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General Section

# Introducing Minkowski normality ☆

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This paper is dedicated to our advisors

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ABSTRACT

We introduce the concept of Minkowski normality, a different type of normality for the regular continued fraction expansion. We use the ordering

$$\frac{1}{2}, \quad \frac{1}{3}, \frac{2}{3}, \quad \frac{1}{4}, \frac{3}{4}, \frac{2}{5}, \frac{3}{5}, \quad \frac{1}{5}, \ldots$$

of rationals obtained from the Kepler tree to give a concrete construction of an infinite continued fraction whose digits are distributed according to the Minkowski question mark measure. To do this we define an explicit correspondence between continued fraction expansions and binary codes to show that we can use the dyadic Champernowne number to prove normality of the constructed number. Furthermore, we provide a generalised construction based on the underlying structure of the Kepler tree, which shows that any construction that concatenates the continued fraction expansions of all rationals, ordered so that the sum of the digits of the

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continued fraction expansion are non-decreasing, results in a number that is Minkowski normal.


## 1. Introduction

Normality as introduced by Borel focuses on integer base expansions and the Lebesgue measure. We say that $x \in [0,1)$ is *normal in base* $b$ if for any block $d = d_1 d_2 \cdots d_k$ of $k$ digits, $d_i \in \{0, 1, \ldots, b-1\}$, we have

$$\lim_{n\to\infty} \frac{1}{n} G_n(x,d) = \lambda(\Delta(d)) = b^{-k}.$$

Here, $\lambda$ is Lebesgue measure, $\Delta(d)$ is the set of numbers whose decimal expansions start with $d$ and $G_n(x,d)$ denotes the number of occurrences of $d$ in the first $n$ digits of the base $b$ expansion of $x$. Borel showed that $\lambda$ almost every $x \in [0,1)$ is normal. However, for common explicit numbers that seem to be normal, for example $\sqrt{2}-1$, $e-2$, $\pi-3$, it is unknown in all cases whether or not they are. The first and most well-known construction of an explicit normal number is due to David Champernowne [6]. He proved that the number that is obtained by concatenating the natural numbers, i.e.

$$\mathcal{C}_{10} = 0.\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ \cdots,$$

is normal in base 10. Later, Copeland and Erdös gave a generalised construction of a normal number [7], which they used to prove the normality of the number that is obtained by concatenating all the primes. A small selection of further generalisations and results include that of Davenport and Erdos [9] and Nakai and Shiokawa [18]. Some similar constructions of normal numbers for $\beta$-expansions (i.e. expansions with respect to a non-integer base $\beta$), determined by the absolutely continuous so-called Parry measure [20], can be found in [12] and [22].

The definition of normality can also be extended to continued fractions. Any real number $x$ can be represented as a — possibly finite — continued fraction expansion

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cfrac{1}{\ddots}}}},$$

where the digits $a_i(x) \in \mathbb{N}$ are the *partial quotients* of $x$, $i \geq 1$. In shorthand, we write $x = [a_1, a_2, a_3, \cdots]$. For any irrational $x$, the continued fraction expansion is infinite and unique [19, Theorem 5.11]. Moreover, any rational has exactly two expressions as a finite continued fraction $[a_1, a_2, \cdots, a_n - 1, 1] = [a_1, a_2, \cdots, a_n]$. We use the convention that any rational continued fraction is written in its *reduced form*: the one on the right, where $a_n \geq 2$.

The type of normality that is related to the continued fraction expansion comes from the *Gauss measure* $\gamma$ that, for any Lebesgue set $A \subset [0, 1)$, is defined by

$$\gamma(A) := \frac{1}{\log 2} \int_A \frac{1}{1+x} dx. \tag{1.1}$$

Therefore, we say that $x \in [0, 1)$ is *continued fraction normal*, if for any $k \geq 1$ and any block $d = d_1, d_2, \cdots, d_k$, $d_i \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \frac{1}{n} G_n(x, d) = \gamma(\Delta(d)),$$

where $\Delta(d) = \{y \in [0, 1) : y = [d_1, d_2, \cdots, d_k, \cdots]\}$ is the cylinder set corresponding to $d$. In the above and henceforth, $G_n(x, d)$ will denote the number of occurrences of $d$ in the first $n$ digits of the continued fraction expansion of $x$. It follows from Birkhoff's Ergodic Theorem, applied to the $\gamma$-preserving, ergodic Gauss map $\mathcal{G}$, that Lebesgue almost all numbers are continued fraction normal.

In contrast to the case of normality for radix base expansions, where there are a large number of explicit constructions of normal numbers, there are relatively few results to date about continued fraction normality. So far, there are at least six construction results. The first is due to Postnikov [21], who used Markov chains to construct a continued fraction normal number. Another construction is due to Adler, Keane and Smorodinsky [1]. They first construct a (sub)sequence of rationals by taking all non-reduced fractions with denominator $n$ in increasing order

$$\frac{1}{2}, \quad \frac{1}{3}, \frac{2}{3}, \quad \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \quad \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \quad \ldots, \frac{n-1}{n}, \quad \frac{1}{n+1}, \cdots. \tag{1.2}$$

Their continued fraction normal number is then obtained by concatenating the - finite - continued fraction expansions of these rationals

$$x_{aks} = [2, \quad 3, 1, 2, \quad 4, 2, 1, 3, \quad 5, 2, 2, 1, 1, 2, 1, 4, \cdots] \approx 0.44034.$$

It took about 30 years before the constructions of Postnikov and Adler, Keane and Smorodinsky were generalized. The generalisation of Postnikov's construction is due to Madritsch and Mance [15]. Both of these works define a sequence of blocks of symbols such that each block better approximates the desired frequency. The (continued fraction) normal number is then obtained by concatenating (repetitions) of the elements of the sequence. This is different from the work of Adler, Keane and Smorodinsky and the generalisation of their work, which is due to Joseph Vandehey [30]. Among

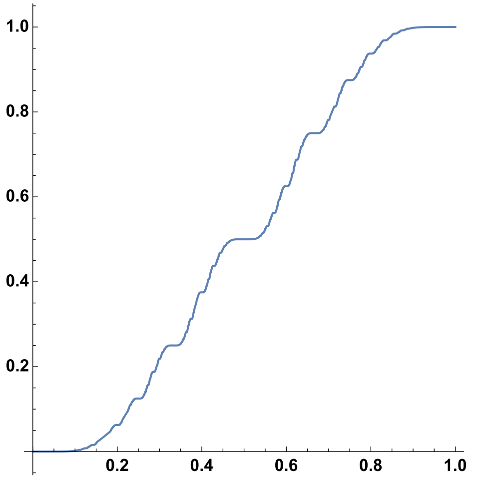


**Fig. 1.** The Minkowski question mark function $?(\cdot)$.

other things, Vandehey proves that some explicit subsequences of (1.2) can be used to construct a continued fraction normal number. For the proof, he uses metrical results to get asymptotics on how many rationals have good small-scale properties. In turn, these asymptotics imply conditions that determine whether the constructed number is continued fraction normal. One of the constructions, for instance, considers the subsequence of rationals that have integer numerators and prime denominators. Another normality result due to Vandehey is the theoretical existence of numbers that are both continued fraction normal and absolutely abnormal [29]. The proof, however, is conditional on the Generalized Riemann Hypothesis. Furthermore, Scheerer constructed a number that is both continued fraction normal and absolutely normal [22]. Becher and Yuhjtman [3] improved on Scheerer's construction, reducing the number of operations to obtain $n$ binary digits from $O(2^{2^n})$ to $O(n^4)$. The key idea in their proof is to construct a sequence of nested intervals that satisfy certain conditions. Most of these conditions are related to discrepancy in the sense that they ensure an arbitrary small bound on the discrepancy of the numbers in that interval. The normal number is then obtained by taking the intersection of all these -sequences of nested- intervals. All known examples of (computable) absolutely normal numbers, are given in the form of an algorithm [16]. Up to the authors' knowledge, the aforementioned constructions are the only constructions of continued fraction normal numbers.

