DIRECTIONAL ERGODICITY AND WEAK MIXING FOR ACTIONS OF $\mathbb{R}^d$ AND $\mathbb{Z}^d$

E. ARTHUR ROBINSON JR, JOSEPH ROSENBLATT, AND AYŞE A. ŞAHİN

Abstract. We define notions of direction L ergodicity, weak mixing, and mixing for a measure preserving $\mathbb{Z}^d$-action T on a Lebesgue probability space (X, μ), where $L \subseteq \mathbb{R}^d$ is a linear subspace. For $\mathbb{R}^d$-actions these notions clearly correspond to the same properties for the restriction of T to L. For $\mathbb{Z}^d$-actions T we define them by using the restriction of the unit suspension $\widetilde{T}$ to the direction L and to the subspace of $L^2(\widetilde{X}, \widetilde{\mu})$ perpendicular to the suspension rotation factor. We show that for $\mathbb{Z}^d$-actions these properties are spectral invariants, as they clearly are for $\mathbb{R}^d$-actions. We show that for weak mixing actions T in both cases, directional ergodicity implies directional weak mixing. For ergodic $\mathbb{Z}^d$-actions T we explore the relationship between directional properties defined via unit suspensions and embeddings of T in $\mathbb{R}^d$-actions. Genericity questions and the structure of non-ergodic and non-weakly mixing directions are also addressed.

1. Introduction

Given an ergodic measure preserving $\mathbb{Z}^d$-action, $T = \{T^{\vec{n}}\}_{\vec{n} \in \mathbb{Z}^d}$, it is natural to ask which dynamical properties of the action are inherited by its subgroup actions. Namely, if T has some dynamical property and $\Lambda \subseteq \mathbb{Z}^d$ is a subgroup, does the restricted action $T^{|\Lambda} := \{T^{\vec{n}}\}_{\vec{n} \in \Lambda}$ have this property? This question is well understood in the case that Λ is a finite index subgroup of $\mathbb{Z}^d$. Any other nontrivial proper subgroup $\Lambda \subseteq \mathbb{Z}^d$ satisfies $\Lambda \approx \mathbb{Z}^e$ for some $0 < e < d$, and there is a linear subspace $L \subseteq \mathbb{R}^d$ such that $\Lambda = L \cap \mathbb{Z}^d \sim \mathbb{Z}^e$. We will refer to linear subspaces $L \subseteq \mathbb{R}^d$ as *e-dimensional directions* in $\mathbb{R}^d$, and if $\Lambda = L \cap \mathbb{Z}^d \sim \mathbb{Z}^e$, we call L a *rational direction*. Using this terminology, for subgroups Λ that correspond to rational directions, the dynamical properties of $T^{|\Lambda}$ are referred to as the *directional dynamics of the action T in the direction L*.

In the 1980's Milnor ([23], [22]) introduced a more general notion of directional dynamics, where he proposed a means of studying the dynamics of a $\mathbb{Z}^d$-action along arbitrary, not necessarily rational, directions. He defined directional entropy, which he used to refine the classification of $\mathbb{Z}^d$-actions of zero entropy up to isomorphism. The study of directional dynamics has led to other productive lines of research, the most notable among these being Boyle and Lind's work on expansive sub-dynamics [4]. In this paper we define and study directional ergodicity, weak mixing, and mixing for $\mathbb{Z}^d$-actions.

There are two approaches in the literature for defining directional properties of $\mathbb{Z}^d$-actions in an arbitrary direction L in $\mathbb{R}^d$. The first approach, which follows Milnor's original definition of directional entropy, is to study a direction L property by studying transformations $\{T^{\vec{n}_i}\}$, for vectors $\vec{n}_i \in \mathbb{Z}^d$ that approximate L in some way. The second approach is to use the unit suspension $\widetilde{T}$ of T, which is an $\mathbb{R}^d$-action (albeit on a different space). Since any direction L corresponds to a subgroup of $\mathbb{R}^d$, one can study the corresponding dynamical property of the L-action $\widetilde{T}^{|L} := \{\widetilde{T}^{\vec{\ell}}\}_{\vec{\ell} \in L}$ obtained by restricting $\widetilde{T}$ to L. These two approaches are

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equivalent for directional entropy [24] and for directional recurrence [14], but there are also directional properties for which they are not equivalent [8]. In this paper we adopt the second approach and define directional ergodicity, directional weak mixing, and directional mixing for an ergodic $\mathbb{Z}^d$-action T using its unit suspension. In order to do so, we first study these directional properties for $\mathbb{R}^d$-actions.

In their 1971 paper [25], Pugh and Shub considered the question of the ergodicity of a single transformation $S = T^{\vec{t_0}}$ from an ergodic measure preserving $\mathbb{R}^d$-action $T = \{T^{\vec{t}}\}_{\vec{t}\in\mathbb{R}^d}$. This is equivalent to studying the ergodicity of the $\mathbb{Z}$-action $S = \{T^{k\vec{t_0}}\}_{k\in\mathbb{Z}}$. They showed that S is not ergodic if and only if the spectral measure of the $\mathbb{R}$ action gives positive measure to the linear subspace $\{\vec{t_0}\}^{\perp}$. In this paper we extend the work in [25] in several ways. We show that the L-action $T^{|L}$, obtained by restricting an $\mathbb{R}^d$-action T to an e-dimensional subspace $L \subseteq \mathbb{R}^d$, is not weak mixing if and only if there exists $\vec{\ell} \in L$ so that the spectral measure of T gives positive measure to the affine subset $L^{\perp} + \vec{\ell}$. It follows, using the case $\vec{\ell} = \vec{0}$, that $T^{|L}$ is not ergodic if and only if L has positive spectral measure.

Our main interest in this paper is to study directional ergodicity and weak mixing of $\mathbb{Z}^d$-actions T, and moreover, to do this by studying the corresponding properties of their unit suspension $\mathbb{R}^d$-actions $\widetilde{T}$. In doing this we encounter a new technical issue, caused by the fact that $\widetilde{T}$ always has a rotation factor: the toral rotation introduced by the suspension. Thus it is not weak mixing in any direction, and not ergodic in any rational direction L. This lack of directional ergodicity in $\widetilde{T}$ is not related to the dynamics of T, so it is possible that T might be ergodic or weak mixing in some of those directions even though $\widetilde{T}$ is not.

We circumvent this technical issue by working in the closed subspace of L^2 orthogonal to the suspension rotation factor, and once this is done we obtain results almost identical to the $\mathbb{R}^d$ case. We show that both directional ergodicity and directional weak mixing are spectral properties of T, even for directions L that are not rational. We show that in a rational direction L, the definitions of directional ergodicity and weak mixing using $\widetilde{T}$ coincide with the usual definitions of ergodicity and weak mixing for the restriction $T^{|\Lambda}$ to the subgroup $\Lambda = L \cap \mathbb{Z}^d$. For those weak mixing $\mathbb{Z}^d$-actions T that embed in an $\mathbb{R}^d$-action $\overline{T}$ we show that T is weak mixing in the direction L if and only if the restriction $\overline{T}^{|L}$ weak mixing. We show a similar result for ergodic actions and ergodic directions.

Pugh and Shub use their spectral characterization of non-ergodicity at time t_0 to show that for an $\mathbb{R}^d$ action T, $T^{\vec{t_0}}$ is ergodic for all $\vec{t_0}$ off a countable union of affine subsets in $\mathbb{R}^d$. Analogous to their work, our spectral characterizations of directional ergodicity and directional weak mixing lead to a description of sets of directions of any dimensions $0 < e < d$ where these properties fail. We also show how to realize an example of a $\mathbb{Z}^d$ or $\mathbb{R}^d$-action that is directionally ergodic or weak mixing for any such set of directions. Finally, we prove the somewhat counterintuitive result that if T is a weak mixing $\mathbb{Z}^d$ (or $\mathbb{R}^d$) action, then T is ergodic in direction L if and only if it is weak mixing in direction L.

In the last part of the paper we show that a strongly mixing $\mathbb{Z}^d$ action is mixing in every direction and turn to questions of genericity. Ryzhikov [27] shows that a generic $\mathbb{Z}^d$-action T is weak mixing and embeds in an $\mathbb{R}^d$-action $\overline{T}$ such that $\overline{T}^{|L}$ is weak mixing for every $L \in \mathbb{G}_d$. Our Corollary 6.6, says that if a weak mixing $\mathbb{Z}^d$-action T embed in an $\mathbb{R}^d$-action $\overline{T}$, then T and $\overline{T}$ have the same weak mixing directions. Combining these two, we see that weak mixing and weak mixing in all directions are generic properties for $\mathbb{Z}^d$ actions. We end the paper with a direct proof of this fact using our spectral characterization of directional weak mixing.

There has been other work in directional recurrence properties of discrete group actions. In [14] Johnson and the third author investigated directional recurrence properties of infinite measure preserving actions of $\mathbb{Z}^d$. That work has been generalized by Danilenko [8], who also investigated directional rigidity properties of infinite measure preserving actions of $\mathbb{Z}^d$ and of the Heisenberg group. In particular, Danilenko [8] establishes a framework for studying these questions for groups Γ that are lattices in simply connected nilpotent Lie groups. Relevant to the questions we address in this paper we note that in the infinite dimensional case the definitions for directional recurrence [14] given by approximants in the $\mathbb{Z}^d$-action and the unit suspension are equivalent. However, they are different for directional rigidity and the question for recurrence in the case of groups Γ as described above remains open [8]. More recently Liu and Xu ([19],[20]) have defined directional properties intrinsically and studied their relationship to spectrum.

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2. Basic definitions and motivating examples

In this section we remind the reader of basic definitions from the spectral theory of ergodic actions, establish notation, and provide details for some basic examples that motivate the results in the paper. We refer the reader to [11] or [15] for omitted details.

2.1. **Group and subgroup actions.** Let G be a second countable locally compact abelian group (from now on we simply call G a *group*, and use additive notation). We will consider measurable and measure preserving G-actions $T = \{T^g\}_{g\in G}$ on a Lebesgue probability space (X, μ). In particular, T consists of a map $(g, x) \mapsto T^g x : G \times X \to X$ that is measurable with respect to the Borel sets in G. Moreover, for each $g \in G$, $T^g : X \to X$ is a measure preserving transformation, the map $g \mapsto T^g$ is continuous in the weak topology, and $T^g\left(T^h(x)\right) = T^h\left(T^g(x)\right) = T^{g+h}(x)$ for all $g, h \in G$. We often just refer to $T = \{T^g\}_{g\in G}$ on (X, μ) as *the G-action T*.

Let $H \subseteq G$ be a closed subgroup. The *restriction* of T to H, denoted by $T^{|H}$, is the H-action on $(X\mu)$ defined by $T^{|H} = \{T^h\}_{h\in H}$.

In this paper, the group G will usually be either $\mathbb{Z}^d$ or $\mathbb{R}^d$ or a nontrivial closed subgroup of one of these. For $G = \mathbb{R}^d$, the subgroups that will mostly interest us will be the the linear subspaces $L \subseteq \mathbb{R}^d$.

Definition 2.1. For $0 < e < d$ an e-dimensional linear subspace $L \subseteq \mathbb{R}^d$ is called an *e-dimensional direction in $\mathbb{R}^d$*.

The set of all e-dimensional directions in $\mathbb{R}^d$ is an $e(d-e)$-dimensional compact manifold, called the *Grassmanian*, and denoted by $\mathbb{G}_{e,d}$ (see [3]). We define $\mathbb{G}_{0,d} = \{\vec{0}\}$ and $\mathbb{G}_{d,d} = \{\mathbb{R}^d\}$, and we write $\mathbb{G}_d := \cup_{e=0}^d \mathbb{G}_{e,d}$.

The nontrivial subgroups $H \subseteq \mathbb{Z}^d$ are all closed, and they all satisfy $H \sim \mathbb{Z}^e$ for some $0 < e \leq d$. Letting $L = \operatorname{span}(H) \subseteq \mathbb{R}^d$ we have that $L \subseteq \mathbb{R}^d$ is an e-dimensional linear subspace (that is, $L \in \mathbb{G}_{e,d}$).

Definition 2.2. We call $L \in \mathbb{G}_{e,d}$ a *rational direction* if there exists $\Lambda \subset \mathbb{Z}^d$ with $\Lambda \sim \mathbb{Z}^e$ so that $\Lambda = L \cap \mathbb{Z}^d$. L is called an *irrational direction* if $L \cap \mathbb{Z}^d = \emptyset$.

Note that in the case $d = 2$ if L is not rational then it is irrational, but this is not true for $d > 2$.

2.2. **Duals.** The *dual group* $\widehat{G}$ of a group G is the set of *characters* of the group, namely the set of all continuous homomorphisms $\gamma : G \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$ with pointwise multiplication. With the compact open topology, $\widehat{G}$ is also a second countable locally compact abelian group.

For a closed subgroup $H \subseteq G$, the *annihilator* of H is defined by $H^\perp := \{\gamma \in \widehat{G} : \gamma(h) = 1 \text{ for all } h \in H\} \subseteq \widehat{G}$. One has $\widehat{H} = \widehat{G}/H^\perp$, and $H^\perp = \widehat{G/H}$.

Definition 2.3. We let $\pi : \widehat{G} \to \widehat{G}/H^\perp = \widehat{H}$ denote the canonical projection.

For a linear subspace $L \subseteq \mathbb{R}^d$, we identify the dual $\widehat{L}$ of with L itself by identifying $\vec{\ell} \in L$ with the character

$$\vec{\ell}(\vec{t}) = e^{2\pi i\,(\vec{\ell}\cdot\vec{t})}. \tag{2.1}$$

In this case $L^\perp$ is just the perpendicular subspace. Similarly, we identify $\widehat{\mathbb{R}}^d = \mathbb{R}^d$.

The dual group of $\mathbb{Z}^d$ is $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d \cong [0, 1)^d$, where $\vec{a} \in \mathbb{T}^d$ is identified with the character

$$\vec{a}\,(\vec{n}\,) = e^{2\pi i(\vec{a}\cdot\vec{n}\,)}. \tag{2.2}$$

If $\Lambda \subseteq \mathbb{Z}^d$ is the maximal subgroup in an e-dimensional rational direction $L \subseteq \mathbb{R}^d$ then (2.2) implies that $\Lambda^\perp = \pi(L^\perp) \subseteq \mathbb{T}^d$, where $\pi : \mathbb{R}^d \to \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is the canonical projection. In this case, $\Lambda^\perp \subseteq \mathbb{T}^d$ is a closed $(d-e)$-dimensional sub-torus, and one has $\widehat{\Lambda} = \widehat{G}/\Lambda^\perp \sim \mathbb{T}^e$. In the non-maximal case $H \subseteq \mathbb{Z}^d$ one has $H^\perp = \pi(L^\perp) + F$ where $L = \mathrm{span}(H)$ and $F \subseteq \pi(L)$ is a finite subgroup. Geometrically, $H^\perp \subseteq \mathbb{T}^d$ is a finite union of closed $(d-e)$-dimensional subtori, and $\widehat{H} = \widehat{\Lambda}/\widehat{F} \sim \mathbb{T}^e$.

