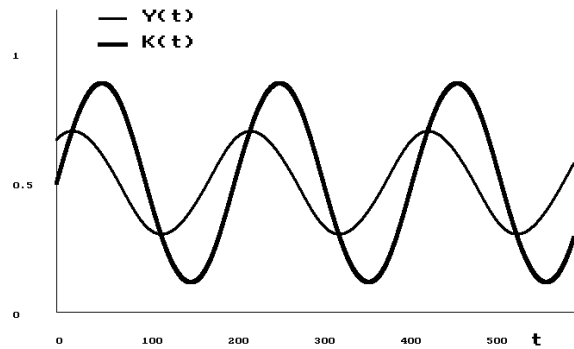with initial conditions inside the limit cycle.



Figure 4b: Simulation of eqs. 14-ab
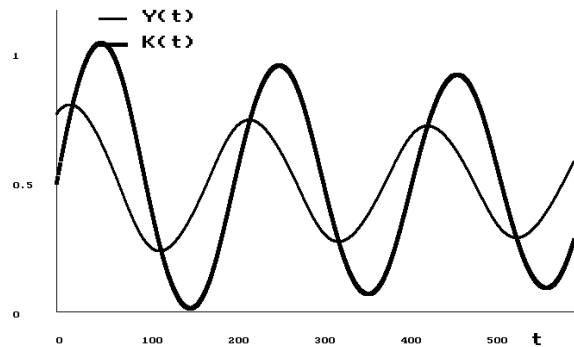with initial conditions on the limit cycle.



Figure 4c: Simulation of eqs. 14-ab
with initial conditions outside the limit cycle.

When the initial conditions are inside the limit cycle, the amplitude increases to the limit cycle, as shown in Fig. 4a. When the initial conditions are on the limit cycle, the amplitude does not change, as shown in Fig. 4b. When the initial conditions are outside the limit cycle, the amplitude decreases to the limit cycle.

The numerical simulations of eqs. 14-abc, with the nonlinear function 15, with a retardation, $\tau = 100$, are given in the following Figures 5-ab.
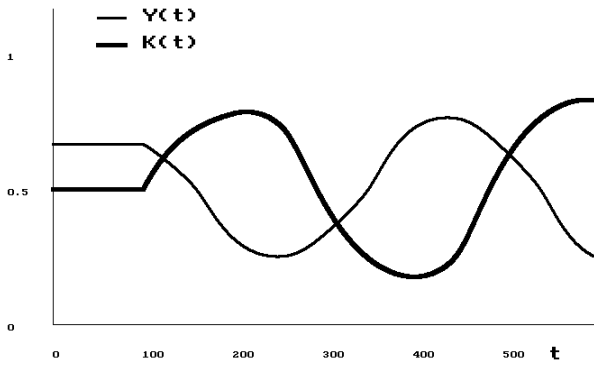


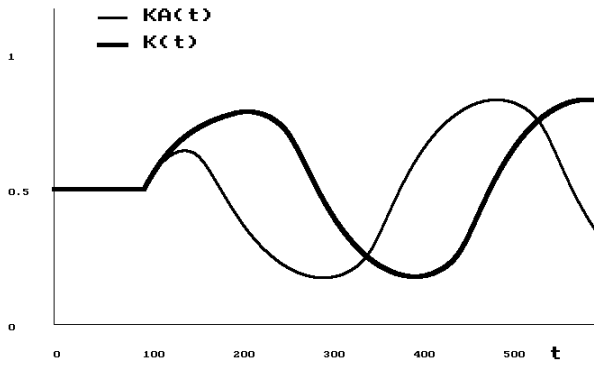Figure 5a: Simulation of eqs. 14-ab with a retardation $\tau = 100$.



Figure 5b: Simulation of eqs. 14-bc with a retardation $\tau = 100$.

## 3.5 Including Anticipated Investment and Saving Functions

When a system is described by two differential difference equations with two variables, it is possible to create an anticipation on one variable and a retardation on the other one, with the method proposed recently (Dubois, 2002).