All the constructions of normal numbers discussed so far are for a distribution of digits according to Lebesgue measure or, in the cases of Gauss measure for regular continued fractions and Parry measure for $\beta$-expansions, absolutely continuous measures. However, in this article we consider a measure that is singular with respect to Lebesgue measure. We consider the *Minkowski question mark measure* $\mu_?$, which is specified by the following distribution function (see Fig. 1)

$$?(x) := 2 \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2^{a_1(x)+a_2(x)+\cdots+a_i(x)}}.$$

Here, $a_i(x)$ comes from the *continued fraction expansion* of $x \in [0, 1)$, $i \geq 1$. In particular, we introduce a different type of normality for regular continued fraction expansions that we call *Minkowski normality*. Informally, we say that a number $x$ is Minkowski normal if its digits $(a_i(x))_{i\geq 1}$ are distributed according to the Minkowski question mark measure.

The idea of a normal number with respect to an ergodic invariant measure that is not either Lebesgue measure or at least an absolutely continuous measure is not new. In [15], Madritsch and Mance construct explicit normal numbers (or *generic point* as they are called in ergodic theory) for an arbitrary ergodic invariant Borel probability measure $\mu$ on a finite or countable alphabet symbolic dynamical system that satisfies a weak version of the specification property. In particular, a symbolic sequence $x$ is generic (or equivalently, the corresponding number $x$ is generic) if every length $k$ block occurs in $x$ with a frequency equal to the measure of the corresponding cylinder set (the measure of any inadmissible block is zero). As usual, an application of the ergodic theorem shows that $\mu$ almost every number is normal.

Various earlier versions of constructions similar to [15], generally much more restrictive in their assumptions and less general their results, can be found, for example, in: [4], [12], [22], [26] and [27]. For continued fractions, the symbolic dynamical system involved is the full shift with (countable) alphabet $\mathbb{N}$. Since this obviously satisfies the specification [15], their construction is general enough to produce both continued fraction normal numbers and Minkowski normal numbers. However, all these constructions involve many repetitions of longer and longer approximately generic blocks (with respect to $\mu$), and the result is a number that is generic yet not easy to write down. A simplified version of this construction is due to Vandehey [28].

The main goal of the article is to construct an explicit Minkowski normal number, whose digits are easy to write down and have a number theoretic description. We construct an infinite continued fraction expansion and show that the corresponding sequence of digits is distributed according to the Minkowski question mark measure. Specifically, we consider the ordering of rationals that is given by the *Kepler tree*. This is a specific binary tree that orders the rationals in the unit interval. The constructed number is obtained by concatenating the continued fraction expansions of the rationals using the Kepler order. For the proof of normality, we show that there is a correspondence between binary codes and rationals in the Kepler tree. Moreover, we show that we can use the dyadic Champernowne number to determine the distribution of the sequence of digits that represent the constructed number. Finally, we use generalised Champernowne numbers to extend normality of the constructed number to more general cases.

## 2. The construction

The crucial factor in determining the limiting distribution of the partial quotients of the constructed number, is the ordering that is chosen. In the case of Adler, Keane and Smorodinsky, the ordering of rationals they use leads to normality with respect to the Gauss measure. Hence, the constructed number is continued fraction normal. In this section, we consider the ordering of the rationals that results from the *Kepler tree*. We use this ordering to construct a number whose partial quotients are distributed according to the Minkowski question mark measure.

The first part of the Kepler tree is found in Johannes Kepler's magnum opus, a book containing his most important work. See [13, p. 163] for an English translation. Though Johannes Kepler starts from 1/1, the binary tree starts from 1/2 and then uses the rule

$$\begin{array}{ccc} & p/q & \\ p/(p+q) & & q/(p+q) \end{array}.$$

As rationals can be represented by finite continued fractions and vice versa, this is equivalent to

$$\begin{array}{ccc} & [a_1, a_2, \cdots, a_n] & \\ [(a_1+1), a_2, \cdots, a_n] & & [1, a_1, a_2, \cdots, a_n] \end{array}.$$

This representation allows us to understand the behaviour of the sequence of digits that is obtained from the construction. Here, note that a left move increases the first digit in the continued fraction by one and does not alter the total number of digits in the continued fraction. A right move however, inserts a 1 as a first digit and thus increases the length of the continued fraction by one. This also means that a left move does not preserve the block of digits that form the continued fraction of the mother node, whereas a right move does preserve the block. Lastly, note that both moves increase the sum of the digits of the continued fraction expansion by one. Hence, the Kepler tree orders the rationals into levels based on the sum of the digits of their continued fraction expansion. The first four levels of the tree are displayed in Fig. 2.

The key idea in proving normality of the constructed number is that we create a one-to-one correspondence between rationals and binary codes. This correspondence is based on the fact that there exists a unique path between the root and any rational in the Kepler tree. In turn, we use this unique path to define a one-to-one correspondence between rationals and binary codes.

The root corresponds to the empty path and therefore to the empty binary code. Moreover, given an arbitrary rational, we can retrace its path as follows. Let $[a_1, a_2, \cdots, a_n]$ denote the continued fraction of an arbitrary rational $p/q$ in the Kepler tree. Then by going $(a_1 - 1)$ steps from the left up, we end up at the rational that corresponds to

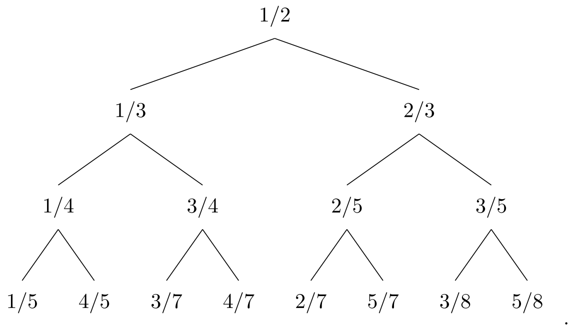


**Fig. 2.** The first 4 levels of the Kepler tree.

$[1, a_2, \cdots, a_n]$. Subsequently, going from the right up we end at $[a_2, a_3, \cdots, a_n]$. By repeating this process for $a_2, a_3, \ldots, a_{n-1}$ and $a_n$ we can find the path to the root. We summarise these steps symbolically by writing $L$ for a left move and $R$ for a right move. Subsequently, we reverse the path and apply the substitution $\{L \mapsto 0, R \mapsto 1\}$ to associate a binary code to $p/q$. Hence,

$$\begin{aligned} p/q \overset{\text{cfe}}{\longleftrightarrow} [a_1, a_2, \cdots, a_n] &\overset{\text{upward path}}{\longleftrightarrow} L^{a_1-1}RL^{a_2-1}R \cdots L^{a_n-2} \\ &\overset{\text{downward path}}{\longleftrightarrow} L^{a_n-2} \cdots RL^{a_2-1}RL^{a_1-1} \\ &\overset{\text{binary code}}{\longleftrightarrow} 0^{a_n-2} \cdots 10^{a_2-1}10^{a_1-1}. \end{aligned}$$

The binary code that is associated to a rational contains a lot of information. It gives the continued fraction expansion of the rational that it represents and its exact location within the tree. Namely, it gives the level in which the rational occurs and the position within that level. The level is given by the total number of 0's and 1's in its binary code and its position within the level can be read from the ordering of the 0's and 1's. The following lemma is an immediate consequence of the binary coding and the concept of retracing paths in the tree.