2.3. **Eigenvalues and the spectral type.** For a G-action T, we say $f : X \to \mathbb{C}$, $f \in L^2(X, \mu)$, is an *eigenfunction* corresponding to *eigenvalue* $\gamma \in \widehat{G}$ if $f(T^g x) = \gamma(g)f(x)$ for all $g \in G$ and μ a.e. $x \in X$. In particular, an *invariant function* $f \in L^2(X, \mu)$, i.e. $f(T^g x) = f(x)$ for all $g \in G$, is an eigenfunction for eigenvalue $0 \in \widehat{G}$. The set $\Sigma \subseteq \widehat{G}$ of eigenvalues of T is called the *point spectrum* of T.

A G-action T is *ergodic* if the constant functions are the only invariant functions. A G-action T is *weak mixing* if the constant functions are the only eigenfunctions. The following is standard.

Lemma 2.4. *Let T be an ergodic G-action. Then*

(1) *all eigenvalues are simple (each eigenfunction f is unique up to a constant multiple),*
(2) *all eigenfunctions f have constant absolute value μ a.e. (up to a constant multiple we may assume $|f| = 1$), and*
(3) *the set $\Sigma \subseteq \widehat{G}$ is an (at most) countable subgroup.*

The *Koopman representation* for a G-action T is the unitary representation of G on the (complex) separable Hilbert space $L^2(X, \mu)$, $U_T = \{U_T^g\}_{g\in G}$, defined by $(U_T^g f)(x) := f(T^g x)$. For any $f \in L^2(X, \mu)$ one defines the correlation function

$$\varphi_f(g) := (U_T^g f, f) = \int_X f(T^g x)\overline{f(x)}\, d\mu. \tag{2.3}$$

The function φ_f is positive definite on G so by Bochner's Theorem, there exists a unique finite Borel measure σ_f on $\widehat{G}$ with φ_f as its Fourier transform (see [15]):

$$\varphi_f(g) = \int_{\widehat{G}} \overline{\gamma(g)} d\sigma_f(\gamma). \tag{2.4}$$

Assuming T is implicitly known, we call σ_f the *spectral measure* for f.

Many of our arguments depend on consequences of the *spectral theorem for unitary representations of* G (see [15]). Most of what we will need is included in the following proposition.

Proposition 2.5. *Suppose T is a G-action on (X, μ) and let U_T be the corresponding Koopman representation of G on $L^2(X, \mu)$. Let $U_T|_{\mathcal{F}}$ be the restriction of U_T to the closed U_T invariant subspace $\mathcal{F} \subseteq L^2(X, \mu)$. Then there exists $f_0 \in \mathcal{F}$ so that for any $f \in \mathcal{F}$ the corresponding spectral measures satisfy $\sigma_f \ll \sigma_{f_0}$. Conversely, for any $\sigma \ll \sigma_{f_0}$ there is an $f \in \mathcal{F}$ so that $\sigma = \sigma_f$. Moreover, if $\sigma_{f_1} \perp \sigma_{f_2}$ for $f_1, f_2 \in \mathcal{F}$ then $U_T^g f_1 \perp f_2$ for all $g \in G$.*

Note that σ_{f_0} is unique up to Radon-Nikodym equivalence. We call f_0 and σ_{f_0}, respectively, a *function* and a *measure* of *maximal spectral type* for $U_T|_{\mathcal{F}}$.

By Proposition 2.5 there exists $f_* \in L^2(X, \mu)$ that is a function of maximal spectral type for U_T on $L^2(X, \mu)$. We call the measure $\sigma^T := \sigma_{f^*}$ (unique up to equivalence) the *spectral measure* for T. Let $L_0^2(X, \mu) = \{1\}^\perp \subseteq L^2(X, \mu)$. By Proposition 2.5 there is also is a function (also unique up to equivalence) $f_0 \in L_0^2(X, \mu)$ of maximal spectral type for the restriction $U_T|_{L_0^2(X,\mu)}$. We call $\sigma_0^T = \sigma_{f_0}$ the *reduced spectral measure* for T.

If two G-actions T and S are metrically isomorphic, then their Koopman representations are U_T and U_S are unitarily conjugate (also called spectrally isomorphic), and thus by the spectral theorem (see [15]), their reduced spectral measures σ_0^T and σ_0^S (and their spectral measures σ^T and σ^S) are equivalent. Thus the spectral measures and reduced spectral measures are isomorphism invariants for G-actions T.

The following results are well known (see e.g., [15]).

Lemma 2.6. *Let T be a G-action. Then $\gamma \in \Sigma$ if and only if $\sigma^T(\{\gamma\}) > 0$ (i.e., γ is an atom).*

Corollary 2.7. *Let T be an ergodic G-action. Then*

1. *$\sigma_0^T = \sigma^T - \sigma^T(\{0\})\delta_0$, where δ_0 denotes unit point mass at $0 \in \widehat{G}$, and*
2. *T is weak mixing if and only if σ_0 is non-atomic.*

Lemma 2.8 (see Theorem 3.16 in [15])**.** *Let T be an ergodic G-action and let $\gamma \in \Sigma$. Define $\tau^\gamma : \widehat{G} \to \widehat{G}$ by $\tau^\gamma(\omega) = \omega + \gamma$. Then $\sigma^T \sim \sigma^T \circ \tau^\gamma$, where $\sim$ denotes the equivalence of measures.*

For notational convenience we will use ς to denote spectral measures of actions of continuous groups and σ to denote spectral measures of actions of discrete groups.

2.4. **Properties of subgroup actions.** It is a straightforward observation for $H \subseteq G$ a closed subgroup that if $T^{|H}$ is ergodic then T is also ergodic (ergodicity passes to supergroup actions). Thus, if T is not ergodic then no restriction $T^{|H}$ of T can be ergodic. The same statements hold for weak mixing. The next two examples, and the discussion in the remainder of the section, indicate that the converse statements are not generally true and motivate the focus of the paper.

Example 2.9. Let T_1 and T_2 be measure preserving $\mathbb{Z}$-actions on (X,μ). Define the *product type* $\mathbb{Z}^2$-action $T_1 \otimes T_2$ by

$$(T_1 \otimes T_2)^{\vec{n}}(x_1, x_2) = (T_1^{n_1} x_1, T_2^{n_2} x_2),$$

$\vec{n} = (n_1, n_2)$. Let $\Lambda_1 = \{(n_1, 0) : n_1 \in \mathbb{Z}\}$ and $\Lambda_2 = \{(0, n_2) : n_2 \in \mathbb{Z}\}$ be the vertical and horizontal subgroups of $\mathbb{Z}^2$. Clearly the restrictions $T_1^{|\Lambda_1}$ and $T_2^{|\Lambda_2}$ are not ergodic since they correspond to the $\mathbb{Z}$-actions $T_1 \times Id$ and $Id \times T_2$. However, it is easy to see that $T_1 \otimes T_2$ is ergodic (or weak mixing) if and only if T_1 and T_2 are ergodic (or weak mixing). Indeed, if $F\left((T_1 \otimes T_2)^{(n_1,n_2)}(x_1, x_2)\right) = e^{2\pi i (n_1 a_1 + n_2 a_2)} F(x_1, x_2)$ then $f_1(x_1) = \int_X F(x_1, x) d\mu(x)$ is an eigenfunction for T_1 with eigenvalue a_1. Sim Similarly $f_2(x_2) = \int_X F(x, x_2) d\mu(x)$ is an eigenfunction for T_2 with eigenvalue a_2. Conversely if $f_1(T_1 x) = e^{2\pi i a_1} f_1(x)$ (or $f_2(T_2 x) = e^{2\pi i a_1} f_2(x)$), then $F(x_1, x_2) = f_1(x_1)$ ($F(x_1, x_2) = f_2(x_2)$) is an eigenfunction for $T_1 \otimes T_2$ with eigenvalue $(a_1, 0)$ ($(0, a_2)$).

Example 2.10. We provide an example here, due to Bergelson and Ward [2], of a weak mixing $\mathbb{Z}^2$-action $\mathring{T}$ so that $\mathring{T}^{|\Lambda}$ is not ergodic for any subgroup of the form $\Lambda = \{k\vec{n} : k \in \mathbb{Z}\}$.

Let T be a weak mixing $\mathbb{Z}$-action on (X,μ) and let $\mathring{X} = \prod_{i=1}^{\infty} X_i$, each $X_i = X$, with product measure $\mathring{\mu}$. Let $\{\vec{m}_i\}_{i\in\mathbb{Z}}$ be an enumeration of $\mathbb{Z}^2$, and define a $\mathbb{Z}^2$-action $\mathring{T}$ on $\mathring{X}$ by

$$\mathring{T}^{\vec{n}}\left(\prod_{i=1}^{\infty} x_i\right) = \prod_{i=1}^{\infty} T^{\vec{n}\cdot\vec{m}_i}(x_i)$$

for $\vec{n} \in \mathbb{Z}^2$. Note that $\mathring{T}$ is weak mixing as in Example 2.9. For each subgroup $\Lambda = \{k\vec{n} : k \in \mathbb{Z}\}$ the generating transformation $\mathring{T}^{\vec{n}}$ is not ergodic because it acts as the identity on X_i for any i with $\vec{m}_i \perp \vec{n}$. This construction can be generalized to provide a variety of examples, all with similar types of directional properties.

Here is a simple way that ergodicity or weak mixing can fail for subgroup actions.

Proposition 2.11. *Let T be a G-action that is not weak mixing and let $\gamma \in \Sigma\backslash\{0\}$. If there exists a subgroup $H \subseteq G$ so that $\gamma \in H^\perp$ then $T^{|H}$ is not ergodic.*

Proof. Let f be an eigenfunction for γ. Then $\gamma \in H^\perp$ means $\gamma(h) = 1$ for all $h \in H$, which implies $f(T^h(x)) = f(x)$ for all $h \in H$. Thus f is an invariant function for $T^{|H}$. □

In the case $G = \mathbb{R}^d$, such a subgroup H always exists. Indeed, if $\vec{\ell} \in \Sigma\backslash\{0\}$ then $H = \langle\vec{\ell}\rangle^\perp$. The situation is quite different for $G = \mathbb{Z}^d$. Consider the case of d rotations on the circle $\mathbb{T}$. Since rotations commute they define a measure preserving $\mathbb{Z}^d$-action T on $\mathbb{T}$ with Lebesgue measure. If the rotations are by irrationals that are rationally independent then T is ergodic. It is clear that any subgroup action is another such rotation and therefore ergodic. To relate this to Proposition 2.11 we note that the eigenvalues Σ satisfy $\mathbb{Z}^d \cap \Sigma = \{\vec{0}\}$ so $H^\perp \cap \Sigma = \{\vec{0}\}$ for all subgroups $H \subseteq \mathbb{Z}^d$. Thus all subgroup actions are ergodic. Contrast this with the case of $\mathbb{Z}^d$ odometer actions (see [7]) which always satisfy $\Sigma \subseteq \mathbb{Q}^d/\mathbb{Z}^d \subseteq \mathbb{T}^d$. Thus for any such action T there are subgroups $H \subseteq \mathbb{Z}^d$ such that $H^\perp \cap \Sigma \neq \{\vec{0}\}$, and there are always non-ergodic subgroup actions.

To go beyond Proposition 2.11 we need to describe the relationship between the spectral measures of T and those of $T^{|H}$.

Proposition 2.12. *Let T be a G-action on (X,μ), with Koopman representation U_T. Let $U_T|_{\mathcal{F}}$ be the restriction of U_T to a closed U_T invariant subspace $\mathcal{F} \subseteq L^2(X,\mu)$. For a closed subgroup $H < G$ let $U_T|_{\mathcal{F}}^H := (U_{T^{|H}})|_{\mathcal{F}}$ be the unitary representation of H on $\mathcal{F}$ obtained by restricting $U_T|_{\mathcal{F}}$ to H. If $f_0 \in \mathcal{F}$ is a function of maximal spectral type for $U_T|_{\mathcal{F}}$ then f_0 is also a function of maximal spectral type for $U_T|_{\mathcal{F}}^H$. Moreover, $\sigma_{f_0}^{T^{|H}} = \sigma_{f_0}^T \circ \pi^{-1}$ where π is given by Definition 2.3 and $\sigma_{f_0}^T$ on $\widehat{G}$ and $\sigma_{f_0}^{T^{|H}}$ on $\widehat{H}$ are the corresponding spectral measures.*

Proof. Note that for all $h \in H$ and $\gamma \in \widehat{G}$, $\gamma(h)$ is constant on the cosets of $H^\perp$ in $\widehat{G}$. Let $f \in \mathcal{F}$. By (2.4) we have

$$\begin{aligned}\widehat{\sigma_f^T}(h) &= \int_{\widehat{G}} \overline{\gamma(h)}\, d\sigma_f^T(\gamma) = \int_{\widehat{G}} \overline{(\gamma + H^\perp)\,(h)}\, d\sigma_f^T(\gamma)\\ &= \int_{\widehat{G}/H^\perp} \overline{(\gamma + H^\perp)\,(h)}\, d(\sigma_f^T \circ \pi^{-1})(\gamma + H^\perp).\end{aligned}$$

That is,

$$\widehat{\sigma_f^T}(h) = \int_{\widehat{G}/H^\perp} \overline{(\gamma + H^\perp)\,(h)}\, d(\sigma_f^T \circ \pi^{-1})(\gamma + H^\perp) = \int_{\widehat{H}} \overline{\alpha(h)}\, d(\sigma_f^T \circ \pi^{-1})(\alpha),$$

where $\alpha \in \gamma + H^\perp$. On the other hand

$$\widehat{\sigma_f^T}(h) = \int_X f(T^{h^{-1}}x)\overline{f(x)}\, dm(x) = \widehat{\sigma_f^{T^{|H}}}(h) = \int_{\widehat{H}} \overline{\alpha(h)}\, d\sigma_f^{T^{|H}}(\alpha).$$

Hence, for any $f \in \mathcal{F}$, $\sigma_f^{T^{|H}}$ is the *push forward* of σ_f^T; that is, $\sigma_f^{T^{|H}} = \sigma_f^T \circ \pi^{-1}$.

