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Research Statement

1. Research topic overview

My current research projects are primarily devoted to studying properties and finding applications of various categorifications of knot invariants. Categorification is a novel approach for constructing new knot invariants, which was developed by Mikhail Khovanov [Kh1] in 2000 and which is attracting significant attention of the mathematical community ever since. The idea of categorification is to replace a known polynomial knot invariant with a family of chain complexes such that the coefficients of the original polynomial are the Euler characteristics of these complexes. The first knot invariant to undergo categorification in [Kh1] was, quite aptly, the Jones polynomial, an invariant that revolutionized the knot theory back in 1983 [J]. The resulting homology is commonly referred to as *Khovanov homology*. Since then, many other knot invariants were categorified.

The importance of the Khovanov homology became apparent after a seminal result by Rasmussen in 2004 [Ra], who used it to give the first purely combinatorial proof of the famous Milnor conjecture about the slice genus of torus knots [M]. This conjecture was originally proved by Kronheimer and Mrowka in 1993 using gauge theory [KM1]. Rasmussen's approach, on the other hand, avoids relying on this complicated theory. Many other applications of the Khovanov homology were found since then, some of them by me (see Section 2.2). It was proved in 2011 by Kronheimer and Mrowka that Khovanov homology detects the unknot [KM2]. This development is a major step towards proving a long-standing conjecture that the Jones polynomial itself detects the unknot.

A major milestone in the development of the Khovanov homology theory happened in 2007 when Ozsváth, Rasmussen, and Szabó introduced its *odd* version [ORS]. Odd Khovanov homology is isomorphic to the original (even) one modulo 2 and, in particular, categorifies the same Jones polynomial. On the other hand, odd homology is often more "useful" than the even one (see Section 2.3).

In 2010, Krzysztof Putyra combined even and odd Khovanov homology into a single unified theory [P1]. It was commonly believed that the unified Khovanov homology is better than both even and odd ones combined at distinguishing knots. In [PSh1, PSh2], Putyra and I were able to confirm this conjecture and, in the process, find new knot invariants with interesting properties. Our construction is related to one of the more recent developments of studying homological operations on knot homology (see Section 2.4). This activity started in 2012 with a seminal work by Lipshitz and Sarkar on a spectrification of Khovanov homology [LS1] that led to the construction of Steenrod squares acting on Khovanov homology over $\mathbb{Z}_2$ [LS2].

2. Past Results

2.1. **Torsion in the Khovanov Homology.** Over the years, I obtained a variety of results about properties of torsion in the Khovanov homology. As it turns out, it is at least as interesting and important as the free part. In [Sh5], I proved that for a special class of links, called *homologically thin* or H-thin links, the torsion is *almost* determined by the free part of the homology, except that one cannot distinguish between 2^k-torsion for different values of k.

A link is said to be *H-thin* if its Khovanov homology is supported on two adjacent diagonals of slope 2. More generally, given a commutative ring with unity R, a link is RH-thin if its Khovanov homology with coefficients in R is supported on two adjacent diagonals. A link L is said to be *homologically slim* or simply *H-slim* if it is $\mathbb{Z}$H-thin and all its homology groups that are supported on the *upper* diagonal have no torsion. A link L that is not RH-thin is said to be *RH-thick*. It follows from the definition that if a link is $\mathbb{Z}$H-thin, then it is $\mathbb{Q}$H-thin as well. Consequently, a $\mathbb{Q}$H-thick link is necessarily $\mathbb{Z}$H-thick. If a link is H-slim, then it is $\mathbb{Z}_m$H-thin for every $m > 1$ by the Universal Coefficient Theorem.

After the work of Eun Soo Lee [Lee], it was known that non-split alternating links are H-slim and that the rational Khovanov homology of H-slim links is determined by their Jones polynomial and signature. I proved in [Sh5] that integer Khovanov homology of such links has no odd torsion and that $\mathbb{Z}_{2^k}$ factors in the canonical decomposition of their Khovanov homology fit nicely into the patterns, called *knight-move pairs*, that were observed by Lee [Lee].