In considering an anticipated investment depending on the anticipated capital stock, such as

$$I(Y(t), K(t)) = I(Y(t)) - \beta K(t + \tau)$$

with an anticipatory saving function S also depending on the anticipated capital stock as

$$S(Y(t), K(t)) = S(Y(t), K(t + \tau))$$

the system 6-abc becomes

$$dY(t)/dt = \alpha[I(Y(t)) - \beta K(t + \tau) - S(Y(t), K(t + \tau))] \quad (16a)$$

$$dK(t)/dt = I(Y(t - \tau)) - (\delta + \beta)K(t) \quad (16b)$$

$$dKA(t)/dt = I(Y(t)) - (\delta + \beta)KA(t) \quad (16c)$$

As KA(t) tends to $K(t + \tau)$, we can replace $K(t + \tau)$ in the first equation 16a by KA(t) as follows

$$dY(t)/dt = \alpha[I(Y(t)) - \beta KA(t) - S(Y(t), KA(t))] \quad (17a)$$

$$dK(t)/dt = I(Y(t - \tau)) - (\delta + \beta)K(t) \quad (17b)$$

$$dKA(t)/dt = I(Y(t)) - (\delta + \beta)KA(t) \quad (17c)$$

The first and third equations 17-a-c form a system defined at the current time t, without any time retardation nor anticipation.

With the assumptions, given above:

$$S(Y(t), K(t)) = \gamma Y(t), \text{ with } \gamma \in (0,1), \text{ and}$$

$$I(Y(t)) = \varepsilon Y(t), \text{ with } \varepsilon > 0$$

the system of eqs. 17-abc becomes:

$$dY(t)/dt = \alpha[\varepsilon Y(t) - \beta KA(t) - \gamma Y(t)] \quad (18a)$$

$$dK(t)/dt = \varepsilon Y(t - \tau) - (\delta + \beta)K(t) \quad (18b)$$

$$dKA(t)/dt = \varepsilon Y(t) - (\delta + \beta)KA(t) \quad (18c)$$

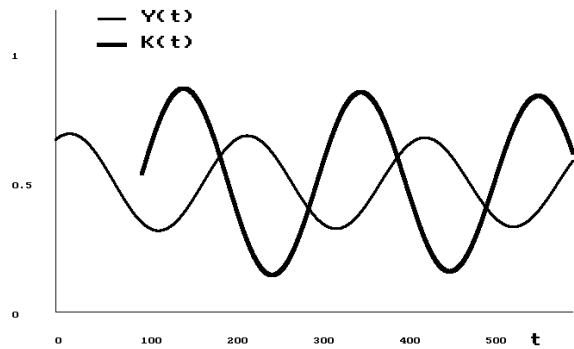Simulation of this equation system 18-abc is given in Figures 6-ab.



Figure 6a : Simulation of the linear extended Kaldor-Kalecki model 18-ab, with an anticipated investment.

The time shift $\tau = 100$ of K(t) is well seen in comparison with figure 2.

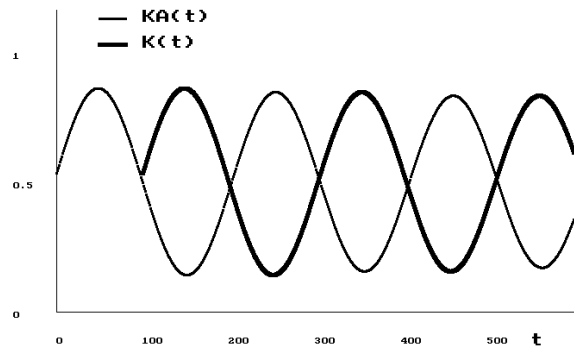

Figure 6b: Simulation of the linear extended Kaldor-Kalecki model 18-bc, with an anticipated investment $KA(t) = K(t + \tau)$.

The effect of the anticipated investment is to change the phase between the oscillations of the capital stock and the investment. It could be important to control such a phase in practical business applications.

It must be pointed out that the phase shift can be chosen within a range of values: the example given here, with only one single fixed anticipatory time $t + \tau$, is just a first attempt to show the feasibility of such an approach.

## 4 Conclusion

This paper deals with the modelling of retardation and anticipation in business models (Dubois, 2003b).

Firstly, the retarded Kalecki model of business, given by one differential difference equation with retardation, describes the evolution of the capital $K(t)$ related to the retarded capital $K(t - \tau)$. The oscillatory behaviour of the Kalecki retarded model is only possible because there is a retardation: without this retardation, the Kalecki model would give rise to a simple decay (the capital would go the zero) and the system would not survive.