Now suppose that $\sigma_{f_0}^T$ is a measure maximal spectral type for T on $\mathcal{F}$. Let $f \in \mathcal{F}$ and suppose $\sigma_{f_0}^{T^{|H}}(E) = 0$ for some $E \subseteq \widehat{H}$. Then by our discussion above $\sigma_{f_0}^T(\pi^{-1}(E)) = 0$. Since f_0 is the function of maximal type for T on $\mathcal{F}$, it follows that $0 = \sigma_f^T(\pi^{-1}(E)) = \sigma_f^{T^{|H}}(E)$, so $\sigma_{f_0}^{T^{|H}}$ is a measure of maximal spectral type of $T^{|H}$ on $\mathcal{F}$. $\square$

Corollary 2.13. *Let σ^T and σ_0^T denote the spectral measure and reduced spectral measure for a G-action T. For a closed subgroup $H < G$ let $\sigma^{T^{|H}}$ and $\sigma_0^{T^{|H}}$ be the same for the restriction $T^{|H}$. Then $\sigma^{T^{|H}} = \sigma^T \circ \pi^{-1}$ and $\sigma_0^{T^{|H}} = \sigma_0^T \circ \pi^{-1}$ where π is given by Defintion 2.3.*

Now we can strengthen Proposition 2.11.

Proposition 2.14. *Let T be a G-action, $H \subseteq G$ a closed subgroup, and $T^{|H}$ the restriction of T to H. Then $\gamma = \nu + H^\perp \in \widehat{H} = \widehat{G}/H^\perp$ is an eigenvalue for $T^{|H}$ for a non-constant eigenfunction if and only if $\sigma_0^T(\nu + H^\perp) > 0$.*

Proof. By Proposition 2.6, for $\gamma \in \widehat{H}$ to be an non-constant eigenvalue for $T^{|H}$ is equivalent to $\sigma_0^T(\{\gamma\}) > 0$. On the other hand by Proposition 2.13 we have $0 < \sigma_0^{T^{|H}}(\{\gamma\}) = \sigma_0^T(\pi^{-1}\{\gamma\}) = \sigma_0^T(\nu + H^\perp)$. $\square$

Corollary 2.15. *Let T be a G-action and let H be a closed subgroup. Then*

(1) *$T^{|H}$ is not ergodic if and only if $\sigma_0^T(H^\perp) > 0$, and*
(2) *$T^{|H}$ is not weak mixing if and only if $\sigma_0^T(\nu + H^\perp) > 0$ for some $\nu \in \widehat{G}$.*

Pugh and Shub showed in [25] that an element T^h of a G-action T is not ergodic if and only if $\sigma_0^T(\{h\}^\perp) > 0$. For $G = \mathbb{R}^d$, $\vec{h} \in \mathbb{R}^d$ define $\Gamma = \{k\vec{h} : n \in \mathbb{Z}\}$. Then $T^{\vec{h}}$ is not ergodic as a $\mathbb{Z}$-action if and only if $T^{|\Gamma}$ is not ergodic. Additionally, since $\vec{\ell} \cdot \vec{h} \in \mathbb{Z}$ if and only if $\vec{\ell} \cdot k\vec{h} \in \mathbb{Z}$ for all $k \in \mathbb{Z}$ we have that $\Gamma^\perp = \{h\}^\perp$ so their result follows from Corollary 2.15. Note that if $L = \{t\vec{h} : t \in \mathbb{R}\}$ is the direction generated by $\vec{h}$, then $L^\perp \subset \{h\}^\perp$, but it is not necessarily true that $L^\perp$ lies in the support of σ_0^T. Therefore $T^{\vec{h}}$ can fail to be ergodic, even though T is ergodic in the direction L.

We will use Corollary 2.15 extensively below. But first we consider a consequence. As we noted above, if $T^{|H}$ is ergodic (or weak mixing) then T is too. Although the converse is generally false, it is true in the following situation, which includes the case where $\Lambda \subseteq \mathbb{Z}^d$ is a finite index subgroup.

Proposition 2.16. *If T is a weak mixing G-action and $H \subseteq G$ is co-compact then $T^{|H}$ is weak mixing, and thus ergodic.*

Proof of Proposition 2.16. The co-compactness of H implies $H^\perp \subseteq \widehat{G}$ is discrete since $H^\perp = \widehat{G/H}$ is the dual of a compact group. If $T^{|H}$ is not weak mixing, then Corollary 2.15 implies $\sigma_0^T(\nu + H^\perp) > 0$ for some $\nu \in \widehat{G}$, but since $H^\perp$ is discrete, $\sigma_0^T(\{\nu + h^\perp\}) > 0$, for some $h^\perp \in H^\perp$. □

3. Directional properties for $\mathbb{R}^d$-actions

Consider an $\mathbb{R}^d$-action $T = \{T^{\vec{t}}\}_{\vec{t}\in\mathbb{R}^d}$ on (X, μ). Since every direction $L \in \mathbb{G}_{e,d}$ corresponds to a subgroup of $\mathbb{R}^d$ it is natural to define directional ergodicity for T in a direction L to mean the ergodicity of the subgroup action $T^{|L}$ of T (see Section 2), and similarly for directional weak mixing.

It is also possible to choose a set of generators for an e-dimensional direction L and to study the ergodicity and weak mixing properties of the resulting $\mathbb{R}^e$-action. We begin the section showing that while different choices of bases can give rise to non-conjugate actions, all choices will yield $\mathbb{R}^e$-actions with the same ergodicity and mixing properties as $T^{|L}$. To prove this, fix $L \in \mathbb{G}_{e,d}$, choose a basis $\{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_e\}$, and let B be the $(d \times e)$-matrix with $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_e$ as columns. One obtains an $\mathbb{R}^e$-action $T \circ B$, defined $(T \circ B)^{(s_1,s_2,\dots,s_e)}x = T^{(s_1\vec{b}_1+s_2\vec{b}_2+\cdots+s_e\vec{b}_e)}x$. Lind and Boyle [4] call such a choice of basis in this context a *frame*.

Lemma 3.1. *Let T be an $\mathbb{R}^d$-action and let $L \in \mathbb{G}_{e,d}$ and B be as above. Then $\vec{\ell} \in L = \widehat{L}$ is an eigenvalue for $T^{|L}$ if and only if $B^t\vec{\ell} \in \mathbb{R}^e = \widehat{\mathbb{R}}^e$ is an eigenvalue for the $\mathbb{R}^e$-action $T \circ B$.*

Proof. For $\vec{s} \in \mathbb{R}^e$ we have $f((T \circ B)^{\vec{s}}x) = f(T^{B\vec{s}}x) = e^{2\pi i\,(\vec{\ell}\cdot B\vec{s})}f(x) = e^{2\pi i\,(B^t\vec{\ell}\cdot\vec{s})}f(x)$. □

In what follows we always work with the subgroup action, rather than choosing a frame.

Definition 3.2. Let T be an $\mathbb{R}^d$-action and let $L \in \mathbb{G}_{e,d}$ be a direction with $1 \le e \le d$.

(1) We say T is *ergodic* in the direction L if the L-action $T^{|L}$ is ergodic. We denote the set of ergodic directions for T by $\mathcal{E}_T \subseteq \mathbb{G}_d$.
(2) We say T is *weak mixing* in the direction L if the L-action $T^{|L}$ being weak mixing. We denote the set of weak mixing directions for T by $\mathcal{W}_T \subseteq \mathbb{G}_d$.

Proposition 2.14 and Corollary 2.15 show that the ergodicity or weak mixing of T in the direction L depends on the invariant functions and eigenfunctions of the L-action $T^{|L}$. For ease of exposition we cast the notation from Section 2 concretely in the case of $G = \mathbb{R}^d$ and $L \in \mathbb{G}_{e,d}$. Note that we implicitly identify the dual $\widehat{L}$ with L (as explained in Section 2.2). A nonzero $f \in L^2(X,\mu)$ is an eigenfunction for $T^{|L}$ with eigenvalue $\vec{\ell}$ if $f(T^{\vec{t}}x) = e^{2\pi i(\vec{\ell}\cdot\vec{t})}f(x)$ for all $\vec{t} \in L$ and μ-a.e. $x \in X$. We call f a *direction L eigenfunction for* T and $\vec{\ell} \in L$ a *directional L eigenvalue* for T. If $\vec{\ell} = \vec{0}$ then we call f a direction L invariant function for T.

Definition 3.2 can now be restated as follows.

Definition 3.3. Let T be an $\mathbb{R}^d$-action and let $L \in \mathbb{G}_{e,d}$ be a direction with $1 \leq e \leq d$.

(1) We say T is *ergodic* in the direction L if it has no non-zero direction L invariant functions f in $L_0^2(X,\mu)$.
(2) We say T is *weak mixing* in the direction L if it has no non-zero direction L eigenfunctions f in $L_0^2(X,\mu)$ (or equivalently, no direction L eigenvalues, $\vec{\ell} \neq \vec{0}$).

In other words, $\vec{\ell}$ is an eigenvalue and f is the corresponding eigenfunction for the L-action $T^{|L}$. A direction L invariant function f for T is a direction L eigenfunction with eigenvalue $\vec{\ell} = \vec{0}$. Note that for $e = d$, both Definition 3.2 and Definition 3.3 are consistent with the usual definitions of the ergodicity or weak mixing of T.

3.1. **Characterizing directional properties spectrally.** In this section we will show how directional ergodicity and directional weak mixing of $\mathbb{R}^d$-actions can be characterized in terms of the existence of positive-measure affine subsets of $\mathbb{R}^d$ perpendicular to the direction in question. Our work here is closely related to the results in [25], and we begin by formalizing the idea of a measure with lower dimensional support.

Definition 3.4. Let ς be a finite Borel measure on $\mathbb{R}^d$. Let $L \in \mathbb{G}_{e,d}$, and let $P = L + \vec{\ell}$ where $\vec{\ell} \in L^\perp$.

(1) If $\varsigma(P) > 0$, we say that P is an e-dimensional *wall* for ς.
(2) A wall P for ς is called *proper* if it contains no lower dimensional walls $Q \subseteq L$ for ς.
(3) We call a measure ς an e-dimensional *wall-measure* if there is an e-dimensional proper wall P for ς that contains its support. We call P the *carrier* of ς.

Note that a 0-dimensional wall measure is an atomic measure with a single atom $\vec{\ell} \in \mathbb{R}^d$ as its carrier, and a d-dimensional wall measure is a (non-zero) measure with carrier $\mathbb{R}^d$.

Theorem 3.5. *Let T be an ergodic $\mathbb{R}^d$-action and $L \in \mathbb{G}_d$.*

(1) *$L \in \mathcal{E}_T$ if and only if $L^\perp$ is not a wall for ς_0^T.*
(2) *$L \in \mathcal{W}_T$ if and only if no affine subset $P = L^\perp + \vec{\ell}$, $\vec{\ell} \in L$ is a wall for ς_0^T. In other words, no affine subset perpendicular to L is a wall for ς_0^T.*

This is just Proposition 2.14 translated into the terminology of this section. Since spectral type is an isomorphism invariant (see the discussion before Lemma 2.6) we have the following immediate corollary.

Corollary 3.6. *Directional ergodicity and directional weak mixing for $\mathbb{R}^d$-actions T are spectral properties (i.e., they depend only on ς_0^T). It follows that they are isomorphism invariants.*

4. Directional properties for $\mathbb{Z}^d$-actions

We can easily define the directional behavior of $\mathbb{Z}^d$-actions for rational directions (see Definition 2.2) analogous to our definitions of directional properties for $\mathbb{R}^d$-actions: by the behavior of the associated subgroup action.

In this section we use the unit suspension $\widetilde{T}$ of a $\mathbb{Z}^d$-action T in order to extend the notions of directional ergodicity and directional weak mixing to arbitrary directions. We show that these are spectral properties for the $\mathbb{Z}^d$-action itself, and that the subgroup definition is consistent with the directional definition if L is a rational direction.

4.1. **The unit suspension.** Consider a $\mathbb{Z}^d$-action $T = \{T^{\vec{t}}\}_{\vec{t}\in\mathbb{R}^d}$ on (X, μ). Let λ denote Lebesgue probability measure on $\mathbb{T}^d = [0,1)^d$ the d-dimensional torus. The *unit suspension* $\widetilde{T}$ of T is the $\mathbb{R}^d$-action on $(\widetilde{X}, \tilde{\mu}) := (X \times [0,1)^d, \mu \times \lambda)$ defined by $\widetilde{T}^{\vec{t}}(x, \vec{r}\,) = (T^{(\lfloor \vec{t}+\vec{r}\rfloor)}x, \{\vec{t}+\vec{r}\})$, where $\lfloor \vec{w} \rfloor \in \mathbb{Z}^d$ denotes the component-wise floor, and $\{\vec{w}\} := \vec{w} - \lfloor \vec{w} \rfloor \in \mathbb{T}^d$.

It is well known that the unit suspension $\widetilde{T}$ of a $\mathbb{Z}^d$-action T is ergodic if and only if T is ergodic. However, as the next result shows, it is never a weak mixing $\mathbb{R}^d$-action.

Proposition 4.1. *Let T be an ergodic $\mathbb{Z}^d$-action and $\widetilde{T}$ its unit suspension. Define the $\mathbb{R}^d$-action $\{\rho^{\vec{t}}\}_{\vec{t}\in\mathbb{R}^d}$ on $[0,1)^d$ by $\rho^{\vec{t}}(\vec{r}) = \{\vec{r} + \vec{t}\}$. Then the projection $p_2 : X \times [0,1)^d \to [0,1)^d$ given by $p_2(x, \vec{r}) = \vec{r}$ is a factor map from $\widetilde{T}$ to ρ. In particular, $\widetilde{T}$ is not weak mixing and its set of eigenvalues, denoted by Σ, satisfies $\mathbb{Z}^d \subseteq \Sigma$.*

Proof. The fact that p_2 is a factor map follows directly from the definition of $\widetilde{T}$. Given $\vec{n} \in \mathbb{Z}^d$ it is easy to check that $f(x, \vec{r}) = e^{2\pi i(\vec{n}\cdot\vec{r})}$ is an eigenfunction for $\widetilde{T}$ with eigenvalue $\vec{n}$. □

This rotation factor is responsible for the presence of many non-ergodic directions for $\widetilde{T}$.