It was a long-standing conjecture that H-slim links do not have torsion of order 2^k for $k > 1$ in their Khovanov homology. I was recently able to settle this conjecture in the affirmative for $\mathbb{Z}_2$H-thin links by using the structure of a Bockstein spectral sequence on Khovanov homology [Sh6]. The conjecture turned out to be false for $\mathbb{Q}$H-thin or even $\mathbb{Z}$H-thin links. Together with the results by Lee, this implies that the integer Khovanov homology of $\mathbb{Z}_2$H-thin links is completely determined by their Jones polynomial and signature.

It is natural to conjecture that if a link has its mod 2 Khovanov homology supported on n adjacent diagonals of slope 2, then its integer Khovanov homology has no torsion of order 2^k for $k \geq n$. Proving this conjecture is still a work in progress.

In [Sh5], I also conjectured and provided evidence that, in fact, torsion alone is sufficient to decide whether a knot is trivial or not. Knowing this would be very helpful since the existence of non-trivial torsion can often be decided from the knot diagram [AP].

[Sh5] is cited in more than 30 research papers.

2.2. **Rasmussen invariant and its applications.** In [Sh2], I used properties of the Rasmussen invariant [Ra] to give a combinatorial proof for the Slice-Bennequin Inequality for knots. This inequality relates the slice genus of a closed braid to the number of its positive and negative crossings. Given a knot K in the 3-sphere S^3 considered as the boundary of the 4-ball D^4, its *slice genus* is defined as the smallest possible genus of a surface F smoothly embedded into D^4 such that $\partial F = K \subset \partial D^4$. The original proof of the Slice-Bennequin Inequality, due to Lee Rudolph [Ru], was based on the Donaldson theory. In contrast, my approach is completely combinatorial.

Also in [Sh2], I searched for knots that are topologically locally-flatly slice but are not (smoothly) slice by listing all prime knots with at most 16 crossings that have their Alexander polynomial equal 1, yet have non-trivial Rasmussen invariant. A knot is slice if its slice genus equals 0, that is, it bounds a properly embedded disk inside D^4. The Rasmussen invariant provides a lower bound on the slice genus. It was proved by M. Freedman [F] that if a knot has trivial Alexander polynomial, then it is topologically locally-flatly slice. In total, I found 82 knots that possess these two properties. Each such knot gives rise to an uncountable family of exotic $\mathbb{R}^4$'s (see [GS, Exercise 9.4.23]). Only one of these examples was previously known.

[Sh2] is cited in more than 35 research papers.

2.3. **Odd Khovanov homology.** In [Sh3, Sh4], I studied properties of the odd Khovanov homology and showed that its applications tend to be much more useful and practical. In particular, I demonstrated that odd Khovanov homology is often better than the usual (even) one at providing an upper bound for the Thurston-Bennequin number of knots, that is, the maximal Thurston-Bennequin number for Legendrian knots of the topological type of this knot. This bound was proved for the original Khovanov homology by Lenhard Ng [Ng]. My computations have shown that the bound obtained from odd Khovanov homology is the best among all currently known ones for all knots with at most 15 crossings. On the other hand, there are 45 knots with at most 15 crossings for which a bound on their Thurston-Bennequin numbers obtained from the Kauffman polynomial is better than that of even Khovanov homology.

In the same papers, I have also shown that odd Khovanov homology is quite useful for detecting quasi-alternating knots. The class of quasi-alternating knots extends alternating ones and is defined recursively. As such, it is often highly non-trivial to show that a given link is quasi-alternating. It is equally challenging to show that it is not. After the work of Champanerkar and Kofman [ChK], there were only two prime knots with at most 10 crossings, 9_{46} and 10_{140} in Rolfsen notation, for which it was not known whether they are quasi-alternating or not. I was able to confirm that in fact they are not, something that could not be proved using other methods at the time.

[Sh3, Sh4] are sited in 7 research papers.

2.4. **Homological operation on Khovanov homology.** In [P1], Putyra defined a unified even/odd version of the Khovanov homology over the ring $\mathbb{Z}_\xi := \mathbb{Z}[\xi]/(1-\xi^2) \simeq \mathbb{Z}\mathbb{Z}_2$, the group ring of $\mathbb{Z}_2$. This unified theory specializes to the even homology for $\xi = 1$ and to the odd one for $\xi = -1$. It was expected that the unified Khovanov homology is better than both even and odd ones combined at distinguishing knots, but a direct verification turned out to be elusive.