**Lemma 2.1.** *There exists a unique path between the root of the Kepler tree that starts at* $1/2$ *and any arbitrary rational* $p/q$*. If we denote* $p/q$ *by its continued fraction expansion* $[a_1, a_2, \cdots, a_n]$*, then the corresponding path is*

$$L^{a_n-2} \cdots RL^{a_2-1}RL^{a_1-1}, \tag{2.1}$$

*which corresponds to the binary code*

$$0^{a_n-2} \cdots 10^{a_2-1}10^{a_1-1}. \tag{2.2}$$

*This path consists of* $a_1 + a_2 + \cdots + a_n - 2$ *moves, which also corresponds to the level in which the rational occurs for the first and only time.*

Apart from providing information about the occurrence of rationals, the concept of retracing paths also tells us how blocks of the form $d = d_1, d_2, \cdots, d_k$ are formed by the Kepler tree, how these blocks are preserved and how we can identify them using binary codes.

For the construction, we order the rationals in the Kepler tree going top-down, left-right. The ordering of the rationals that result from this procedure is

$$\frac{1}{2}, \quad \frac{1}{3}, \frac{2}{3}, \quad \frac{1}{4}, \frac{3}{4}, \frac{2}{5}, \frac{3}{5}, \quad \frac{1}{5}, \cdots. \tag{2.3}$$

If we concatenate the corresponding binary codes of these rationals in the given order, we obtain an infinite sequence of binary digits. This infinite sequence corresponds to the dyadic Champernowne number

$$\mathcal{C}_2 := 0.\ 0\ 1\ 00\ 01\ 10\ 11\ 000\ \cdots, \tag{2.4}$$

which is known to be normal in base 2. This and other properties of $\mathcal{C}_2$ can for instance be found in [10] or [25]. For our construction of a Minkowski normal number, we concatenate the continued fraction expansions of the rationals in the ordering that results from the Kepler tree. We obtain an infinite continued fraction, which corresponds to a unique irrational number [8, Proposition 4.1.1]. This number is given by

$$\mathcal{K} := [2, \quad 3, \quad 1, 2, \quad 4, \quad 1, 3, \quad 2, 2, \quad 1, 1, 2, \quad 5, \quad \cdots] \approx 0.44031. \tag{2.5}$$

## 3. Minkowski normality

So far, different types of normality correspond to different number expansions. Next, however, we use the Minkowski question mark measure to define another type of normality for the continued fraction expansion. We define Minkowski normality for continued fractions as follows.

**Definition 3.1** *(Minkowski normal number).* We say that $x = [a_1, a_2, a_3, \cdots] \in [0, 1)$ is *Minkowski normal*, if for any $k \geq 1$ and any block $d = d_1, d_2, \cdots, d_k$, with $d_i \in \mathbb{N}$, we have that

$$\lim_{n \to \infty} \frac{1}{n} G_n(x, d) = \mu_?(\Delta(d)) = 2^{-(d_1 + d_2 + \cdots + d_k)}. \tag{3.1}$$

**Theorem 3.2.** *$\mu_?$ almost every number in $[0, 1)$ is Minkowski normal.*

**Proof.** The Gauss map $\mathcal{G}$, defined $\mathcal{G}(x) = 1/x \mod 1$, is known to be ergodic under the Minkowski question mark measure $\mu_?$. This follows from the fact that the Minkowski acts on cylinders as a product measure, which implies that we have an isomorphism with a Bernoulli shift. Therefore, the Gauss map with the Minkowski question mark measure is ergodic (and mixing and Bernoulli). Let $x \in [0,1)$. Then for any $k \geq 1$ and any block $d = d_1, d_2, \cdots d_k$, $d_i \in \mathbb{N}$, it follows from Birkhoff's Ergodic Theorem that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{\Delta(d)}(\mathcal{G}^i x) = \mu_?(\Delta(d)) = 2^{-(d_1+d_2+\cdots+d_k)} \qquad \mu_? \text{ a.e.} \quad \square$$

We note that $\lambda$ almost every number is continued fraction normal and $\mu_?$ almost every number is Minkowski normal. This is possible because Lebesgue measure and Minkowski question mark measure are singular. Furthermore, it is interesting to note that the set of Minkowski normal numbers satisfies the following properties typical for sets of normal numbers.

- Its complement has full Hausdorff dimension. This follows from the fact that the set of Minkowski normal numbers in $[0,1)$ has full measure, hence has Lebesgue measure 0. Therefore the complement has measure 1, which in turn implies that it has full Hausdorff dimension [23].
- Its complement is a winning set in the sense of Schmidt's game. As the set of non-Minkowski normal numbers contains the set of badly approximable numbers, it follows from [24, Theorem 3] that this set is $\alpha$-winning for all $\alpha < 1/2$.
- It is first category (see the comment after Proposition 4.6).

The rest of this section is dedicated to proving the Minkowski normality of $\mathcal{K}$. To do this, we identify explicit binary codes that correspond to different types of occurrences of an arbitrary block $d$. Consequently, we use the base 2 normality of $\mathcal{C}_2$ to determine the frequency that corresponds to these types of occurrences. We then distinguish the following four types of occurrences of a block $d$ in $\mathcal{K}$.

- The block $d$ occurs at the start of a continued fraction expansion of a rational in $\mathcal{K}$;
- The block $d$ occurs in the middle of the continued fraction expansion of a rational in $\mathcal{K}$;
- The block $d$ occurs at the end of the continued fraction expansion of a rational in $\mathcal{K}$;
- The block $d$ occurs in $\mathcal{K}$ as a result of concatenating the continued fraction expansions of different rationals. We refer to this type of occurrences as *divided occurrences*.

**Lemma 3.3.** *Let $d = d_1, d_2, \cdots, d_k$ be an arbitrary block of length $k$, $d_i \in \mathbb{N}$. The asymptotic frequency of divided occurrences of $d$ in $\mathcal{K}$ is equal to 0.*

**Proof.** The $l$-th level of the Kepler tree consists of $2^l$ rationals. Hence, there are $2^l - 1$ concatenations. As $d$ consists of $k$ digits, there is a maximum of $k-1$ positions where $d$ can be divided. Therefore, the number of divided occurrences can be bounded from above by $k2^l$.