Proposition 4.2. *If $L \in \mathbb{G}_{d-1,d}$ is such that $L^\perp \cap \mathbb{Z}^d \neq \{\vec{0}\}$ then $L \in \mathcal{E}^c_{\widetilde{T}}$.*

Proof. This follows directly from Propositions 4.1 and 2.11. □

4.2. **Definition of directional properties.** The presence of the non-ergodic directions for $\widetilde{T}$ identified in Proposition 4.2 means that we cannot define directional properties of T by applying Definition 3.2 directly to $\widetilde{T}$. However, since this directional non-ergodicity is a result only of the rotation factor ρ defined in Proposition 4.1 it has no bearing on the dynamical properties of T. We overcome this complication by defining directional properties in analogy to Definition 3.3 by restricting our attention to eigenfunctions of $\widetilde{T}$ from the closed $\widetilde{T}$ invariant subspace $L^2_{\mathcal{O}}(\widetilde{X}, \tilde{\mu}) := \mathcal{O}^\perp \subseteq L^2(\widetilde{X}, \tilde{\mu})$ defined by

$$\mathcal{O} := \left\{F(x, \vec{s}) = f(\vec{s}) : f \in L^2\left([0,1)^d, \lambda\right)\right\} \subseteq L^2(\widetilde{X}, \tilde{\mu}). \tag{4.1}$$

Definition 4.3. Let T be a $\mathbb{Z}^d$-action on (X, μ), $\widetilde{T}$ its unit suspension, and let $L \in \mathbb{G}_{e,d}$ be a direction with $1 \leq e \leq d$.

(1) We say T is *ergodic* in the direction L if $\widetilde{T}$ has no non-zero direction L invariant functions in $L^2_{\mathcal{O}}(X, \mu)$. We denote the set of ergodic directions for T by $\mathcal{E}_T \subseteq \mathbb{G}_d$.

(2) We say T is *weak mixing* in the direction L if $\widetilde{T}$ has no non-zero direction L eigenfunctions in $L^2_{\mathcal{O}}(X, \mu)$. We denote the set of weak mixing directions for T by $\mathcal{W}_T \subseteq \mathbb{G}_d$.

4.3. **Characterizing directional properties spectrally.**
By Theorem 3.5 directional ergodicity and directional weak mixing are defined for a $\mathbb{Z}^d$-action T, implicitly, in terms of the spectral measure for its unit suspension $\widetilde{T}$ restricted to $L^2_{\mathcal{O}}(\widetilde{X}, \tilde{\mu})$. Since the subspace $L^2_{\mathcal{O}}(\widetilde{X}, \tilde{\mu})$ is $\widetilde{T}$ invariant,

Proposition 2.5 implies that there exists $\tilde{f}_0 \in L^2_{\mathcal{O}}(\widetilde{X}, \tilde{\mu})$ so that $\varsigma_{\tilde{f}} \ll \varsigma_{\tilde{f}_0}$ for all $\tilde{f} \in L^2_{\mathcal{O}}(\widetilde{X}, \tilde{\mu})$. To parallel the notation for a reduced spectral measure we make the following definition.

Definition 4.4. The measure $\varsigma_{\mathcal{O}}^{\widetilde{T}}$ is called the *aperiodic spectral measure of* $\widetilde{T}$.

The aperiodic spectral measure is unique up to equivalence. The following is immediate.

Lemma 4.5. *Let T be a measure preserving $\mathbb{Z}^d$-action and $\widetilde{T}$ its unit suspension. Then the aperiodic spectral measure of $\widetilde{T}$, $\varsigma_{\mathcal{O}}^{\widetilde{T}}$, differs from $\varsigma_0^{\widetilde{T}}$, the reduced spectral measure of $\widetilde{T}$, by omitting the nonzero point spectra that come from the rotation factor ρ, i.e. the eigenfunctions in $\mathcal{O}$.*

In this section we will show that the aperiodic spectral measure is in fact determined by σ_0^T, the spectral measure for the $\mathbb{Z}^d$ action T, and therefore directional ergodicity and directional weak mixing for T in an arbitrary direction L is a spectral property of the $\mathbb{Z}^d$-action itself.

The following is a result from [9], adapted to our context. In the lemma, $\pi : \mathbb{R}^d \to \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ denotes the canonical projection, defined $\pi(\vec{t}) = \vec{t} \mod \mathbb{Z}^d$, and $\sim$ denotes equivalence of measures..

Lemma 4.6. *Let T be a measure preserving $\mathbb{Z}^d$-action and let $\widetilde{T}$ be its unit suspension. Then $\varsigma_{\mathcal{O}}^{\widetilde{T}} \circ \pi^{-1} \sim \sigma_0^T$.*

Proof. Proposition 2.1 in [9] states that $\varsigma^{\widetilde{T}} \circ \pi^{-1} \sim \sigma^T$. This, combined with Lemma 4.5 yields the result. □

By identifying $\mathbb{T}^d = [0,1)^d$ we can think of σ_0^T as a measure on $[0,1)^d$ or as finite measure on $\mathbb{R}^d$ with support $[0,1)^d$. Taking the latter view, we define a finite Borel measure on $\mathbb{R}^d$ by

$$\widetilde{\sigma_0^T} = \sum_{\vec{n}\in\mathbb{Z}^d} a_{\vec{n}} \cdot (\sigma_0^T \circ \tau^{\vec{n}}), \tag{4.2}$$

where $a_{\vec{n}}$ is an arbitrary positive sequence with $\sum_{\vec{n}\in\mathbb{Z}^d} a_{\vec{n}} = 1$ and $\tau^{\vec{n}}(\vec{t}) := \vec{t} - \vec{n}$ is translation on $\mathbb{R}^d$.

Corollary 4.7. $\varsigma_{\mathcal{O}}^{\widetilde{T}} \ll \widetilde{\sigma_0^T}$.

Proof. For $\vec{n} \in \mathbb{Z}^d$ let $Q_{\vec{n}} = [0,1)^d + \vec{n} \subseteq \mathbb{R}^d$. Then using Lemma 4.6 we have $\varsigma_{\mathcal{O}}^{\widetilde{T}}|_{Q_{\vec{n}}} \circ \tau^{-\vec{n}} \ll \sigma_0^T$. The result follows from the fact that $\varsigma_{\mathcal{O}}^{\widetilde{T}} = \sum_{\vec{n}\in\mathbb{Z}^d} \varsigma_{\mathcal{O}}^{\widetilde{T}}|_{Q_{\vec{n}}}$. □

For equivalence of the two measures we need ergodicity.

Theorem 4.8. *Let T be an ergodic $\mathbb{Z}^d$-action and let $\widetilde{T}$ be its unit suspension. Then $\varsigma_{\mathcal{O}}^{\widetilde{T}} \sim \widetilde{\sigma_0^T}$.*

Proof. Since T is ergodic, so is $\widetilde{T}$. Additionally, letting Σ denote the set of eigenvalues of $\widetilde{T}$, by Proposition 4.1 we have $\mathbb{Z}^d \subseteq \Sigma$. By Lemma 2.8, for all $\vec{n} \in \mathbb{Z}^d$ $\varsigma_{\mathcal{O}}^{\widetilde{T}}|_{Q_{\vec{n}}} \sim \varsigma_{\mathcal{O}}^{\widetilde{T}}|_{Q_{\vec{0}}}$. By the definition of π and Lemma 4.6 this means for all $\vec{n} \in \mathbb{Z}^d$ $\varsigma_{\mathcal{O}}^{\widetilde{T}}|_{Q_{\vec{n}}} \sim \sigma_0^T \circ \tau^{-\vec{n}}$ therefore $\varsigma_{\mathcal{O}}^{\widetilde{T}} = \sum_{\vec{n}\in\mathbb{Z}^d} \varsigma_{\mathcal{O}}^{\widetilde{T}}|_{Q_{\vec{n}}} \sim \widetilde{\sigma_0^T}$. □

We are now ready to define directional behavior for a $\mathbb{Z}^d$-action in terms of its spectral measure. We begin by adapting the definition of a wall measure, Definition 3.4, to the case of measures on $\mathbb{T}^d$ with $\pi : \mathbb{R}^d \to \mathbb{T}^d = [0,1)^d$ defined $\pi(\vec{t}) = \vec{t} \mod \mathbb{Z}^d$. For $L \in \mathbb{G}_{e,d}$ the projection $\pi(L^\perp)$ (of its perpendicular subspace) is a $(d-e)$-dimensional (and not always closed) subgroup of $\mathbb{T}^d$. As in the $\mathbb{R}^d$ case, failure of weak mixing in a direction L can be detected by the presence of a wall measure with support perpendicular to that direction. For $\vec{\ell} \in L$, we refer to $P = \pi(L^\perp + \vec{\ell}) = \pi(L^\perp) + \pi(\vec{\ell})$ as an affine coset in $\mathbb{T}^d$ perpendicular to L.

Definition 4.9. Let σ be a finite Borel measure on $\mathbb{T}^d$. Let $L \in \mathbb{G}_{e,d}$ so that $L^\perp \in \mathbb{G}_{d-e,d}$.

(1) Let $P = \pi(L^\perp) + \vec{\ell}$ be an affine subset $\mathbb{T}^d$. If $\sigma(P) > 0$, we say that P is a $(d-e)$-dimensional *wall* for σ. The wall P is said to be in the direction $L^\perp$, or perpendicular to L.
(2) A wall P for σ is called *proper* if P contains no lower dimensional walls $Q \subseteq P$ for σ.
(3) We call a measure σ an e-dimensional *wall-measure* if there is an e-dimensional proper wall P for σ that contains its support. We call P the *carrier* of σ.

A 0-dimensional wall measure on $\mathbb{T}^d$ is an atomic measure with a single atom $\vec{\ell} \in \mathbb{T}^d$ as its carrier, and a d-dimensional wall measure is a (non-zero) measure with carrier $\mathbb{T}^d$.

The following theorem is the $\mathbb{Z}^d$ analogue of Theorem 3.5.

Theorem 4.10. *Let T be an ergodic $\mathbb{Z}^d$-action and $L \in \mathbb{G}_{e,d}$. Then*

(1) *$L \in \mathcal{E}_T$ if and only if $\pi(L^\perp)$ is not a wall for σ_0^T, and*
(2) *$L \in \mathcal{W}_T$ if and only if no affine subset $P = \pi(L^\perp) + \pi(\vec{\ell}) \subseteq \mathbb{T}^d$, $\vec{\ell} \in L$, is a wall for σ_0^T.*

Corollary 4.11. *Directional ergodicity and directional weak mixing for $\mathbb{Z}^d$-actions T are spectral properties (i.e., they depend only on σ_0^T). It follows that they are isomorphism invariants.*

Proof of Theorem 4.10. Definition 4.3 and Theorem 3.5 together imply that $L \notin \mathcal{W}_T$ if and only if there is an affine subset $P = L + \vec{\ell}$ that is a wall for $\varsigma_{\mathcal{O}}^{\widetilde{T}}$. The result then follows immediately from Theorem 4.8. A similar argument holds for $L \notin \mathcal{E}_T$. □

The next result shows that the subgroup definitions of directional ergodicity and weak mixing agree with Definition 4.3 in the case of rational directions.

Proposition 4.12. *Let T be an ergodic $\mathbb{Z}^d$-action and let $L \in \mathbb{G}_{e,d}$ be a rational direction, with $0 < e < d$. Let $\Lambda = \mathbb{Z}^d \cap L$. Then $T^{|\Lambda}$ is an ergodic (weak mixing) subgroup action if and only if T is ergodic (weak mixing) in the direction L.*

Proof. Suppose that T is not weak mixing in the direction L. Then by Definition 4.3, there is a non-zero eigenfunction $F \in L^2_{\mathcal{O}}(\widetilde{X}, \widetilde{\mu})$ with directional eigenvalue $\vec{\ell} \in L$. In particular, $\widetilde{T}^{\vec{v}}(x, \vec{r}) = e^{2\pi i(\vec{\ell}\cdot\vec{v})} F(x, \vec{r})$ for all $\vec{v} \in L$. Since F is non-zero, there exists $\vec{r} \in \mathbb{T}^d$ such that $F|_{X \times \{\vec{r}\}}$ is non-zero. Define $f : X \to \mathbb{C}$ by $f(x) = F(x, \vec{r})$. It is easy to check that f is a non-zero eigenfunction for $T^{|\Lambda}$ with eigenvalue $\pi(\vec{\ell}) \in \pi(L) \subseteq \mathbb{T}^d$. Thus if $T^{|\Lambda}$ is weak mixing then T is weak mixing in the direction L.

This argument, in the case $\vec{\ell} = \vec{0}$ shows that $T|^\Lambda$ ergodic implies T is ergodic in direction L.

While it is possible to prove the converse by extending the eigenfunctions for $T|^\Lambda$ to eigenfunctions for $\widetilde{T}^{|L}$, here we show instead that the result is an immediate corollary of our previous work. Suppose that $T^{|\Lambda}$ is not weak mixing. Then by Theorem 3.5 there is $\vec{\ell} \in L$ such that $\sigma_0^T(\vec{\ell} + L^\perp) > 0$. By Theorem 4.10 it follows that $L \notin \mathcal{W}_T$, i.e. T is not weak mixing in the direction L.

Setting $\vec{\ell} = \vec{0}$ we have the analogous statement for ergodicity.

□

5. The structure and realizations of $\mathcal{E}_T$ and $\mathcal{W}_T$

In this section we turn to the question of what sets can arise as $\mathcal{E}_T$ and $\mathcal{W}_T$. Since every direction corresponds to a subgroup of $\mathbb{R}^d$, it follows $\mathcal{E}_T = \mathcal{W}_T = \emptyset$ for $\mathbb{R}^d$-actions T that are not ergodic (see Section 2.4). Since rational directions correspond to subgroups of $\mathbb{Z}^d$, analogously we have that if a $\mathbb{Z}^d$-action T is not ergodic then $\mathcal{E}_T$, and therefore $\mathcal{W}_T$, can contain no rational directions. More generally we note that if T is not ergodic, then (1) of Corollary 2.7 does not hold. Indeed, σ_0^T has an atom at $\vec{0}$ due to the presence of a non-constant invariant function for the action. Thus, for every direction $L \in \mathbb{G}_d$, $L^\perp$ is trivially a wall and $\mathcal{E}_T = \mathcal{W}_T = \emptyset$. Therefore for the remainder of the section we focus our attention on ergodic actions. We use the spectral characterizations of directional properties that we provided in previous sections to prove structure theorems analogous to the results in [25].

5.1. The case $d = 2$.

Theorem 5.1. *If T is an ergodic (or weak mixing) $\mathbb{R}^2$ or $\mathbb{Z}^2$-action, then the set of non-ergodic (or non weak mixing) 1-dimensional directions is at most countable.*

Proof. By Theorems 3.5 (for $\mathbb{R}^2$) and 4.10 (for $\mathbb{Z}^2$), every non-parallel $L, L' \in \mathcal{E}_T^c$ are walls for the spectral type of T. They intersect only at the point $\vec{0}$ but since T is ergodic $\sigma_0^T(\{\vec{0}\}) = 0$, and therefore the respective wall measures are mutually singular. By Proposition 2.5, the corresponding invariant functions f and f' are orthogonal in $L^2(X, \mu)$. Since $L^2(X, \mu)$ is separable there can only be countably many such functions.

If T is weak mixing, a similar argument holds using the fact that since σ_0^T is non-atomic, it gives zero measure to any point of intersection of the supports of the walls. □

5.2. The case $d > 2$.

We have the following immediate result.