In [PSh1], Putyra and I were able to confirm this conjecture indirectly by investigating homological operations between even and odd Khovanov homology that arise from the unified theory. More specifically, since even and odd Khovanov homology agree over $\mathbb{Z}_2$, there are two Bockstein homomorphisms on $\mathbb{Z}_2$-homology: β_e and β_o. Each of them squares to the zero map, but computations show that they do not commute with each other. Therefore, the compositions $\beta_e\beta_o$ and $\beta_o\beta_e$ are degree 2 operations on Khovanov homology, which appear to be different from the second Steenrod square Sq^2 defined in [LS2], yet related to it. In fact, the first knot with different ranks of $\beta_e\beta_o + \beta_o\beta_e = (\beta_e + \beta_o)^2$ and Sq^2 has 14 crossings.

In [PSh1], we provided evidence for a conjecture that β_e and β_o generate an infinite-dimensional subalgebra of the graded Hopf algebra of homological operations. It would be interesting to investigate how Sq^2 fits into this algebraic structure.

The unified Khovanov homology can be used to define integral lifts of β_e and β_o. Namely, in [PSh1] we constructed homological operations $\varphi_{oe} : \mathcal{H}_o \to \mathcal{H}_e$ and $\varphi_{eo} : \mathcal{H}_e \to \mathcal{H}_o$ that reduce to β_e and β_o, respectively, modulo 2. Both operations take values in the 2-torsion subgroups of the corresponding homology groups. They can be composed to obtain degree-2 operations on even and odd Khovanov homology. All of these operations are non-trivial. In fact, there are many examples of knots that have the same even and odd Khovanov homology, but different actions of these homological operations.

Finally, it turned out that the ranks of φ_{oe}, φ_{eo} and their compositions are knot invariants with very interesting properties. There are several pairs of knots (K_1, K_2) such that these invariants are the same for K_1 and K_2, yet are distinct for the mirror images of K_1 and K_2. It follows that knowing the values of these invariants for a knot does not allow one to deduce them for the mirror image of this knot. Very few knot invariants possess such a property since taking the mirror image is equivalent to reversing the orientation of the ambient space.

2.5. **Computations of the unified Khovanov homology.** After our indirect confirmation in [PSh1] that the unified Khovanov homology theory is a finer knot invariant than the odd and even ones combined, it became increasingly interesting to learn how to compute this unified homology explicitly. In order to do this, we switched to a pullback presentation of the unified homology. Note that $\mathbb{Z}_\xi = \mathbb{Z}\mathbb{Z}_2 \simeq \{(a,b) \in \mathbb{Z}^2 \,|\, a \equiv b \mod 2\}$. More generally, we define a pullback ring $R := (R_1 \xrightarrow{\pi_1} \overline{R} \xleftarrow{\pi_2} R_2)$ as $\{(r_1, r_2) \in R_1 \oplus R_2 \,|\, \pi_1(r_1) = \pi_2(r_2)\}$, where $\overline{R}$ is a field, $R_{1,2}$ are Dedekind domains, and $\pi_{1,2}$ are epimorphisms. In our case $\overline{R} = \mathbb{Z}_2$, $R_1 = R_2 = \mathbb{Z}$ and $\pi_{1,2}$ are reductions mod 2.

In [Le1, Le2], Lawrence S. Levy developed an explicit algorithmic classification of $\mathbb{Z}_\xi$-modules that have a pullback representation as R-diagrams. Every $\mathbb{Z}_\xi$-module can be represented by an R-diagram, but it is in general not clear how to find one. In [P2], Putyra described how to construct R-diagrams for $\mathbb{Z}_\xi$-modules that have separated pullback presentations. All quotients of free $\mathbb{Z}_\xi$-modules, in particular those that arise in the process of computing the unified homology, have such presentations.