Secondly, the more general Kaldor-Kalecki model of business cycle is studied in view of showing its anticipatory capabilities. The instrumental relationship was to take into account a constant retardation, $\tau$, between the investment decision and installation of investment goods. So the Kaldor-Kalecki model is represented by two functional differential equations of the gross product $Y(t)$, at current time, and $Y(t - \tau)$, with retardation, and the capital stock $K(t)$, at the current time. In this paper, this model is extended to include a new equation giving a computational anticipated capital stock $KA(t)$, at the current time. This anticipated capital stock is directly related to the future potential value of the current capital stock, that is $KA(t) = K(t + \tau)$. Numerical simulations show the feasibility of such a method that can be applied to many other similar problems in all areas of science.

The anticipated capital $K(t + \tau)$ and the retarded investment $Y(t - \tau)$ play conjugated roles. As explained in Dubois (2002), an anticipated event can be computed, at the strong sense, at the condition that another event is defined with a time retardation. The time retardation may be interpreted as a memory that is a necessary condition for computing a strong anticipation. Recall that a strong anticipation is an anticipation that is computed by the system itself without a predictive model. Recall that an anticipatory system, as defined by Robert Rosen (1985), is built on a predictive model of the system, and is thus a weak anticipation. So, the strong anticipation is not simply an extrapolation of the past to the future, but a dynamical behaviour of the system that takes explicitly into account past, present, and future events. Mathematical foundations of discrete and functional systems with strong and weak anticipations appeared in (Dubois, 2003a).

Strong anticipation can be modelled with functional retarded differential equations coupled to an anticipative synchronized equation (e.g. Voss, 2000, Dubois, 2001). This paper demonstrates that functional differential equations with both retardation and anticipation are a useful tool for modelling anticipatory systems in a closed form.

## Acknowledgments

## References

[Dubois, 2000] Dubois Daniel M., Review of Incursive, Hyperincursive and Anticipatory Systems - Foundation of Anticipation in Electromagnetism. In Computing Anticipatory Systems: CASYS'99 - Third International Conference. American Institute of Physics, AIP Conference Proceedings 517, pp. 3-30, 2000.

[Dubois, 2001] Dubois Daniel M., Theory of Incursive Synchronization and Application to the Anticipation of a Chaotic Epidemic. International Journal of Computing Anticipatory Systems, volume 10, pp. 3-18, 2001.

[Dubois, 2002] Dubois Daniel M., Theory of Computing Anticipatory Systems Based on Differential Delayed-Advanced Difference Equations. In Computing Anticipatory Systems: CASYS 2001 - Fifth International Conference, American Institute of Physics, AIP Conference Proceedings 627, pp. 3-16, 2002.

[Dubois, 2003a] Dubois Daniel M., "Mathematical foundations of discrete and functional systems with strong and weak anticipations", in Anticipatory Behavior in Adaptive Learning Systems, State-of-the-Art Survey, Edited by Martin Butz et al, Lecture Notes in Artificial Intelligence, Springer, LNAI 2684, pp. 110-132, 2003. ISSN 0302-9743 ISBN 3-540-40429-5

[Dubois, 2003b] Dubois Daniel M., "Extension of the Kaldor–Kalecki model of business cycle with a computational anticipated capital stock", International Journal of Organisational Transformation and Social Change, OTSC 1 (1) 63–80 © Intellect Ltd 2003, ISSN 1477-9633

[Gabisch and Lorenz, 1987] Gabisch G. and Lorenz H.W., Business Cycle Theory: A survey of methods and concepts. Springer-Verlag 1987 (1989, second edition), Berlin.

[Hale and Verduyn Lunel, 1993] Hale Jack K. and Verduyn Lunel Sjoerd M., Introduction to Functional Differential Equations, Applied Mathematical Sciences, 99, Springer-Verlag, 1993.

[Kaldor, 1940] Kaldor N., A Model of the Trade Cycle, Economic Journal , V.50, 78 –92,1940.

[Kalecki, 1935] Kalecki M., A Macrodynamic Theory of Business Cycles, Econometrica, V.3, 327 –344, 1935.

[Krawiec and Szydlowski, 2001] Krawiec A. and Szydlowski M., The Kaldor-Kalecki Model of Business Cycle, Journal of Nonlinear Mathematical Physics, V.8 Supplement, Proceedings NEED'99, 266-271, 2001.

[Rosen, 1985] Rosen Robert, Anticipatory Systems, Pergamon Press, New York, 1985.

[Voss, 2000] Voss Henning U., Anticipating Chaotic Synchronization, Physical Review E, volume 61, number 5, pp. 5115-5119, 2000.