Lemma 5.2. *Let T be an ergodic $\mathbb{Z}^d$ or $\mathbb{R}^d$-action. If $L \in \mathcal{E}_T^c$ then $L' \in \mathcal{E}_T^c$ for any direction $L' \subseteq L$. Similarly, if $L \in \mathcal{W}_T^c$ then $L' \in \mathcal{W}_T^c$ for any direction $L' \subseteq L$.*

This redundancy shows that any result analogous to Theorem 5.1 in higher dimensions will require more care. We start by introducing some terminology that will allow us to describe non-ergodic and non-weak mixing directions efficiently.

Definition 5.3. Call a set of directions $\mathcal{L} \subseteq \mathbb{G}_d$ *concise* if for all $L, L' \in \mathcal{L}$ with $L \subseteq L'$ implies $L = L'$.

Definition 5.4. Let $\mathcal{L} \subseteq \mathbb{G}_d$. A direction $L \in \mathbb{G}_d$ is *subordinate* to $\mathcal{L}$ if $L \subseteq L'$ for some $L' \in \mathcal{L}$.

Our main result here is the following.

Theorem 5.5. *Let T be an ergodic $\mathbb{R}^d$ or $\mathbb{Z}^d$-action. Then there exist unique, and at most countable, concise sets $\mathcal{NE}_T \subseteq \mathbb{G}_d$ and $\mathcal{NW}_T \subseteq \mathbb{G}_d$ so that $L \in \mathcal{E}_T^c$ if and only if L is subordinate to $\mathcal{NE}_T$ and $L \in \mathcal{W}_T^c$ if and only if L is subordinate to $\mathcal{NW}_T$.*

The proof of Theorem 5.5 requires an extension of the Lebesgue decomposition theorem to wall measures. This idea is implicit in [25].

Proposition 5.6. *Any finite Borel measure σ on $\mathbb{R}^d$ or $\mathbb{T}^d$ can be written as a sum $\sigma = \sigma_0 + \sigma_1 + \cdots + \cdots + \sigma_d$ where for each $e = 0, 1, \ldots, d$, the measure σ_e is either zero, or a sum of at most countably many e-dimensional wall-measures with distinct (but not necessarily disjoint) carriers. This decomposition is unique.*

Proof. Using Lebesgue decomposition we write $\sigma = \sigma_0 + \sigma_1'$ where σ_0 is the discrete part of σ. A standard separability argument yields that σ_0 is a (possibly trivial) countable sum of atoms. In other words, σ_0 is a countable sum of 0-dimensional wall measures.

We proceed in a similar manner by induction. Let $\sigma = \sigma_0 + \sigma_1 + \cdots + \sigma_{e-1} + \sigma_e'$, where $e < d$, and such that for each $0 \le k \le e-1$, σ_k is either the zero measure or a countable sum of k-dimensional wall measures with distinct carriers, and such that σ_e' has no walls of dimension $0 \le k < e$.

Suppose $\mathcal{A}_e$ is the set of e-dimensional walls for σ_e'. If $\mathcal{A}_e$ is empty then σ_e is the zero measure. If not, we show $\mathcal{A}_e$ is (at most) countable. For if we suppose not, then by the pigeonhole principle, there exists $\epsilon > 0$ and an uncountable subset $\mathcal{A}_e' \subseteq \mathcal{A}_e$ with the property that $\sigma_e'(P) \ge \epsilon$ for all $P \in \mathcal{A}_e'$. Then for any countable subset $\mathcal{A}_e'' \subseteq \mathcal{A}_e'$, $\sigma_e'\left(\bigcup_{P\in\mathcal{A}_e''} P\right) = \sum_{P\in\mathcal{A}_e''} \sigma_e'(P) = \infty$, where equality follows from $\sigma_e'(P_1 \cap P_2) = 0$ for $P_1, P_2 \in \mathcal{A}_e''$ distinct. This is because σ_e' has no k-dimension walls for any $k < e$. But this contradicts the fact that σ_e' is a finite measure.

Now we let σ_e be the sum of the wall-measures whose distinct carriers are the walls $P \in \mathcal{A}_e$. By the above, it is a countable sum.

□

Proof of Theorem 5.5. Use Proposition 5.6 to write $\sigma_0^T = \sigma_0 + \sigma_1 + \cdots + \cdots + \sigma_d$, and for $0 \le i \le d$, let $\mathcal{A}_i$ denote the (possibly empty) set of distinct carriers $P = K + \vec{\ell}$, $K \in \mathbb{G}_{i,d}$ and $\vec{\ell} \in K^\perp$, for σ_i. Define the subset of $\mathbb{G}_{d-i,d}$ given by the orthocomplements of the subspaces in $\mathcal{A}_i$, namely $\mathcal{A}_i^\perp = \{L = P^\perp : P \in \mathcal{A}_i\} \subseteq \mathbb{G}_{d-i,d}$. This is the set of non-ergodic $(d-i)$-dimensional directions for T and using the same argument as Theorem 5.1 we know that this is a countable set. We now define $\mathcal{NE}_T = \bigcup_{i=1}^d \left(\mathcal{A}_i^\perp \setminus \bigcup_{0\le k<i} \mathcal{A}_k^\perp\right)$. This set is the countable union of all the countable non-ergodic directions with redundancies, which are a consequence of Lemma 5.2, having been removed.

We construct $\mathcal{NW}_T$ in a similar way. Uniqueness of both sets follows from the uniqueness of the orthocomplements restricted to $\mathbb{G}_d$.

□

Corollary 5.7. *If T is ergodic then $\mathcal{E}_T \neq \emptyset$.*

The natural question to ask now is which countable, concise subsets of $\mathbb{G}_d$ can be realized as $\mathcal{NE}_T$ for some ergodic T? We begin with the following result which generalizes our discussion from Section 2.4.

Proposition 5.8. *If T is an ergodic $\mathbb{R}^d$ or $\mathbb{Z}^d$-action that is not weak mixing then $\mathcal{NE}_T \neq \emptyset$ and $\mathcal{NW}_T = \{\mathbb{R}^d\}$.*

Proof. The proof for $\mathbb{R}^d$-actions is given in Section 2.4 since every direction L corresponds to a subgroup action of $\mathbb{R}^d$.

Let T be a $\mathbb{Z}^d$-action and $L \in \mathbb{G}_{e,d}$ an arbitrary direction. Recall that the nonzero atoms of σ_0^T correspond to the nonzero eigenvalues of T, and these exist since T is not weak mixing. Each atom must lie in $\pi(K) \subseteq \mathbb{T}^d$ for some $K \in \mathbb{G}_{d-1,d}$. Thus $\pi(K)$ is a wall for σ_0^T and by Theorem 4.10 this implies $L = K^\perp \in \mathbb{G}_{1,d}$ is a non-ergodic direction for T. Moreover, for any atom $\vec{b} \in \mathbb{T}^d$, and any $L \in \mathbb{G}_d$, there is an $\vec{\ell} \in L$ so that $\vec{b} \in \pi(L^\perp + \vec{\ell})$. Then $\pi(\vec{\ell}) \in \mathbb{T}^d$ is an eigenvalue for T in the direction L.

□

5.3. **Realization.** In this section we show that any countable concise set $\mathcal{L} \subseteq \mathbb{G}_d$ is the set of non-weak mixing (and therefore non-ergodic) directions of a weak mixing action of $\mathbb{Z}^d$ and $\mathbb{R}^d$.

Theorem 5.9. *For any finite or countably infinite concise set $\mathcal{L} \subseteq \mathbb{G}_d$, there is a weak mixing $\mathbb{R}^d$-action T and a weak mixing $\mathbb{Z}^d$-action S with the property that $\mathcal{L} = \mathcal{NE}_T = \mathcal{NW}_T = \mathcal{NE}_S = \mathcal{NW}_S$.*

Proof. We first construct the $\mathbb{R}^d$ example. Let $\mathcal{L} = \{L_1, L_2, \cdots\}$ be a countable concise set of directions in $\mathbb{R}^d$. Define the set $\mathcal{K} = \{K_1, K_2, \ldots\}$ where $K_i = L_i^\perp$ for each i. Note that $\mathcal{K}$ is also a countable concise set of subspaces of $\mathbb{R}^d$. We will construct an $\mathbb{R}^d$-action T with the property that for any wall P for σ_0^T there exists $K \in \mathcal{K}$ so that $K \subseteq P$. Since $K^\perp = L_j$ for some j this will imply that $L_j \supseteq P^\perp$, and therefore $P^\perp$ is subordinate to $\mathcal{L}$. Thus we will have that $\mathcal{L} = \mathcal{NE}_T$.

We will use the *Gaussian Measure Space Construction* (GMC), starting with a finite Borel measure σ on $\mathbb{T}^d$ to obtain a $\mathbb{Z}^d$-action T on a Lebesgue probability space (X, μ) with $\sigma_T = \exp(\sigma) = \sum_{i=0}^{\infty} \frac{\sigma^{(n)}}{n!}$ (see [6]). Here $\sigma^{(n)} = \sigma * \sigma * \ldots * \sigma$ denotes the n-fold convolution power of σ, $n \geq 1$ and $\sigma^0 = \delta_{\vec{0}}$.

If $\dim(K_i) = r_i$ let σ_{K_i} denote r_i-dimensional Lebesgue measure on K_i and let σ be any finite measure equivalent to $\sum_i \sigma_{K_i}$. The action T will be the Gaussian action with $\sigma_0^T = \exp(\sigma) - \delta_{\vec{0}}$. Note that since σ is non-atomic, T is weak mixing.

We now identify all proper walls for σ_0^T. Each one has to be a proper wall for some $\sigma^{(n)} \sim \sum_{i_1, i_2, \ldots, i_n} \sigma_{K_{i_1}} * \sigma_{K_{i_2}} * \ldots * \sigma_{K_{i_n}}$ thus proper walls for $\sigma^{(n)}$ must be proper walls for some $\sigma_{K_{i_1}} * \sigma_{K_{i_2}} * \ldots * \sigma_{K_{i_n}}$. It is easy to see that the subspace $K = K_{i_1} + K_{i_2} + \ldots + K_{i_n}$ is the only proper wall for $\sigma_{K_{i_1}} * \sigma_{K_{i_2}} * \ldots * \sigma_{K_{i_n}}$ because K_j is the only proper wall for σ_{K_j}. Thus P must be of the form $K_{i_1} + K_{i_2} + \ldots + K_{i_n}$. Therefore P contains every subspace in $\{K_{i_1}, K_{i_2}, \ldots, K_{i_n}\}$ and $P^\perp = L_{i_1}^\perp \cap L_{i_2}^\perp \ldots \cap L_{i_n}^\perp \subseteq L_{i_j}$ for $j = 1, \ldots n$.

We now set $S = T|^{\mathbb{Z}^d}$, the restriction of T to $\mathbb{Z}^d$. By Proposition 2.13 we have that $\sigma_0^S = \sigma_0^T \circ \pi^{-1}$. Note that the proper walls for σ_0^S are projections of the proper walls for σ_0^T and the result follows for $\mathbb{Z}^d$. Alternatively our construction above can be given using measures on $\mathbb{T}^d$ to obtain a $\mathbb{Z}^d$-action with the same properties. $\square$

6. Some additional results

6.1. **Directional ergodicity implies directional weak mixing.** It follows from Corollary 5.7 and Proposition 5.8 that if an ergodic $\mathbb{Z}^d$ or $\mathbb{R}^d$-action is not weak mixing, then there exists $L \in \mathbb{G}_d$ so that $L \in \mathcal{E}_T$ and $L \in \mathcal{NW}_T$. The main result of this section is to show that this is not possible for weak mixing actions T.

Theorem 6.1. *Let T be a weak mixing $\mathbb{R}^d$ or $\mathbb{Z}^d$-action and let $L \in \mathbb{G}_{e,d}$. Then $\mathcal{E}_T = \mathcal{W}_T$.*

Since it is clear that weak mixing implies ergodicity, we need only prove that $\mathcal{E}_T \supseteq \mathcal{W}_T$. The next lemma is a key ingredient in the proof. The proposition following the lemma establishes the $\mathbb{R}^d$ case, and a little more.

Lemma 6.2. *Let T be a weak mixing G-action and $H \subseteq G$ a closed subgroup. Suppose $\gamma \in \widehat{H} \backslash \{0\}$ is a non-trivial eigenvalue for $T^{|H}$. Then the eigenspace*

$$\mathcal{F}_\gamma := \{f \in L^2(X, \mu) : f\left((T^{|H})^h x\right) = \gamma(h) f(x)\}, \tag{6.1}$$

is closed and U_T-invariant.

Proof. It is clear that the eigenspace is closed and $U_{T^{|H}}$ invariant. Let $f \in \mathcal{F}_\gamma$, $g \in G$, and $h \in H$, and note $(f \circ T^g)(T^h x) = f\left(T^h\left(T^g\left(x\right)\right)\right) = \gamma(h)(f \circ T^g)(x)$, so $f \circ T^g \in \mathcal{F}_\gamma$. $\square$

Several special cases of Theorem 6.1 follow from the next proposition.

Proposition 6.3. *Let T be a weak mixing action of a locally compact abelian group G and $H \subseteq G$ a closed subgroup. Then $T^{|H}$ is weak mixing if and only if $T^{|H}$ is ergodic.*

Proof. If $T^{|H}$ is weak mixing then it is ergodic. Conversely, suppose $T^{|H}$ is ergodic. Suppose $\gamma \in \widehat{H}\backslash\{0\}$ is a non-trivial eigenvalue for $T^{|H}$, and denote its eigenspace by $\mathcal{F}_\gamma$. We show that every $f \in \mathcal{F}_\gamma$ is constant, which will imply that $\gamma = 1$. By Lemmas 2.4 and 6.2 $\mathcal{F}_\gamma$ is one-dimensional and U_T invariant. Therefore for $f \in \mathcal{F}_\gamma$, and every $g \in G$ there is a constant $\alpha(g)\mathbb{C}$ such that $f \circ T^g = \alpha(g) f$. Clearly $|\alpha(g)| = 1$ so $\alpha : G \to S^1$. In addition $\alpha(g_1 + g_2) f = f \circ T^{g_1+g_2} = f \circ T^{g_1} \circ T^{g_2} = \alpha(g_1)\alpha(g_2) f$. This, together with the fact that $g \to T^g$ is continuous in the weak topology, gives us that $\alpha \in \widehat{G}$, namely α is an eigenvalue for T. But T is a weak mixing G-action so $\alpha = 1$ and f is constant. $\square$

The proof of Theorem 6.1 for $\mathbb{R}^d$-actions follows directly from Proposition 6.3 since by Definition 4.3, direction L weak mixing for an $\mathbb{R}^d$-action T is equivalent to weak mixing for $T^{|L}$. Similarly the $\mathbb{Z}^d$ case where L is a rational direction follows from Proposition 6.3 since in this case by Proposition 4.12, direction L weak mixing for a weak mixing $\mathbb{Z}^d$-action T is equivalent to weak mixing for $T^{|\Lambda}$.