Each rational in the $l$-th level of the tree is formed by $i$ left moves and $l-i$ right moves, where $i$ varies between 0 and $l$. A left move does not alter the number of digits and a right move increases the number of digits by 1. As we start off with one digit at level 0, we find that the total number of digits in level $l$ is given by

$$\sum_{i=0}^{l}(i+1)\binom{l}{i} = (l+2)2^{l-1}, \quad l \geq 0.$$

Suppose that the $n$-th digit of $\mathcal{K}$ occurs within the $L$-th level of the Kepler tree. The number of divided occurrences in the first $n$ digits of $\mathcal{K}$ is then bounded from above by

$$\sum_{l=0}^{L-1} k2^l + \mathcal{O}(2^L) = k(2^L - 1) + \mathcal{O}(2^L).$$

Furthermore, the total number of possible occurrences of $d$ in the first $n$ digits of $\mathcal{K}$ is

$$\sum_{l=0}^{L-1}(l+2)2^{l-1} - k + 1 + \mathcal{O}(2^L) = L2^{L-1} - k + 1 + \mathcal{O}(2^L).$$

When we consider the asymptotic frequency of occurrences, we note that $n \to \infty$ implies that $L \to \infty$. Therefore the asymptotic frequency of this type of occurrences is

$$\lim_{L\to\infty} \frac{k(2^L - 1) + \mathcal{O}(2^L)}{L2^{L-1} - k + 1 + \mathcal{O}(2^L)} = 0. \quad \square$$

**Theorem 3.4.** *The number $\mathcal{K}$, defined in (2.5), is Minkowski normal.*

**Proof.** Let $d = d_1, d_2, \cdots, d_k$ be an arbitrary block of length $k$, $d_i \in \mathbb{N}$. In order to determine the frequency of $d$ in $\mathcal{K}$ it is sufficient to count the binary blocks $10^{d_k-1}\cdots 10^{d_2-1}10^{d_1-1}1$ and $10^{d_k-1}\cdots 10^{d_2-1}10^{d_1-1}0$ in $\mathcal{C}_2$. We argue this by considering the four different types of occurrences.

It follows from Lemma 3.3 that the frequency of divided occurrences of $d$ tends to 0.

Now, let $p/q$ be an arbitrary rational in the Kepler tree that corresponds to the continued fraction $[a_1, a_2, \cdots, a_n]$. By Lemma 2.1, the path from 1/2 to $p/q$ is unique and given by

$$L^{a_n-2}\cdots RL^{a_2-1}RL^{a_1-1}.$$

Similarly, there exists a unique path to the rational $[d_1, d_2, \cdots, d_k, a_1, a_2, \cdots, a_n]$. By (2.1), this path is

$$\boldsymbol{L^{a_n-2}\cdots RL^{a_2-1}RL^{a_1-1}}RL^{d_k-1}\cdots RL^{d_2-1}RL^{d_1-1}.$$

Considering the latter path, we see that it passes through the rational $p/q$, of which the path is marked in bold. As this path and that to $p/q$ are unique, we conclude that there exists a unique subpath from $p/q$ to $[d_1, d_2, \cdots, d_k, a_1, a_2, \cdots, a_n]$ that is given by

$$RL^{d_k-1} \cdots RL^{d_2-1}RL^{d_1-1}.$$

Therefore, the following binary code corresponds to $d$ occurring at the start of a continued fraction expansion

$$10^{d_k-1} \cdots 10^{d_2-1}10^{d_1-1}. \tag{A}$$

The binary code associated to occurrences of $d$ in the middle of a continued fraction expansion is similar. The difference with (A) is that another right move is needed in the Kepler tree. This preserves the block forever and causes it to occur in the middle. Therefore, the binary code associated to this type of occurrence is the same as that in (A) with a 1 appended. Hence

$$10^{d_k-1} \cdots 10^{d_2-1}10^{d_1-1}1. \tag{B}$$

Lastly we consider what happens when $d$ occurs at the end of a continued fraction. Due to the fact that the Kepler rule alters the start of continued fraction expansions, these types of occurrences are descendants from the rational $[d_1, d_2, \cdots, d_k]$. In order to preserve the block $d$, another right move is needed. Using this and Lemma 2.1 we find that the corresponding binary code is

$$0^{d_k-2} \cdots 10^{d_2-1}10^{d_1-1}1, \tag{C}$$

where the last 1 results from the extra right move. However, occurrences of this binary code in $\mathcal{C}_2$ do not always correspond to an occurrence of $d$ in $\mathcal{K}$. This is due to the fact that the digit 2 is used to form $d_k$. That is, $d_k$ is formed from the digit 2, whereas in the other type of occurrences, the block $d$ is formed from scratch. Hence for the binary code in (C) to correspond to an occurrence of $d$ in $\mathcal{K}$, this occurrence of $d$ should originate from a rational of the form $[2, b_2, \cdots, b_{j-1}, b_j]$. By Lemma 2.1, this corresponds to rationals that have a binary code given by

$$0^{b_j-2} \cdots 10^{b_2-1}10.$$

In other words, for (C) to correspond to an occurrence of $d$ in $\mathcal{K}$, we need to consider occurrences of $d$ that originate from rationals whose corresponding binary code ends in 10. If $d$ is formed through a subpath that starts from such a rational, the binary code that is associated to this subpath is appended to that of the rational it originates from. We conclude that we can count these occurrences by looking at the frequency of the block

$$100^{d_k-2}\cdots 10^{d_2-1}10^{d_1-1}1 = 10^{d_k-1}\cdots 10^{d_2-1}10^{d_1-1}1. \tag{C*}$$

This is similar to (B). Moreover by counting the blocks in (A), we count (B) and (C*) as well. In order to prevent double counts, we append a 0 to the code in (A). In conclusion, in order to find the frequency of $d$ in $\mathcal{K}$, it is sufficient to consider the asymptotic frequencies of $10^{d_k-1}\cdots 10^{d_2-1}10^{d_1-1}1$ and $10^{d_k-1}\cdots 10^{d_2-1}10^{d_1-1}0$ in $\mathcal{C}_2$. Both blocks occur with relative frequency

$$2^{-(d_1+d_2+\cdots+d_k+1)}.$$

This results from the fact that the binary codes are of length $d_1+d_2+\cdots+d_k+1$ and that $\mathcal{C}_2$ is normal in base 2. Adding these frequencies gives the desired result

$$\frac{1}{2^{d_1+d_2+\cdots+d_k+1}}+\frac{1}{2^{d_1+d_2+\cdots+d_k+1}} = 2^{-(d_1+\cdots+d_k)}.$$