Together with Putyra, we were able to fill the remaining gaps and implement Levy's algorithm in `KhoHo` (see below). We also proved that, similarly to the case of even and odd Khovanov homology (see Section 2.1), the unified homology of non-split alternating links is completely determined by their Jones polynomial and signature. Being $\mathbb{Z}_2$H-thin is not enough in this case since our proof relies on the spanning tree description of the Khovanov chain complex [W], adapted to the unified setting. A paper with these results is in a final stage of preparation [PSh2]

2.6. `KhoHo`**, a program for computing and studying Khovanov homology.** As is often the case with new theories, the initial discovery is led by experiments. This frequently results in a necessity to use specialized software. This is especially true for Khovanov homology, since it can be computed by hand only for a very limited family of knots. One of my major contributions to the realm of Khovanov homology was developing `KhoHo` [Sh1], a program for computing and studying Khovanov homology. `KhoHo` is currently one of the main experimental tools for investigating Khovanov homology and is actively used by many researchers around the world. In 2008 `KhoHo` was extended to work with odd Khovanov homology, in 2014 I taught it how to compute ranks of the homological operations discussed in Section 2.4, and in 2017 I was finally able to implement the Levy's algorithm from Section 2.5.

`KhoHo` was instrumental in observing many properties of the Khovanov homology mentioned above. In [DGShT], `KhoHo` was used to demonstrate non-invariance of the Khovanov homology under genus-2 mutation. This type of mutation is a generalization of the Conway one and can be proved to preserve such knots invariants as the colored Jones polynomial and hyperbolic volume. It is still an open conjecture that even Khovanov homology is preserved under Conway mutation of knots. On the other hand, odd Khovanov homology is known not to change under Conway mutation [B]. Interestingly enough, it also takes the same values on all known examples of genus-2-mutant knots that are distinguished by the even homology. This phenomenon currently has no explanation.

In total, `KhoHo` is cited in more than 30 research papers.

3. Future Research Plans

I plan to continue working in my current area of expertise for the foreseeable future. All my research projects are building upon the previously obtained results.

3.1. **Odd and unified Khovanov homology for tangles.** I had a very successful collaboration with Krzysztof Putyra over the past several years, which was partially supported by a grant from the Simons Foundation. Together, we wrote one paper about the relations between even and odd Khovanov homology theories [PSh1], and have two more in the pipeline [PSh2, PSh3]. One of our main research efforts focuses on developing a "local theory" for the odd and, consequently, unified Khovanov homology theories, that is, their extension to tangles and their cobordisms, similar to the approach pioneered by Bar-Natan for even homology [BN1, BN2]. Having such a local theory should speed all computations up tremendously.

We are searching for an algebraic structure on odd Khovanov homology for tangles that is similar in nature to the one from [Kh2]. Our approach is based on Putyra's description [P1] of odd Khovanov homology in terms of chronological cobordisms, that is, knot cobordisms with fixed ordering of its critical points. In the case of even homology, one starts with rings H_n for each n that are generated by pairs of crossingless matchings on $2n$ points. The multiplication is induced by certain cobordisms after applying Khovanov's TQFT functor. For the odd homology, we define a quasi-ring OH_n with multiplication coming from the 2-functor of the unified homology applied to Putyra's chronologies.

The intrinsic non-associativity of chronologies implies that this new structure would have to be non-associative. One of the more appropriate choices for such a non-associative structure appears to be G-graded quasi-algebras introduced by Albuquerque and Majid in [AM]. Given an abelian group G, a G-graded non-associative algebra A over a field k is called a *G-graded quasi-algebra* if the multiplication in A satisfies the following anti-associativity condition: $(ab)c = \varphi(|a|, |b|, |c|) \cdot a(bc)$, where $a, b, c \in A$ are homogeneous, $|x| \in G$ is the grading of a homogeneous element $x \in A$, and $\varphi : G \times G \times G \to k^\times$ is a cocycle. The latter condition ensures that φ satisfies the pentagon axiom.

In our construction we have to go even further: the quasi-rings OH_n are graded not by groups but by certain groupoids. We extend this definition to quasi-modules and quasi-bimodules over OH_n as well. In this framework, we associate to a crossingless tangle T with $2n$ endpoints on the bottom and $2m$ on the top an (OH_m, OH_n)-quasi-bimodule $C(T)$, similarly to [Kh2]. The graded tensor product of these bimodules is compatible with composition of tangles.

Project 1. Extend the construction above to tangles with crossings. Describe a divide-and-conquer algorithm to compute odd Khovanov homology using the quasi-rings OH_n.

Project 2. Combine divide-and-conquer algorithms for even and odd Khovanov homology into the one for the unified homology.