If $d = 2$ and L is not rational, then L is irrational, as is $L^\perp$ (see Definition 2.2), and we can provide a quick proof of Theorem 6.1 in this case as well. If T is a weak mixing $\mathbb{Z}^2$-action and T is ergodic but not weak mixing in the direction L, then by Definition 4.3 $L^\perp$ is not a wall for $\varsigma_{\mathcal{O}}^{\widetilde{T}}$. Since $L^\perp$ is irrational, it misses all the atoms of $\varsigma_0^{\widetilde{T}}$ so it is also not a wall for $\varsigma_0^{\widetilde{T}}$. This implies that $\widetilde{T}^{|L}$ is ergodic. By Proposition 6.3 we have that $\widetilde{T}^{|L}$ is weak mixing. We conclude by Definition 4.3 that T is weak mixing in the direction L.

For $d > 2$ however it is possible for $L^\perp \cap \mathbb{Z}^d \neq \emptyset$ even if L is not rational. This means that $\widetilde{T}^{|L}$ fails to be ergodic, but with the non-ergodicity entirely due to the rotation factor. Take, for example, $d = 3$, and $L = \{t(0, 1, \sqrt{2}) : t \in \mathbb{R}\}$, which is irrational since $L \cap \mathbb{Z}^3 = \{\vec{0}\}$, but $L^\perp \cap \mathbb{Z}^3 \sim \mathbb{Z}$ is neither rational or irrational. Write $\mathbb{T}^3 = [0, 1)^3$ and note that sets of the form $X \times [0, \epsilon) \times [0, 1)^2$ are invariant under $\widetilde{T}^{|L}$. Thus we can no longer assume that eigenvalues of $\widetilde{T}^{|L}$ are simple and the proof of Proposition 6.3 above fails. For this reason, we provide a different proof of Theorem 6.1 for $\mathbb{Z}^d$ in general.

Proof of Theorem 6.1 *for $\mathbb{Z}^d$-actions.* Let T be a weak mixing $\mathbb{Z}^d$-action, $L \in \mathbb{G}_{e,d}$ for some $0 < e < d$, and suppose T is ergodic but not weak mixing in direction L. By Definition 4.3 this implies that there is an eigenfunction $F \in L^2_{\mathcal{O}}(\widetilde{X}, \widetilde{\mu})$ for $\widetilde{T}^{|L}$ with eigenvalue $\vec{\ell} \in \widehat{L} = L$, $\vec{\ell} \neq 0$. For $\vec{v} \in \mathbb{R}^d$ set $G_{\vec{v}} = F \circ \widetilde{T}^{\vec{v}}$. By Lemma 6.2 we have that $G_{\vec{v}} \in \mathcal{F}_{\vec{\ell}}$ and therefore $|F|$ and $|G_{\vec{v}}|$ are both $\widetilde{T}^{|L}$ invariant functions.

The ergodicity of T in the direction L, using Definition 4.3, allows us to conclude that any $\widetilde{T}^{|L}$ invariant functions must lie in $\mathcal{O}$. Therefore by (4.1), for all $\vec{v} \in \mathbb{R}^d$ there exist $\omega_{\vec{v}}$ and $\omega : \mathbb{T}^d \to \mathbb{R}$ so that $|F(x, \vec{s})| = \omega(\vec{s})$ and $|G_{\vec{v}}(x, \vec{s})| = \omega_{\vec{v}}(\vec{s})$.

Let $Z = \{\vec{s} \in \mathbb{T}^d : \omega(\vec{s}) = 0\} \subseteq \mathbb{T}^d$. Since F is non-trivial, the set $Z^c = \mathbb{T}^d \setminus Z$ has positive measure. Furthermore if $\vec{n} \in \mathbb{Z}^d$, then $\widetilde{T}^{\vec{n}}(x, \vec{s}) = (T^{\vec{n}} x, \vec{s})$, so $G_{\vec{n}}(x, \vec{s}) = F(\widetilde{T}^{\vec{n}}(x, \vec{s})) =$

$F(T^{\vec{n}}x, \vec{s})$. Thus for all $\vec{n} \in \mathbb{Z}^d$ the function $\omega_{\vec{n}}(\vec{s}) = 0$ if and only if $\vec{s} \in Z$, and we can define

$$H_{\vec{n}}(x, \vec{s}) = \begin{cases} \frac{F(x,\vec{s})}{G_{\vec{n}}(x,\vec{s})} & \vec{s} \notin Z \\ 0 & \vec{s} \in Z. \end{cases}$$

Since F and $G_{\vec{n}}$ are both eigenfunctions for $\widetilde{T}^{|L}$ with eigenvalue $\vec{\ell}$, $H_{\vec{n}}$ is an invariant function for $\widetilde{T}^{|L}$. Therefore $H_{\vec{n}}$ is an element of $\mathcal{O}$ allowing us to conclude $H_{\vec{n}}(x, \vec{s}) = c_{\vec{n}}(\vec{s})$ for some function $c_{\vec{n}} : \mathbb{T}^d \to \mathbb{C}$, and $G_{\vec{n}}(x, \vec{s}) = c_{\vec{n}}(\vec{s})F(x, \vec{s})$.

We claim that there is a subset of $Z' \subseteq Z^c$ of positive measure so that for any $\vec{s} \in Z'$, the function $f_{\vec{s}} : X \to \mathbb{C}$ defined by $f_{\vec{s}}(x) = F(x, \vec{s})$ is not constant. Indeed, suppose for a.e $\vec{s} \in Z^c$, $f_{\vec{s}}(x) = a_{\vec{s}}$. Then we would have for a.e. $x \in X$ $F(x, \vec{s}) = 0$ if $\vec{s} \in Z$ and $a_{\vec{s}}$ if $\vec{s} \notin Z$, contradicting the fact that $F \in L^2_{\mathcal{O}}(\widetilde{X}, \mu)$.

Fix $\vec{s} \in Z'$, define $f(x) = F(x, \vec{s})$, and note that the function $g_{\vec{n}}(x) = G(x, \vec{s}) = c_{\vec{n}}(\vec{s})F(x, \vec{s}) = c_{\vec{n}}(\vec{s})f(x)$ is also not constant. Furthermore, $f(T^{\vec{n}}x) = g_{\vec{n}}(x) = c_{\vec{n}}(\vec{s})f(x)$. Since T is a measure preserving action we have $\|c_{\vec{n}}(\vec{s})\| = 1$, i.e. $c_{\vec{n}}(\vec{s}) = e^{2\pi i\alpha_{\vec{n}}}$ for some $\alpha_{\vec{n}} \in [0, 1)$. Arguing as in the proof of Proposition 6.3, we have that f is a non-trivial eigenfunction for the $\mathbb{Z}^d$-action T with eigenvalue $\vec{\alpha}$, a contradiction. □

6.2. **Restrictions and embeddings.** Let $T = \{T^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^d}$ be a measure preserving $\mathbb{Z}^d$-action on (X, μ). If $\overline{T} = \{\overline{T}^{\vec{v}}\}$ is a measure preserving $\mathbb{R}^d$-action, also on (X, μ), so that $T = \overline{T}|^{\mathbb{Z}^d}$, we call $\overline{T}$ an *embedding* of T. As noted in the introduction, while not every $\mathbb{Z}^d$-action has an embedding, the property is generic so there is a dense G_δ set of $\mathbb{Z}^d$-actions that do (see [27]). In this section we restrict our attention to such $\mathbb{Z}^d$-actions and study the relationship between directional properties of T in a direction L, as given by Definition 4.3, and those of $\overline{T}^{|L}$ as a subgroup action. We begin with the following observation relating embeddings of T to the unit suspension of T.

Lemma 6.4. *Let T be a measure preserving, ergodic $\mathbb{Z}^d$-action on (X, μ) and let $\overline{T}$ be an embedding of T. Then $\varsigma_0^{\overline{T}} \ll \varsigma_0^{\widetilde{T}}$ where $\widetilde{T}$ is the unit suspension of T.*

Proof. Let $T, \widetilde{T}$, and $\overline{T}$ be as defined in the statement of the lemma. We identify $\vec{s} \in \mathbb{T}^d$ with $(\vec{s}) \in [0, 1)^d \in \mathbb{R}^d$. It is easy to check that the map $\Phi : \widetilde{X} \to X$ defined by $\Phi(x, \vec{s}) = \overline{T}^{\vec{s}}(x)$ is a factor map between $\widetilde{T}$ and $\overline{T}$. It then follows that the Koopman representation $U_{\overline{T}}$ is unitarily equivalent to a restriction of $U_{\widetilde{T}}$ to an invariant subspace. By Proposition 2.5 then we have $\varsigma^{\overline{T}} \ll \varsigma^{\widetilde{T}}$. Since T is ergodic, so are $\widetilde{T}$ and $\overline{T}$ so the reduced maximal spectral types are obtained by eliminating the mass at $\vec{0}$ for both measures, so $\varsigma_0^{\overline{T}} \ll \varsigma_0^{\widetilde{T}}$. □

Theorem 6.5. *Let T be an ergodic $\mathbb{Z}^d$-action and let the $\mathbb{R}^d$-action $\overline{T}$ be an embedding of T. Let $L \in \mathbb{G}_{e,d}$. Then $\overline{T}^{|L}$ is ergodic if and only if T is ergodic in the direction L.*

Proof. Let $T, \overline{T}$, and L be as in the statement of the theorem. Note that $\overline{T}$ must be ergodic as an $\mathbb{R}^d$-action. Suppose $L \notin \mathcal{E}_{\overline{T}}$. By Theorem 3.5 we have that $L^\perp$ is a wall for $\sigma_0^{\overline{T}}$, namely $\sigma_0^{\overline{T}}(L^\perp) > 0$. By Lemma 6.4, then, it is also wall for $\varsigma_0^{\widetilde{T}}$ and by Theorem 3.5, we have that $L \notin \mathcal{E}_T$.

For the converse, suppose $L \notin \mathcal{E}_T$. Then $\sigma_0^T\left(\pi\left(L^\perp\right)\right) > 0$, and since $T = \overline{T}^{|\mathbb{Z}^d}$, letting $\pi : \mathbb{R}^d \to \mathbb{T}^d$ be the canonical projection map, we have by Proposition 2.13 $\sigma_0^T = \varsigma_0^{\overline{T}} \circ \pi^{-1}$. Therefore $\sigma_0^{\overline{T}}\left(L^\perp\right) > 0$ and by Theorem 3.5, $L \notin \mathcal{E}_{\overline{T}}$. □

If T is weak mixing as a $\mathbb{Z}^d$-action, any embedding of T must be weak mixing as an $\mathbb{R}^d$-action. In this case by Theorem 6.1 ergodicity in a direction is equivalent to weak mixing and we have the following immediate corollary of Theorem 6.5.

Corollary 6.6. *Let T be a weak mixing $\mathbb{Z}^d$-action and let $\overline{T}$ denote an embedding of T. Then T is weak mixing in a direction $L \in \mathbb{G}_{e,d}$ if and only if $\overline{T}^{|L}$ is weak mixing.*

We conclude this section by showing the consequences of removing the ergodicity hypothesis from Theorem 6.5. Let T denote the identity $\mathbb{Z}^d$-action on $\mathbb{T}^d$. Note that T embeds both into the identity $\mathbb{R}^d$-action $\overline{T}_1$ on $\mathbb{T}^d$, as well as the rotation action of $\mathbb{R}^d$ on $\mathbb{T}^d$ given by $\overline{T}_2^{\vec{v}}(\vec{x}) = \{\vec{x} + \vec{v}\}$. Notice that $\widetilde{T}$, the unit suspension of T is an uncountable product of $\mathbb{R}^d$ rotations on $\mathbb{T}^d$. Further, for any $A \subseteq \mathbb{T}^d$ with $\mu A > 0$, the function $\chi_{A\times\mathbb{T}^d} \in L^2_{\mathcal{O}}(\widetilde{X}, \widetilde{\mu})$ is an L invariant function for $\widetilde{T}$ for any direction L. Therefore T is not ergodic in the direction L, for any L, according to Definition 4.3.

Turning to the embeddings we see that while $\overline{T}_1$ is not ergodic in any direction L, it is the case that $\overline{T}_2$ is ergodic in any irrational direction.

6.3. **Strong mixing.** A $G = \mathbb{R}^d$ or $\mathbb{Z}^d$-action T (or a subgroup of one of these) is *strong mixing* if for each $f \in L^2_0(X, \mu)$ its correlation function satisfies $\widehat{\sigma}_f(\vec{g}) = (U_T^{\vec{g}} f, f) \to 0$ as $\vec{g} \to \infty$. As above, the correlation function $\widehat{\sigma}_f(\vec{g})$ is the Fourier transform of the spectral measure $\widehat{\sigma}_f$ on $\widehat{G}$. A measure like $\widehat{\sigma}_f$, which has a Fourier transform that vanishes at infinity is called a *Rajchman measure*. In particular, a G-action T is strong mixing if and only if its reduced spectral measure σ_0^T is Rajchman (see [15]). No measure with an atom can be a Rajchman measure, so strong mixing implies weak mixing, which implies ergodicity. By the Riemann-Lebesgue lemma, if T has Lebesgue spectrum, which means σ_0^T is absolutely continuous, then σ_0^T is a Rajchman measure, so T is strong mixing. In particular, Bernoulli actions are strong mixing, as are certain entropy zero actions.

It is easy to see that if T is a strong mixing $\mathbb{R}^d$-action, then the restriction to $T^{|L}$ to any direction $L \in \mathbb{G}_d$ is also strong mixing. Indeed, strong mixing passes to subgroups, unlike ergodicity and weak mixing and ergodicity, which pass to supergroups. Thus, strong mixing $\mathbb{R}^d$ actions are ergodic and weak mixing in every direction. The same argument holds for rational directions L if T is a $\mathbb{Z}^d$-action. Our goal now is to prove a the same result for $\mathbb{Z}^d$-actions in arbitrary directions. To this end we need the following lemma.

Lemma 6.7. *A wall measure σ on $\widehat{G} = \mathbb{R}^d$ or $\mathbb{T}^d$ is not a Rajchman measure.*

Proof. Call a subset $I \subseteq \widehat{G}$, for $\widehat{G} = \mathbb{R}^d$ or $\mathbb{T}^d$ a *flat disc* if it is a single point, or is the intersection of an e-dimensional hyperplane in G, $0 < e < d$, with an open disc $B \subseteq G$. If σ is a wall measure then $\sigma(I) > 0$ for some flat disc I, and it will suffice to replace σ with the wall measure $\sigma'(E) = \sigma(E \cap I)$. Clearly, if σ' is atomic it is not Rajchman, so we assume σ' is non-atomic.