We conclude that $\mathcal{K}$ is Minkowski normal. $\square$

## 4. Extending Minkowski normality

When constructing a normal number, it is the ordering that is chosen that determines the distribution. Apparently, ordering the rationals based on their denominator leads to the distribution given by the Gauss measure, e.g. see Vandehey [30]. Although the sequence of rationals in (1.2) is distributed according to the Lebesgue measure and not the Gauss, it is not that surprising that the number constructed by Adler, Keane and Smorodinsky is continued fraction normal. When we consider the frequency of occurrences of an arbitrary block $d = d_1, d_2, \cdots, d_k$ starting at the $n$-th position of a continued fraction expansion of a number in a uniformly distributed sequence, this frequency is given by the Lebesgue measure of the set $\mathcal{G}^{-n}\Delta(d)$ [1], where $\mathcal{G}$ denotes the Gauss map. Gauss showed that, as $n \to \infty$, $\lambda(\mathcal{G}^{-n}\Delta(d))$ converges in distribution to $\gamma(\Delta(d))$. In a similar manner, we can argue that $\mathcal{K}$ should be Minkowski normal. Namely, the sequence of rationals that is obtained by ordering the rationals in the Kepler tree top-down left-right, see (2.3), is distributed according to the Minkowski question mark. Then it follows that the frequency of occurrences of $d$, starting at the $n$-th position of a continued fraction expansion of a number in a Minkowski question mark distributed sequence, is given by the Minkowski measure of $\mathcal{G}^{-n}\Delta(d)$. As $\mu_?$ is $\mathcal{G}$-invariant, this measure is simply $\mu_?(\Delta(d))$. The fact that the sequence in (2.3) is distributed according to $\mu_?$ has implicitly been proved by Viader, Paradís and Bibiloni [31]. In the article, they first define a one-to-one correspondence $q : \mathbb{N} \to (0, 1)$. The first few terms of $q$ are

$$\begin{aligned} q(1) &= [2] = 1/2 & \qquad q(5) &= [1,3] = 3/4 \\ q(2) &= [3] = 1/3 & q(6) &= [2,2] = 2/5 \\ q(3) &= [1,2] = 2/3 & q(7) &= [1,1,2] = 3/5 \end{aligned}$$

$$q(4) = [4] = 1/4 \qquad q(8) = [5] = 1/5,$$

which result from the following definition. If $n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k}$ with $0 \leq a_1 < a_2 < \cdots < a_k$, then

$$q(n) := \begin{cases} [k+2] & \text{if } n = 2^k, \\ [a_1 + 1, a_2 - a_1, a_3 - a_2, \cdots, a_k - a_{k-1} + 1] & \text{otherwise.} \end{cases} \tag{4.1}$$

Among other things, Viader, Paradís and Bibiloni prove that, for any $x \in [0, 1]$,

$$\lim_{n \to \infty} \frac{\#\{q(i) \leq x : 1 \leq i \leq n\}}{n} = ?(x), \tag{4.2}$$

see [31, Theorem 2.7]. Here $\#A$ denotes the cardinality of the set $A$. We next show that the sequence of rationals in (2.3) is distributed according to the Minkowski question mark. More specifically, we prove that this sequence coincides with the sequence $(q(i))_{i \geq 1}$. Let the sequence in (2.3) be represented by $(k_i)_{i \geq 1}$. That is, $k_i$ denotes the $i$-th rational in (2.3).

**Lemma 4.1.** *The sequence $(k_i)_{i \geq 1}$ is distributed according to the Minkowski question mark measure. That is, for any $x \in [0, 1]$, we have that*

$$\lim_{n \to \infty} \frac{\#\{k_i \leq x : 1 \leq i \leq n\}}{n} = ?(x),$$

*where $\#A$ denotes the cardinality of the set $A$.*

**Proof.** We prove that $q(n) = k_n$ for all $n \in N$. It is clear that $q(1) = k_1 = 1/2$. We next show that the Kepler rule coincides with

$$\begin{array}{ccc} & q(n) & \\ q(2n) & & q(2n+1) \end{array},$$

which concludes the proof. Let $n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k}$ with $0 \leq a_1 < a_2 < \cdots < a_k$. Suppose that $n = 2^l$ for some $l$. Then $2n = 2^{l+1}$ and $2n + 1 = 2^0 + 2^{l+1}$. Using (4.1), we find

$$\begin{array}{cc} \multicolumn{2}{c}{q(n) = [l+2]} \\ q(2n) = [(l+1)+2] = [(l+2)+1] & q(2n+1) = [0+1, (l+1) - 0 + 1] = [1, l+2]. \end{array}$$

Next, assume that $n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k} \neq 2^l$. Then $q(n) = [a_1 + 1, a_2 - a_1, a_3 - a_2, \cdots, a_k - a_{k-1} + 1]$, and

$$\begin{aligned} 2n &= 2^{a_1+1} + 2^{a_2+1} + \cdots + 2^{a_k+1}; \\ 2n+1 &= 2^0 + 2^{a_1+1} + 2^{a_2+1} + \cdots + 2^{a_k+1}. \end{aligned}$$

Applying (4.1) to the above, we get

$$\begin{aligned} q(2n) &= [(a_1+1)+1, (a_2+1)-(a_1+1), (a_3+1)-(a_2+1), \cdots, (a_k+1) \\ &\quad - (a_{k-1}+1)+1] \\ &= [(a_1+1)+1, a_2-a_1, a_3-a_2, \cdots, a_k - a_{k-1}+1]; \\ q(2n+1) &= [0+1, (a_1+1)-0, (a_2+1)-(a_1+1), (a_3+1) \\ &\quad - (a_2+1), \cdots, (a_k+1)-(a_{k-1}+1)+1] \\ &= [1, (a_1+1), a_2-a_1, a_3-a_2, \cdots, a_k-a_{k-1}+1]. \end{aligned}$$

We conclude that $(q(i))_{i\geq 1}$ coincides with $(k_i)_{i\geq 1}$. Therefore, there is an equivalence between the statement in (4.2) and the limit in Lemma 4.1. $\square$

Thus, the sequence in (2.3) is distributed according to $\mu_?$. Apart from this fact, there is an important underlying structure in the sequence that causes normality. We discuss this structure and show that it can be used to construct a class of Minkowski normal numbers. Moreover, we provide an explicit example using the Farey tree.

The continued fraction normality of $x_{aks}$ results from the ordering of rationals based on their denominator. This ordering causes the sequence of rationals in (1.2) to be distributed uniformly and hence $x_{aks}$ to be continued fraction normal. Minkowski normality of $\mathcal{K}$, however, results from a completely different underlying structure. The underlying structure in this case comes from fact that the rationals are ordered increasingly, based on the sum of the digits of their continued fraction expansion. That is, the $l$-th level of the Kepler tree contains all possible rationals that have a continued fraction expansion whose sum of digits is equal to $l+2$. By ordering these top-down, left-right, the ordering is done as claimed. To see that the Kepler tree has this structure, we start by considering the root. The root of the tree, which corresponds to level 0, is given by $1/2 = [2]$. Then, every next level, the sum of digits of the continued fraction expansion is increased by 1 through the Kepler rule. Furthermore, the $l$-th level of the Kepler tree contains $2^l$ rationals, which is exactly the number of distinct[2] rationals that have a continued fraction expansion whose digits sum up to $l+2$.