3.2. **Relations between Turner and Bockstein spectral sequences on Khovanov homology with $\mathbb{Z}_2$ coefficients.** In order to prove that the Khovanov homology of $\mathbb{Z}_2$H-thin links has no 4-torsion [Sh6], I relied on the fact that the Turner spectral sequence [Tu] on the Khovanov homology with coefficients in $\mathbb{Z}_2$ collapses at the second page for such links. This spectral sequence arises from a structure of a double complex that the Khovanov complex can be equipped with. The first differential in this double complex is the original Khovanov one of bidegree $(1,0)$, while the second differential was defined by Paul Turner in [Tu] and has bidegree $(1,2)$. In [Sh6], I proved that $d_T^* = \beta\nu^* + \nu^*\beta$ on mod-2 Khovanov homology groups, where d_T is the Turner differential, β is the Bockstein differential, and ν is yet another differential of bidegree $(0,2)$ that I have defined in [Sh5]. Both ν and ν^* are acyclic. In particular, ν^* establishes an isomorphism between the two non-trivial diagonals of the Khovanov homology over $\mathbb{Z}_2$ for $\mathbb{Z}_2$H-thin links. Because of this relation on d_T^* and β, the Bockstein spectral sequence collapses at the second page for $\mathbb{Z}_2$H-thin links as well.

But the story does not end here. d_T^* and β are differentials on the first pages in the corresponding spectral sequence. One can ask whether there is a relation between differentials on higher pages as well. Over the last couple of years, I have made several strides in this direction, closing in on a formula that relates second-page differentials in these spectral sequences, but a general formulation is still elusive.

Project 3. Find an algebraic relation between Turner and Bockstein differentials on higher pages in the corresponding spectral sequences. Prove that if the Turner spectral sequence collapses at the n-th page, then the Bockstein one does so as well.

As a corollary, one would confirm that if a link has its mod-2 Khovanov homology supported on n adjacent diagonals, then its integer homology has no torsion of order 2^k for $k \geq n$. More importantly, though, this would significantly advance our understanding of the algebraic structure of Khovanov homology.

3.3. **Odd Khovanov homology and transversely non-simple knots.** Consider $\mathbb{R}^3$ equipped with the standard contact structure $dz - y\,dx$. A knot in $\mathbb{R}^3$ is said to be *transverse* if it is everywhere transverse to the contact planes of this contact structure. Given a transverse knot K, one defines its *self-linking number*, $sl(K)$, as the linking number of K with its push-off K' along the vector field $v = \frac{\partial}{\partial y}$. This vector field always lies in the contact planes. Finally, a topological knot K is said to be *transversely non-simple* if it has two transverse representatives K_1 and K_2 such that $sl(K_1) = sl(K_2)$, but $K_1 \neq K_2$ as transverse knots.

It is surprisingly hard to find transversely non-simple knots, with the first examples discovered only in 2003. Birman and Menasco found a family of transversely non-simple 3-braids [BM1, BM2], while Etnyre and Honda proved that the $(2,3)$-cable of the trefoil is transversely non-simple as well [EH]. More recently, several families of transversely non-simple knots were found using knot Floer homology [B, NOT, V].

It turns out that many of these transversely non-simple knots have very special odd Khovanov homology. Namely, it has only torsion and no free part in the homological grading 0. We call such knots *zero-omitting*. In [Sh3], I have listed 10 such knots with 12 crossings or less. Seven of them are known to be transversely non-simple. On the other hand, the

(2, 3)-cable of the trefoil, a transversely non-simple knot, is not zero-omitting. It is worth pointing out that even Khovanov homology always has free part in the homological grading 0 by the results of Lee [Lee]. In fact, this is why the Rasmussen invariant could be defined in the first place.

Project 4. Prove the conjecture that all zero-omitting knots are transversely non-simple.

This project is admittedly ambitious. One way to approach it would be to try to find a relation between the θ-invariant on the knot Floer homology, used in [B, NOT, V] to prove that a certain knot is transversely non-simple, and the analog of the Rasmussen invariant for odd Khovanov homology.

3.4. **Closing remarks.** Khovanov homology is a young and quickly growing field. It lies on the crossroads of research in 3- and 4-dimensional topology, symplectic topology, homological algebra, and representation theory. The recent explosion of interest in knot homology, its structure and applications, is likely to continue in the foreseeable future.

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