Without loss of generality, we can translate I so it goes through $\vec{0}$. In the case $\widehat{G} = \mathbb{T}^d$ we can identify $\mathbb{T}^d = [-1/2, 1/2)^d \subseteq \mathbb{R}^d$ and assume $I \subseteq [-1/2, 1/2)^d$. Thus in either case, we view σ' as a measure on $\mathbb{R}^d$. We can write any $\vec{t} \in \mathbb{R}^d$ uniquely as $\vec{t} = \vec{t}_L + \vec{t}_{L^\perp}$ where $\vec{t}_L \in L = \mathrm{span}(I)$ and $\vec{t}_{L^\perp} \in L^\perp$. Then since σ is supported in I,

$$\widehat{\sigma}(\vec{t}) = \int_{\widehat{G}} e^{-2\pi i(\vec{a}\cdot\vec{t})} d\sigma(\vec{a}) = \int_I e^{-2\pi i(\vec{a}\cdot(\vec{t}_L + \vec{t}_{L^\perp}))} d\sigma(\vec{a}) = \int_I e^{-2\pi i(\vec{a}\cdot\vec{t}_L)} d\sigma(\vec{a}) = \widehat{\sigma|_L}(\vec{t}_L).$$

Here $\sigma|_L$ is the restriction of σ to L, and its Fourier transform $\widehat{\sigma|_L}(\vec{t}_L)$ is a positive definite function on L. Thus $\widehat{\sigma}(\vec{t})$ is constant on each coset $L^\perp + \vec{t}_L$ and in particular, is not 0 at infinity. $\square$

Corollary 6.8. *If T is a strong mixing $G = \mathbb{R}^d$ or $\mathbb{Z}^d$-action then T is weak mixing in every direction, and thus ergodic in every direction.*

Proof. Having already discussed $G = \mathbb{R}^d$, we just need to prove the case $G = \mathbb{Z}^d$. Since T is strong mixing, σ_0^T is a Rajchman measure on $\widehat{G} = \mathbb{T}^d$. If T is not weak mixing in the direction L, then L is a wall and the corresponding wall measure σ' satisfies $\sigma' \ll \sigma_0^T$. But this is impossible since the Rajchman property is preserved by absolute continuity (Proposition 2.5 in [15]). $\square$

For an $\mathbb{R}^d$-action T and a direction $L \in \mathbb{G}_{e,d}$, we say T is *strong mixing in the direction* L if $T^{|L}$ is mixing. Following our usual practice, we say a $\mathbb{Z}^d$-action T is *strong mixing in a direction* $L \in \mathbb{G}_{e,d}$ for any $f \in L^2_{\mathcal{O}}(\widetilde{X}, \widetilde{\mu})$, the unit suspension correlation function satisfies $\widehat{\varsigma}^{\widetilde{T}}_f(\vec{t}) := (U^{\vec{t}}_{\widetilde{T}} f, f) \to 0$ as $||\vec{t}|| \to \infty$ in L.

Proposition 6.9. *If T is a strong mixing $G = \mathbb{R}^d$ or $\mathbb{Z}^d$-action, then T is strong mixing in every direction.*

Proof. We noted above that if T is strong mixing $\mathbb{R}^d$-action then each $T^{|L}$ is strong mixing, thus T is strong mixing in every direction. Now assume T is a strong mixing $\mathbb{Z}^d$-action. Then σ_0^T is a Rajchman measure on $\widehat{G} = \mathbb{T}^d$. To show that T is mixing in any direction $L \in \mathbb{G}_{e,d}$ we show the stronger result that for any $f \in L^2_{\mathcal{O}}(X, \mu)$, $\varsigma^{\widetilde{T}}_f$ is a Rajchman measure, since then $\widehat{\varsigma}^{\widetilde{T}}_f(\vec{t}) \to 0$ for any $||\vec{t}|| \to \infty$. Proposition 2.5 in [15] shows that the Rajchman property is preserved by absolute continuity. Since $\varsigma^{\widetilde{T}}_f \ll \varsigma^{\widetilde{T}}_{\mathcal{O}}$ and by Theorem 4.8, $\varsigma^{\widetilde{T}}_{\mathcal{O}} \sim \widetilde{\sigma_0^T}$ it suffices to show $\widetilde{\sigma_0^T}$ is a Rajchman measure on $\mathbb{R}^d$. Taking the Fourier transform of (4.2) over $\mathbb{R}^d$, we have

$$(\widetilde{\sigma_0^T})\widehat{\ }(\vec{t}) = \sum_{\vec{n} \in \mathbb{Z}^d} a_{\vec{n}} (\sigma_0^T \circ \tau^{-\vec{n}})\widehat{\ }(\vec{t}) = \sum_{\vec{n} \in \mathbb{Z}^d} a_{\vec{n}} \left(e^{2\pi i (\vec{n} \cdot \vec{t})}\, \widehat{\sigma_0^T}(\vec{t}) \right) = \left(\sum_{\vec{n} \in \mathbb{Z}^d} a_{\vec{n}}\, e^{2\pi i (\vec{n} \cdot \vec{t})} \right) \widehat{\sigma_0^T}(\vec{t}),$$

where

$$\widehat{\sigma_0^T}(\vec{t}) = \int_{\mathbb{R}^d} e^{-2\pi i (\vec{\ell} \cdot \vec{t})} d\sigma_0^T(\vec{\ell}) = \int_{[0,1)^d} e^{-2\pi i (\vec{\ell} \cdot \vec{t})} d\sigma_0^T(\vec{\ell}).$$

is the Fourier transform of σ_0^T over $\mathbb{R}^d$, so we need to show that σ_0^T is a Rajchman measure on $\mathbb{R}^d$. By the mixing assumption we know that σ_0^T is a Rajchman measure on $\mathbb{T}^d = [0,1)^d$, which means

$$\widehat{\sigma_0^T}(\vec{n}\,) = \int_{[0,1)^d} e^{-2\pi i (\vec{\ell} \cdot \vec{n})} d\sigma_0^T(\vec{\ell}) \to 0 \text{ as } ||\vec{n}|| \to \infty \text{ in } \mathbb{Z}^d.$$

We need to show the same happens for $||\vec{t}|| \to \infty$ in $\mathbb{R}^d$.

Take a sequence $\vec{t} \in \mathbb{R}^d$ with $||\vec{t}_j|| \to \infty$ and suppose $|\widehat{\sigma_0^T}(\vec{t}_j)| \geq K > 0$. Write $\vec{t}_j = \vec{n}_j + \vec{\omega}_j$ where $\vec{n}_j \in \mathbb{Z}^d$ and $\vec{\omega}_j \in [0,1)^d$. By passing to a subsequence we can assume $\vec{\omega}_j \to \vec{\omega} \in [0,1]^d$. Define a measure σ' by setting $d\sigma'(\vec{\ell}) = e^{-2\pi i (\vec{\ell} \cdot \vec{\omega})} d\sigma_0^T(\vec{\ell})$. Note that $\sigma' \ll \sigma_0^T$ and

$$\widehat{\sigma}'(\vec{n}_j) = \int_{[0,1)^d} e^{-2\pi i (\vec{\ell} \cdot \vec{n}_j)} d\sigma'(\vec{\ell}) = \int_{[0,1)^d} e^{-2\pi i (\vec{\ell} \cdot (\vec{n}_j + \vec{\omega}))} d\sigma_0^T(\vec{\ell}).$$

Let $\epsilon < \frac{K}{2}$, then

$$\begin{aligned}
|\widehat{\sigma'}(\vec{n}_j) - \widehat{\sigma_0^T}(\vec{t}_j)| &= \left| \int_{[0,1)^d} e^{-2\pi i(\vec{\ell}\cdot(\vec{n}_j+\vec{\omega}))} - e^{-2\pi i(\vec{\ell}\cdot\vec{t}_j)} d\sigma_0^T(\vec{\ell}) \right| \\
&= \left| \int_{[0,1)^d} e^{-2\pi i(\vec{\ell}\cdot\vec{t}_j)} \left(e^{-2\pi i(\vec{\ell}\cdot(\vec{n}_j+\vec{w}-\vec{t}_j))} - 1 \right) d\sigma_0^T(\vec{\ell}) \right| \\
&= \left| \int_{[0,1)^d} e^{-2\pi i(\vec{\ell}\cdot\vec{t}_j)} \left(e^{-2\pi i(\vec{\ell}\cdot(\vec{w}-\vec{w}_j))} - 1 \right) d\sigma_0^T(\vec{\ell}) \right| \\
&\leq \max_{\vec{\ell}\in[0,1)^d} \left| e^{-2\pi i(\ell\cdot(\vec{w}-\vec{w}_j))} - 1 \right| \sigma_0^T([0,1)^d) < \epsilon
\end{aligned}$$

for j sufficiently large. It follows that $|\widehat{\sigma}'(n_j)| \geq \frac{K}{2}$ for j large enough so it is not Rajchman on $\mathbb{Z}^d$, but yet $\sigma' \ll \sigma_0^T$, a contradiction. □

A similar argument shows the following.

Proposition 6.10. *If T is strong mixing in a direction L then it is weak mixing and ergodic in the direction L, and in particular, T is ergodic.*

6.4. **Rigidity and Genericity.** In this section we investigate the prevalence of directional weak mixing and its relationship to rigidity.

Definition 6.11. A $\mathbb{Z}^d$-action T is *rigid* if there exists a sequence $\vec{n}_k \in \mathbb{Z}^d$, called a *rigidity sequence* for T, so that $T^{\vec{n}_k} \to \mathrm{Id}$ in the weak topology, i.e., $||T^{\vec{n}_k} f - f||_2 \to 0$ for all $f \in L^2(X, \mu)$. For rational directions L, we say T is *rigid in the direction L* if we can choose all the $\vec{n}_k \in \Lambda = L \cap \mathbb{Z}^d$.

We begin by showing that rigidity in a rational direction is compatible with weak mixing in the direction. For convenience we construct a $\mathbb{Z}^d$-action with $d = 2$, but the result holds with the same argument for all $d \geq 2$.

Proposition 6.12. *There is a rigid weak mixing $\mathbb{Z}^2$-action T that is rigid in the vertical, horizontal and diagonal directions, and is weak mixing in all directions.*

The proof relies on the following well known characterization of rigidity (see for example Proposition 2.5, (5) and Corollary 2.6 in [1]), and some classical results about Kronecker sets.

Proposition 6.13. *A sequence $\{\vec{n}_k\} \subseteq \mathbb{Z}^2$ is a rigidity sequence for T if and only if $e^{2\pi i(\vec{n}_k\cdot\vec{\ell})} \to 1$ in $L^2(\mathbb{T}^2, \sigma_0^T)$.*

Definition 6.14. A compact subset $K \subseteq \mathbb{R}$ is a *Kronecker set* if every continuous function $\varphi : K \to \mathbb{C}$ with $|\varphi(a)| = 1$ can be uniformly approximated on K by characters.

We state, without proof, the following result due to Wik [28].

Proposition 6.15. *There is a Kronecker set $K \subseteq [0, 1]$ with full Hausdorff dimension.*

For more details on Kronecker sets see [26] and [17] for a different proof of Proposition 6.15.

Proof of Proposition 6.12. We will use the Gaussian Measure Construction (GMC), as in the proof of Theorem 5.9, to obtain the required $\mathbb{Z}^2$-action T. We begin by describing the measure σ that we use in the construction.

Let $K \subseteq [0, 1]$ be a Kronecker set of full Hausdorff dimension and consider the product $K \times K$. The Hausdorff dimension of $K \times K$ is 2 (see for example [13]). By Frostman's

Lemma [21] this means that for any $\delta > 0$ there is a finite Borel measure σ on $K \times K$ such that for all $\vec{a} \in \mathbb{R}^2$ and $r > 0$, $\sigma(\{\vec{b} \in \mathbb{R}^2 : \|\vec{b} - \vec{a}\|_2 \leq r\}) \leq r^{2-\delta}$. Letting $\delta = \frac{1}{2}$ it follows that σ vanishes on all line segments. But then, the same holds for any convolution power $\sigma^{(k)} = \sigma * \sigma * \cdots * \sigma$, k times, with $k \geq 2$.

Now just as in Theorem 5.9, the GMC with measure σ yields an ergodic $\mathbb{Z}^2$-action T on a Lebesgue probability space (X, μ) such that $\sigma_0^T = \sum_{n=1}^{\infty} \sigma^{(n)}/n!$ is the restricted spectral measure for T. Since σ_0^T vanishes on all line segments, by Theorem 4.10 T is weak mixing in all directions.

Fix $\epsilon > 0$. By Definition 6.14, there exist $m \in \mathbb{Z}$ so that for all $a \in K$, $|e^{ima} - 1| \leq \epsilon/2$. Hence, we can choose vectors $\vec{n}_k \in \mathbb{Z}^2$ of the form $(0, \pm n_k)$, $(\pm n_k, 0)$, or $(\pm n_k, \pm n_k)$ such that $e^{2\pi i(\vec{n}_k \cdot \vec{a})} \to 1$ as $k \to \infty$ for all $\vec{a} = (a_1, a_2) \in K \times K$. It follows that $e^{2\pi i(\vec{n}_k \cdot \vec{a})} \to 1$ in $L^2(\mathbb{T}, \sigma)$ and thus in $L^2(\mathbb{T}, \sigma_0^T)$.Therefore by Proposition 6.13, T has rigidity sequences $\{\vec{n}_k\}$ of the form $\{(0, \pm n_k)\}$, $\{(\pm n_k, 0)\}$, and $\{(\pm n_k, \pm n_k)\}$. $\square$

We note that because we use the GMC, the $\mathbb{Z}^2$-action constructed in the proof of Proposition 6.12 embeds in a rigid, weak mixing $\mathbb{R}^2$-action $\overline{T}$ which is, itself, weak mixing in all directions. We note that it is not known if there is a rank one action that satisfies Proposition 6.12.

Next, we consider directional weak mixing and rigidity for $\mathbb{Z}^d$-actions in the Baire category sense. In particular, we will consider the *weak topology* on the set $\mathcal{G}_d$ of all measure preserving $\mathbb{Z}^d$-actions on a Lebesgue probability space (X, μ). We will also need a second topology called the uniform topology.