**Lemma 4.2.** *There exist exactly $2^l$ distinct rationals that have a continued fraction expansion of which the sum of the digits equals $l+2$, $l \geq 0$. That is,*

[2] We say that two rationals p/q and r/s are distinct if and only if $ps \neq qr$.

$$\#\Big\{\frac{p}{q}\in[0,1):\frac{p}{q}=[a_1,a_2,\cdots,a_n],\sum_{i=1}^{n}a_i=l+2\Big\}=2^l,$$

*where $\#A$ denotes the cardinality of the set $A$.*

We omit a proof, as it follows directly from [31, p. 215]. Due to this lemma, we conclude that $\mathcal{K}$ is a concrete example of a number that is obtained by concatenating the (reduced) continued fraction expansions of all rationals based on the sum of their digits, in increasing order. That is, one first concatenates the continued fraction expansions of rationals that have a continued fraction expansion of which the digits sum up to 2, then those that sum up to 3, etc. It turns out that all such constructions are Minkowski normal. In order to prove this, we use the fact that generalised Champernowne numbers are normal. That is, if we take $\mathcal{C}_2$ and rearrange the blocks of the same length in any order, the resulting number is normal in base 2 [10]. Due to the structure that underlies our construction, we can use this to extend our results. Again, the key idea is the unique correspondence between binary codes of length $l$ and continued fractions whose digits sum up to $l+2$. Let $[a_1,a_2,\cdots,a_n]$ be such that $\sum_{i=1}^{n}a_i=l+2$, then recall that this correspondence is given by

$$[a_1,a_2,\cdots,a_n]\overset{\text{binary code}}{\longleftrightarrow}\underbrace{0^{a_n-2}\cdots10^{a_2-1}10^{a_1-1}}_{\text{binary code of length l}}. \tag{4.3}$$

The proof of Theorem 3.4 shows that we can count arbitrary blocks in $\mathcal{K}$ through binary codes and explains why and how by referring to the structure of the Kepler tree. However, it is the coding that is important. Moreover, it is the explicit one-to-one correspondence between continued fraction expansions and binary codes that allows us to obtain frequencies and extend our results. This is due to the fact that divided occurrences are negligible and that the binary codes used in the proof result from the coding that is used. That is, if we convert a continued fraction expansion $[a_1,a_2,\cdots,a_n]$ to its binary code $0^{a_n-2}\cdots10^{a_2-1}10^{a_1-1}$, we can use the binary codes in the proof to obtain the frequency of occurrences of $d$ in $[a_1,a_2,\cdots,a_n]$. As such, we can extend the normality of $\mathcal{K}$ to more general cases.

**Theorem 4.3.** *Let the constructed number $\mathcal{K}$ be denoted by*

$$\mathcal{K}=[\kappa_1^1,\ \kappa_2^1,\ \kappa_1^2,\ \kappa_2^2,\ \kappa_3^2,\ \kappa_4^2,\ \kappa_1^3,\ \cdots],$$

*where $\kappa_1^l,\kappa_2^l,\cdots,\kappa_{2^l}^l$ is the concatenation of the continued fraction expansions of the rationals in the $l$-th level of the Kepler tree, ordered from left to right. Furthermore, for all $l\in\mathbb{N}$, let $\pi_l$ be a permutation of $\{1,2,\ldots,2^l\}$. Then*

$$\mathcal{K}^\pi:=[\kappa^1_{\pi_1(1)},\ \kappa^1_{\pi_1(2)},\ \kappa^2_{\pi_2(1)},\ \kappa^2_{\pi_2(2)},\ \kappa^2_{\pi_2(3)},\ \kappa^2_{\pi_2(4)},\ \kappa^3_{\pi_3(1)},\ \cdots]$$

*is Minkowski normal.*

$$[a_1, a_2, \cdots, a_n]$$

$$[a_1, a_2, \cdots, (a_n + 1)] \qquad [a_1, a_2, \cdots, (a_n - 1), 2]$$

(a)

$$[a_1, a_2, \cdots, a_n]$$

$$[a_1, a_2, \cdots, (a_n - 1), 2] \qquad [a_1, a_2, \cdots, (a_n + 1)]$$

(b)

**Fig. 3.** The rule of the Farey tree for (a) $n$ is odd and (b) $n$ is even.

**Proof.** Let $\mathcal{C}_2$ be denoted by

$$\mathcal{C}_2 = 0.\ c_1^1\ c_2^1\ c_1^2\ c_2^2\ c_3^2\ c_4^2\ c_1^3\ \cdots,$$

where $c_1^l c_2^l \cdots c_{2^l}^l$ denotes the concatenation of all binary codes in the $l$-th level of the binary Kepler tree, ordered from left to right. It follows from [10] and [25] that

$$\mathcal{C}_2^\pi := 0.\ c_{\pi_1(1)}^1\ c_{\pi_1(2)}^1\ c_{\pi_2(1)}^2\ c_{\pi_2(2)}^2\ c_{\pi_2(3)}^2\ c_{\pi_2(4)}^2\ c_{\pi_3(1)}^3\ \cdots$$

is normal in base 2. Let $d = d_1, d_2, \cdots, d_k$ be an arbitrary block of length $k$. Note that $\mathcal{C}_2^\pi$ corresponds to the concatenation of the binary codes of the continued fraction expansions that are concatenated in $\mathcal{K}^\pi$. As these binary codes and continued fraction expansions are (uniquely) related by the correspondence in (4.3), we can count the number of occurrences of $d$ in $\mathcal{K}^\pi$ by considering the frequency of $10^{d_k-1} \cdots 10^{d_2-1}10^{d_1-1}1$ and $10^{d_k-1} \cdots 10^{d_2-1}10^{d_1-1}0$ in $\mathcal{C}_2^\pi$. The rest of the proof is analogous to the proof of Theorem 3.4. We conclude that $\mathcal{K}^\pi$ is Minkowski normal. □

In particular, Theorem 4.3 proves Minkowski normality of the number that is obtained by concatenating the continued fraction expansions of the rationals in the Farey tree top-down, left-right. The tree starts with $1/2 = [2]$ at the root and forms new rationals according to the tree rule displayed in Fig. 3, see [5]. Consequently, the ordering of the rationals that is obtained by this, is

$$\frac{1}{2}, \quad \frac{1}{3}, \frac{2}{3}, \quad \frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{3}{4}, \quad \frac{1}{5}, \cdots.$$

It was implicitly shown by Kessebömer and Stratmann [14] that this sequence is distributed according to $\mu_?$. Therefore it should not be surprising that the following holds.

**Proposition 4.4.** *The number that is obtained by concatenating the continued fraction expansions of the rationals in the Farey tree top-down left-right is Minkowski normal.*

**Proof.** It can be seen from the tree rules that, regardless of whether $n$ is even or odd, the Farey tree rule increases the sum of the digits of the continued fraction expansion by 1 each next level. Therefore, the underlying structure of the tree is similar to that of the

Kepler tree. Namely, the rationals are ordered increasingly, based on the sum of the digits of their continued fraction expansion. Hence, the $l$-th level of the Farey tree contains all possible rationals that have a continued fraction expansion whose sum of digits is equal to $l+2$. By concatenating the continued fraction expansions of the rationals in the Farey tree top-down, left-right, we obtain a permutation of $\mathcal{K}$ that satisfies the conditions in Theorem 4.3. Therefore, we conclude that the number that is obtained by concatenating the continued fraction expansions of the rationals in the Farey tree top-down left-right is Minkowski normal. $\square$

**Remark.** The extension in Theorem 4.3 is based on work of Shiokawa and Uchiyama [25], which extends normality of the dyadic Champernowne number. Moreover, our extension is based on a specific case of [25, Lemma 4]. This extension exploits the underlying structure of the Kepler tree to extend the Minkowski normality of $\mathcal{K}$ to more general cases. As such, we preserve the underlying structure and hence - in some way - preserve normality. We have not been able to prove a full analogue of Shiokawa and Uchiyama's result. One of the reasons that we cannot extend normality to this general case, is that we can no longer use the normality of $\mathcal{C}_2^\pi$ to count frequencies. That is, our extension allows one to reorder the continued fraction expansions of rationals that have a continued fraction expansion of which the partial quotients sum up to the same number. A full analogue of the work of Shiokawa and Uchiyama would allow one to break up the continued fraction expansion of the same rationals into smaller parts and reorder these arbitrarily. However, when we break up continued fraction expansions into smaller parts, one creates subblocks of which the sum of its digits will vary and the composition of binary codes will change. Consider for instance the continued fraction $[2,1,1,3]$, which corresponds to the binary code 01110. Suppose we break this up into $[2]$ and $[1,1,3]$. Then these correspond to the binary codes $\emptyset$ and 011 respectively. Conversely, break up 01110 into the blocks 011 and 10. These binary codes correspond, respectively, to the continued fraction expansions $[1,1,3]$ and $[1,3]$. This shows that the underlying structure is not preserved when breaking up continued fraction expansions into smaller parts. However, it should be possible to find a similar extension.

We now use the normality of $\mathcal{K}$ to explicitly prove the existence of some other normal numbers. For this, we first consider the Farey map $\mathcal{F}$ on $[0,1]$, defined

$$\mathcal{F}(x) := \begin{cases} \frac{x}{1-x} & \text{if } x \in [0,1/2), \\ \frac{1-x}{x} & \text{if } x \in [1/2,1]. \end{cases}$$

This map has an infinite ergodic absolutely continuous invariant measure (the density is $1/x$), but no finite absolutely continuous invariant measure. However, the Minkowski question mark measure is an ergodic invariant measure for the Farey map. By the definition of $\mu_?$, it is the push forward of Lebesgue measure

$$(?^*\lambda)([0,x)) = \lambda([0,?(x)) = ?(x) = \mu_?([0,x)). \tag{4.4}$$

In turn, this implies that

$$(\mathcal{F}\circ ?)(x) = (? \circ \mathcal{T})(x), \text{ for } x \in [0,1], \tag{4.5}$$

where $\mathcal{T}(x) = 1-2|x-1/2|$ is the tent map on $[0,1]$, see [17]. The tent map is ergodic and measure preserving for the Lebesgue measure, which is the measure of maximal entropy. Note that it is essentially the $(1/2, 1/2)$-Bernoulli measure of the 2-shift. Since (4.5) is a topological conjugacy, it follows that $\mu_?$ is the measure of maximal entropy for $\mu_?$ [17]. Therefore, we claim the following.

**Proposition 4.5.** *The number $\mathcal{K}$ is normal for the Farey map with the Minkowski question mark measure.*

**Proof.** The Gauss map $\mathcal{G}$ is the "jump" of the Farey map, see [17]. That is, for $x = [a_1, a_2, a_3, \cdots] \in [0,1]$ one has

$$\mathcal{G}(x) = \mathcal{F}^{a_1-1}(x). \tag{4.6}$$

The *binary Farey expansion* of $x$ is given by $f(x) = f_1 f_2 f_3 \cdots$ where $f_n = 0$ if $\mathcal{F}^{n-1}(x) \in [0, 1/2)$ and $f_n = 1$ if $\mathcal{F}^{n-1}(x) \in [1/2, 1]$. It follows from (4.6) that $f(x) = 0^{a_1-1}10^{a_2-1}10^{a_3-1}1\cdots$. Thus $f(\mathcal{F}(x)) = \mathcal{S}(f(x))$, where $\mathcal{S}$ denotes the left shift. When viewed as a map from continued fraction expansions to Farey expansions, it is clear that $f$ respects concatenation. Thus, using the notation of Theorem 4.3, we have that

$$\begin{aligned} f(\mathcal{K}) &= f([\kappa_1^1,\ \kappa_2^1,\ \kappa_1^2,\ \kappa_2^2,\ \kappa_3^2,\ \kappa_4^2,\ \kappa_1^3,\ \cdots]) \\ &= f(\kappa_1^1)\ f(\kappa_2^1)\ f(\kappa_1^2)\ f(\kappa_2^2)\ f(\kappa_3^2)\ f(\kappa_4^2)\ f(\kappa_1^3)\ \cdots \\ &= \ 1c^1_{\pi_1(1)}\ 1c^1_{\pi_1(2)}\ 1c^2_{\pi_2(1)}\ 1c^2_{\pi_2(2)}\ 1c^2_{\pi_2(3)}\ 1c^2_{\pi_2(4)}\ 1c^3_{\pi_3(1)}\ \cdots . \end{aligned}$$

Here $\pi_l$ is the permutation on $\{1, 2, \ldots, 2^l\}$ such that $\pi_l(1 + b_0 + b_1 2 + \cdots + b_{l-1}2^{l-1}) = 1 + b_{l-1} + b_{l-2}2 + \cdots + b_1 2^{l-1}$ where $b_k \in \{0,1\}$. Notice that $f(\mathcal{K})$ is just $\mathcal{C}_2^\pi$ with some extra 1s between the blocks $c^j_{\pi_j(k)}$. As these 1s have density zero, it follows that all the binary blocks in $f(\mathcal{K})$ occur with the same frequency as in $\mathcal{C}_2^\pi$. □

Now consider the two generalized Lüroth (GLS) maps $\mathcal{L}_1$ and $\mathcal{L}_2$ on $(0, 1]$, defined for $x \in (1/2^n, 2/2^n]$, $n \geq 1$,

$$\mathcal{L}_1(x) = 2 - 2^{-n}x \quad \text{and} \quad \mathcal{L}_2(x) = 2^n x - 1.$$

Like all GLS maps, both of these preserve Lebesgue measure, and it is easy to see that both are ergodic. The latter follows from the fact that both are isomorphic to a one-sided shift $\mathcal{S}$ on a countable alphabet, with a Bernoulli (product) measure. Let

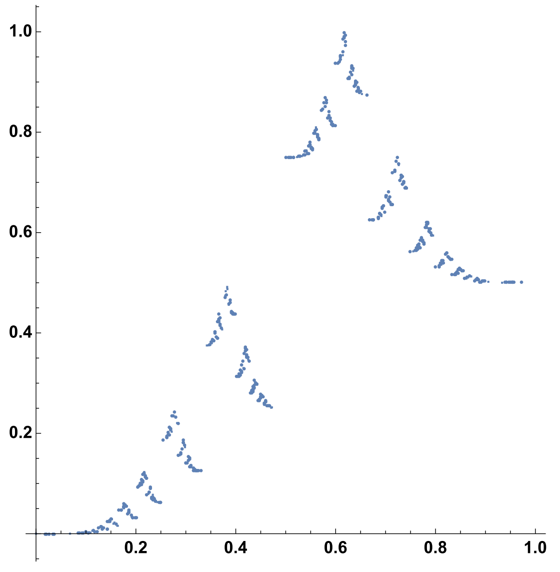


**Fig. 4.** The function $\phi(\cdot)$.

$$\phi(y) = \sum_{i=1}^{\infty} \frac{1}{2^{a_1+a_2+\cdots+a_i}},$$

for $y = [a_1, a_2, a_3, \cdots]$ (see Fig. 4). We then have the following.