Definition 6.16. The *weak topology* on the set $\mathcal{G}_d$ is defined by $T_n \to T$ if $U_{T_n}^{\vec{n}} f \to U_T^{\vec{n}} f$ for all $\vec{n} \in \mathbb{Z}^d$ and $f \in L^2(X, \mu)$, or equivalently, $\mu(T_n^{\vec{n}} E \triangle T^{\vec{n}} E) \to 0$ for all measurable $E \subseteq X$. The *uniform topology* on the set $\mathcal{G}_d$ is defined by $T_n \to T$ if $\lim_n \sup_{E \subseteq X} \mu(T_n^{\vec{n}} E \triangle T^{\vec{n}} E) = 0$ for all $\vec{n} \in \mathbb{Z}^d$.

Clearly, uniform convergence implies weak convergence.

Lemma 6.17 (see [18])**.** *The weak topology on $\mathcal{G}_d$ is Polish (i.e., completely metrizable and separable). Thus $\mathcal{G}_d$ is a Baire space in the weak topology. The uniform topology on $\mathcal{G}_d$ is completely metrizable.*

Metrics that give the weak and strong topologies are given in [18]. Both depend on the corresponding metrics for the weak and strong topologies in the case of $\mathbb{Z}$-actions, or equivalently invertible measure preserving transformations, which were discussed in detail by Halmos, [12]. In particular, $\mathcal{G} := \mathcal{G}_1$ is the group of all invertible measure preserving transformations R on (X, μ). For the weak topology, let $\{E_k : k \in \mathbb{N}\}$ be a dense collection measurable sets in X for the (complete, separable) metric $\mu(E \triangle E')$ on sets mod 0. For $T, S \in \mathcal{G}$ the weak topology is given by the metric $\rho_1(T, S) = \sum_k (\mu(TE_k \triangle SE_k) + \mu(T^{-1}E_k \triangle S^{-1}E_k))$. The uniform topology is given by the metric

$$\delta_1(T, S) = \mu\{x \in X : Tx \neq Sx\}, \tag{6.2}$$

(see [12]). For $\vec{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$ define $||\vec{n}||_\infty = \max_{i=1}^m |n_i|$. Then for $T, S \in \mathcal{G}_d$, the weak topology is given by the complete metric $\rho_d(T, S) = \sum_{\vec{n} \in \mathbb{Z}^d} 2^{-||\vec{n}||_\infty} \rho_1(T^{\vec{n}}, S^{\vec{n}})$ and the uniform topology is given by the complete metric

$$\delta_d(T, S) = \sum_{\vec{n} \in \mathbb{Z}^d} 2^{-||\vec{n}||_\infty} \delta_1(T^{\vec{n}}, S^{\vec{n}}), \tag{6.3}$$

(see [18]).

The idea is that $\mathcal{G}_d$ is essentially a closed subset of $\mathcal{G}^{\mathbb{Z}^d}$ with the product topology, where, $\mathcal{G}$ is given either the weak or the uniform topology. Here, $\mathcal{G}^{\mathbb{Z}^d}$ denotes the set of all functions $\mathbb{Z}^d \to \mathcal{G}$, and $\mathcal{G}_d$ is those functions given by $\vec{n} \mapsto T^{\vec{n}}$. Since the product is countable, the nice properties of $\mathcal{G}$ transfer to $\mathcal{G}_d$. For the case of $\mathbb{R}^d$-actions, the technicalities here and in the arguments that follow are more complicated. We hope to address them in a later paper.

We say a property of a $\mathbb{Z}^d$-action is *generic* (or that a generic $\mathbb{Z}^d$-action *has* the property) if the property holds for a dense G_δ set of actions $T \in \mathcal{G}_d$. Ryzhikov [27] shows that a generic $\mathbb{Z}^d$-action T is weak mixing and embeds in an $\mathbb{R}^d$-action $\overline{T}$ such that $\overline{T}^{|L}$ is weak mixing for every $L \in \mathbb{G}_d$. Our Corollary 6.6, says that if a weak mixing $\mathbb{Z}^d$-action T embeds in in an $\mathbb{R}^d$-action $\overline{T}$, then T and $\overline{T}$ have the same weak mixing directions. Combining these two, we see that weak mixing and weak mixing in all directions are generic properties for $T \in \mathcal{G}_d$. Our goal now is to give a direct proof of this:

Theorem 6.18. *Let $\mathcal{WG}_d$ be the set of all weakly mixing $T \in \mathcal{G}_d$ that are weakly mixing in all directions. Then $\mathcal{WG}_d$ is a dense G_δ subset of $\mathcal{G}_d$ in the weak-topology.*

The remainder the paper consists of the proof of Theorem 6.18. Recall that a $\mathbb{Z}^d$-action T is said to be *free* if $\mu\{x : T^{\vec{n}}x = x\} = 0$ for all $\vec{n} \in \mathbb{Z}^d \backslash \{\vec{0}\}$, and let $\mathcal{G}_d' \subseteq \mathcal{G}_d$ be the set of free $\mathbb{Z}^d$-actions. The proof of Theorem 6.18 relies on the following, which is well know for $\mathbb{Z}$-actions (see e.g., [12] or [15]).

Note that for $T \in \mathcal{G}_d$ (a $\mathbb{Z}^d$-action) and $R \in \mathcal{G}$ (a measure preserving transformation), the *conjugate action*, defined $RTR^{-1} = \{RT^{\vec{n}}R^{-1}\}_{\vec{n}\in\mathbb{Z}^d}$, is isomorphic to T. We define $\mathcal{C}_T := \{RTR^{-1} : R \in \mathcal{G}\} \subseteq \mathcal{G}_d$ to be the set of all conjugates of T.

Proposition 6.19. *Let $\mathcal{G}_d$ be the set of all measure preserving $\mathbb{Z}^d$-actions T on (X, μ). If $T \in \mathcal{G}_d'$ (i.e., T is aperiodic) then the set $\mathcal{C}_T$ is dense in $\mathcal{G}_d$ in the weak topology.*

The proof of Proposition 6.19 uses the Rokhlin lemma. For $m \geq 0$ let $Q_m = \{\vec{n} \in \mathbb{Z}^d : ||\vec{n}||_\infty \leq m\}$. A set $B \subseteq X$ is called a shape-Q_m *Rokhlin tower base* for a $\mathbb{Z}^d$-action T if the *levels* $T^{\vec{n}}B$ for $\vec{n} \in Q_m$ are pairwise disjoint. The *Rokhlin tower* is $T^{Q_m}B$.

Lemma 6.20 (Katznelson-Weiss, [16], Conze, [5])**.** *Let $T \in \mathcal{G}_d'$. Then for any $m \geq 0$ and $\epsilon > 0$ there is a shape Q_m Rokhlin tower base B so that so that $\mu(T^{Q_m}B) > 1 - \epsilon$.*

The proof of Proposition 6.19 also uses the following:

Lemma 6.21 (Glasner-King, [10])**.** *The set $\mathcal{G}_d' \subseteq \mathcal{G}_d$ of free actions $\mathbb{Z}^d$-actions is dense G_δ in $\mathcal{G}_d$ in the weak topology.*

Proof of Lemma 6.19. Fix $\delta > 0$. Choose K so large that

$$\sum_{||\vec{n}||_\infty > K} 2^{-||\vec{n}||_\infty} < \delta/2.$$

Let $\gamma < (\delta/2)/(2K+1)^d$. Let m be large enough that $\rho := (2m - dK)/(2m+1) > 1 - \gamma/2$ and let $\epsilon < \gamma/2$.

For $T, S \in \mathcal{G}_d'$, let A, B be shape Q_m Rokhlin tower bases for T and S respectively, so that $\mu(T^{Q_m}A) = \mu(S^{Q_m}B) \geq 1 - \epsilon$. Since $\mu(A) = \mu(B) = (1-\epsilon)/(2m+1)^d$, we can define a map $R(B) = A$, bijective measure preserving, but otherwise arbitrary. We extend R to a measure preserving map $R : S^{Q_m}B \to T^{Q_m}A$ by $R(S^{\vec{n}}x) = S^{\vec{n}}(Rx)$ for $x \in B$ and $\vec{n} \in Q_m$. Finally, we define $R : (S^{Q_m}B)^c \to (S^{Q_m}A)^c$, again bijective and measure preserving but otherwise arbitrary. Clearly $R \in \mathcal{G}$. Let $T_1 = RTR^{-1}$. Then by the definition of R, B is a shape Q_m Rokhlin tower base for both S and T_1.

Now suppose $||\vec{n}||_\infty \leq K$. Then $(Q_m \cap (Q_m - \vec{n})) + \vec{n} \subseteq Q_m$. Thus if $x \in S^{Q_m \cap (Q_m - \vec{n})}B$, then $S^{\vec{n}}x = T_1^{\vec{n}}x$ since B is a shape Q_m Rokhlin tower base for both S and T_1. By (6.2) it follows that

$$\delta_1(S^{\vec{n}}, T_1^{\vec{n}}) \leq 1 - \mu(S^{Q_m \cap (Q_m - \vec{n})}B). \tag{6.4}$$

Now $\#(Q_m) = (2m+1)^d$, and by the binomial theorem

$$\begin{aligned} \#\left(Q_m \cap (Q_m - \vec{n})\right) &\geq (2m+1)^{d-1}(2m - d||\vec{n}||_\infty) \\ &\geq (2m+1)^{d-1}(2m - dK). \end{aligned}$$

Thus

$$\begin{aligned} \mu(S^{Q_m \cap (Q_m - \vec{n})}B) &\geq \mu(B)(2m+1)^{d-1}(2m - dK) \\ &\geq (1-\epsilon)\rho > (1-\gamma/2)^2 > 1 - \gamma. \end{aligned}$$

By (6.4), $\delta_1(S^{\vec{n}}, T_1^{\vec{n}}) < \gamma$ for all $||\vec{n}||_\infty \leq K$. So (6.3) implies

$$\begin{aligned} \delta_d(S, RTR^{-1}) = \delta_d(S, T_1) &= \sum_{\vec{n} \in \mathbb{Z}^d} 2^{-||\vec{n}||_\infty} \delta_1(S^{\vec{n}}, T_1^{\vec{n}}) \\ &\leq \delta/2 + \sum_{||\vec{n}||_\infty \leq K} \delta_1(T^{\vec{n}}, T_1^{\vec{n}}) \\ &\leq \delta/2 + (2K+1)^d \gamma < \delta. \end{aligned}$$

This shows $\mathcal{C}_T$ is dense in the uniform topology, but since uniform convergence implies weak convergence, it is dense in the weak topology as well. □

In what follows, for $T \in \mathcal{G}_d$ and $f \in L^2(X, \mu)$, we write σ_f^T for the spectral measure of f with respect to the action T, a measure on $\mathbb{T}^d = \widehat{\mathbb{Z}^d}$. Also, let $\mathcal{I}$ denote the set of all the set of all flat discs in $\mathbb{T}^d$ (see the proof of Lemma 6.7).

Lemma 6.22. *Let $f \in L_0^2(X, \mu)$ and $k \geq 1$. Define*

$$\mathcal{B}_k(f) = \{T \in \mathcal{G}_d : \sigma_f^T(I) \geq 1/k \text{ for some } I \in \mathcal{I}\}.$$

Then $\mathcal{B}_k(f)$ is a closed subset of $\mathcal{G}_d$ with no interior.

Proof. Let $T_n \in \mathcal{B}_k(f)$ so that $T_n \to T$ in $\mathcal{G}_d$ in the weak topology. Equivalently, $U_{T_n}^{\vec{n}} f \to U_T^{\vec{n}} f$ for all $f \in L_0^2(X, \mu)$. Since $\widehat{\sigma_f^T}(\vec{n}\,) = (U_T^{\vec{n}} f, f)$ it follows that

$$\int_{\mathbb{T}^d} e^{-2\pi i(\vec{n} \cdot \vec{t})} d\sigma_f^{T_n} \to \int_{\mathbb{T}^d} e^{-2\pi i(\vec{n} \cdot \vec{t})} d\sigma_f^T,$$

which, by the Stone-Weierstrass theorem implies $\sigma_f^{T_n} \to \sigma_f^T$ in the weak* topology.

Since $T_n \in \mathcal{B}_k(f)$, we have that $\sigma_f^{T_n}(I_n) \geq 1/k$ for some $I_n \in \mathcal{I}$. By passing to a subsequence, we can assume that there is a flat disc I with the property that for all $U \supseteq I$ open, there is an open set U' with $I \subseteq U' \subseteq U$ so that $I_n \subseteq U'$ for all sufficiently large n. Thus for such n we have $\sigma_f^{T_n}(U') \geq 1/k$. Let $h \in C(\mathbb{T}^2)$ satisfy $h = 1$ on U' and $h = 0$ on U^c. Then for all $\epsilon > 0$, and sufficiently large n, we have

$$\frac{1}{k} \leq \sigma_f^T(U') \leq \int_{\mathbb{T}^d} h \, d\sigma_f^T \leq \int_{\mathbb{T}^d} h \, d\sigma_f^T + \epsilon.$$

Letting ϵ go to zero and noting that $\chi_U \geq h$ we have

$$\sigma_f^T(U) \geq \int_{\mathbb{T}^d} h \, d\sigma_f^T \geq 1/k.$$

Using the outer regularity of σ, we conclude that $\sigma_f^T(I) \geq 1/k$, so $T \in \mathcal{B}_k(f)$.

To see that $\mathcal{B}_k(f)$ has no interior, we note that the strong mixing actions T are dense in $\mathcal{G}_d$. In particular, if T is strong mixing then T is free (i.e., $T \in \mathcal{G}_d'$), so by Proposition 6.19, $\mathcal{C}_T = \{RTR^{-1} : R \in \mathcal{G}\}$ is dense in $\mathcal{G}_d$. But every $T' \in \mathcal{C}_T$ is isomorphic to T, so T' is strong mixing. Thus there is a dense set of $T \in \mathcal{G}_d$ so that $T \notin \mathcal{B}_k(f)$ for any k. □

Proof of Theorem 6.18. Let $f_n \in L_0^2(X,\mu)$ be a dense sequence. Suppose $T \notin \mathcal{WG}_d$, i.e. T is not weakly mixing in some direction. Then there is $f \in L_0^2(X,\mu)$ and $k \geq 1$ such that the spectral measure σ_f^T has $\sigma_f^T(I) \geq 1/k$ for some $I \in \mathcal{I}$. Approximating f by the dense sequence, we have that there is some f_n with $\sigma_{f_n}^T(I) \geq 1/(2k)$ for some $I \in \mathcal{I}$. In particular, $\mathcal{WG}_d^c \subseteq \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{B}_k(f_n)$. But also clearly, $\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{B}_k(f_n) \subseteq \mathcal{WG}_d^c$. By Lemma 6.22, $\mathcal{WG}_d$ is a dense G_δ. □

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(E. A. Robinson) Department of Mathematics, George Washington University, Washington, DC 20052

(J. Rosenblatt) Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801

(A. Şahin) Department of Mathematics and Statistics, Wright State University, Dayton, OH 45435