**Proposition 4.6.**

*(i)* *One has* $(\mathcal{G}\circ ?)(x) = (? \circ \mathcal{L}_1)(x)$ *and* $(\mathcal{G} \circ \phi)(x) = (\phi \circ \mathcal{L}_2)(x)$ *for* $x \in [0,1]$, *with* $(?^*\lambda) = (\phi^*\lambda) = \mu_?$. *Thus* $\mathcal{L}_1$ *with Lebesgue measure and* $\mathcal{L}_2$ *with Lebesgue measure are both isomorphic to* $\mathcal{G}$ *with* $\mu_?$. *It follows that* $\mathcal{G}$ *with* $\mu_?$ *is ergodic.*

*(ii)* *Let* $\mathcal{K} \in [0,1]$ *(i.e., view* $\mathcal{K}$ *as the continued fraction expansion of the sequence (2.5)). Then* $x_1 = ?(\mathcal{K})$ *is normal for* $\mathcal{L}_1$ *and* $x_2 = \phi(\mathcal{K})$ *is normal for* $\mathcal{L}_2$.

*(iii)* *As a sequence,* $\mathcal{K}$ *(2.5) is the* $\mathcal{L}_1$ *generalized Lüroth expansion of* $x_1$ *and also the* $\mathcal{L}_2$ *generalized Lüroth expansion of* $x_2$.

**Proof.** The first conjugacy is well known (see [17]) and the push forward is (4.4). It follows that $?(\mathcal{K})$ is normal, and the sequence $\mathcal{K}$ is the $\mathcal{L}_1$ expansion of this number.

For $\mathcal{L}_2$, we show that

$$\Delta_{\mathcal{L}_2}(a_1, a_2, \cdots, a_n) = \phi(\Delta_{\mathcal{G}}(a_1, a_2, \cdots, a_n)). \tag{4.7}$$

Since $\mathcal{L}_2(x) = 2^{a_1}x - 1$, it follows that $x = \frac{1}{2^{a_1}} + \frac{\mathcal{L}_2(x)}{2^{a_1}}$. Continuing by induction, any $x \in \Delta_{\mathcal{L}_2}(a_1, a_2, \cdots, a_n)$ has the form

$$x = \frac{1}{2^{a_1}} + \frac{1}{2^{a_1+a_2}} + \cdots + \frac{1}{2^{a_1+a_2+\cdots+a_n}} + \frac{\mathcal{L}_2^n(x)}{2^{a_1+a_2+\cdots+a_n}}$$

$$= \sum_{i=1}^{\infty} \frac{1}{2^{a_1+a_2+\cdots+a_i}} = \phi([a_1, a_2, a_3, \ldots, a_n, \ldots]).$$

Since $\mathcal{L}_2^n$ varies linearly from 0 to 1 on this (fundamental) interval, we see that

$$\Delta_{\mathcal{L}_2}(a_1, a_2, \cdots, a_n) = \Big( \sum_{i=1}^{n} \frac{1}{2^{a_1+a_2+\cdots+a_i}}, \sum_{i=1}^{n} \frac{1}{2^{a_1+a_2+\cdots+a_i}} + \frac{1}{2^{a_1+a_2+\cdots+a_n}} \Big].$$

It follows that the endpoints of $\Delta_{\mathcal{L}_2}(a_1, a_2, \cdots, a_n)$ are $\phi(y_1)$ and $\phi(y_2)$ where $y_1 = [a_1, a_2, \cdots, a_n]$ and $y_2 = [a_1, a_2, \cdots, a_n + 1]$. Since $y_1$ and $y_2$ are the endpoints of $\Delta_{\mathcal{G}}(a_1, a_2, \cdots, a_n)$ — possibly in reversed order — one has

$$\lambda(\Delta_{\mathcal{L}_2}(a_1, a_2, \cdots, a_n)) = 2^{-(a_1+a_2+\cdots+a_n)} = \mu_?(\Delta_{\mathcal{G}}(a_1, a_2, \cdots, a_n)),$$

and everything follows. □

**Comment.** The proof shows that $\mathcal{G}$ is conjugate to the countable-alphabet one-sided shift $\mathcal{S}$ on $\mathbb{N}^{\mathbb{N}}$ and $\mu_?$ pushes forward to a Bernoulli (product) measure $\pi$. For any $\mathcal{S}$-invariant measure $\mu$ on $\mathbb{N}^{\mathbb{N}}$ (there are many of them) let $G_\mu$ be the set of points $\mathcal{X} \in \mathbb{N}^{\mathbb{N}}$ that are normal (generic) for $\mu$. That is, the frequency of any finite sequence of digits in $\mathcal{X}$ is the $\mu$-measure of the corresponding cylinder. The quasi-regular points are defined $Q_{\mathcal{S}} = \cup G_\mu$, where the union is taken over all ergodic $\mathcal{S}$-invariant measures. It follows from [11], Theorem 1.1, that $Q_{\mathcal{S}}$ is of first category (since $\mathbb{N}^{\mathbb{N}}$ is Polish, and being a full shift, $\mathcal{S}$ satisfies a very strong specification property), so $G_\pi \subset Q_{\mathcal{S}}$ is also of the first category. In a similar way, [2] show $G_\pi$ is $\Pi_3^0$-complete.

**Final remark.** Similar to the case of the Kepler tree, it follows that the number that is obtained by concatenating the continued fraction expansions of the rationals in the Calkin-Wilf tree top-down left-right is Minkowski normal. This tree starts with 1/1 at the root and uses the tree rule

$$\begin{array}{ccc} & p/q & \\ p/(p+q) & & (p+q)/q \end{array}.$$

Notice that the left tree rule is similar to that of the Kepler tree and that the right one is different. Furthermore, note that the Kepler tree contains all rationals in $(0, 1)$, whereas the Calkin-Wilf tree contains all rationals in $\mathbb{R}$. Due to this difference, we distinguish two cases when considering the tree rule of the Calkin-Wilf tree using continued fraction expansions. Let $[a_0; a_1, a_2, \cdots]$ denote the continued fraction expansion of an arbitrary rational $p/q \in \mathbb{R}$. Then we distinguish the cases $a_0 = 0$ and $a_0 \neq 0$. For the first case, the left and right children are given by

$$[0; a_1 + 1, a_2, \cdots, a_n] \quad \text{and} \quad [1; a_1, a_2, \cdots, a_n],$$

respectively. Alternatively, for $a_0 \neq 1$, the left and right children are, respectively, given by

$$[0; 1, a_0, a_1 a_2, \cdots, a_n] \quad \text{and} \quad [a_0 + 1; a_1, a_2, \cdots, a_n].$$

Neglecting the 0s, which are inadmissible in a continued fraction expansion, we see that in both cases either the first digit is increased by 1 or a 1 is inserted as a first digit. Thus, from a symbolical perspective, the Kepler tree and the Calkin-Wilf tree have a similar behaviour when forming blocks of digits. Therefore, if we concatenate the (non-zero) digits of the continued fraction expansions of the rationals in the Calkin-Wilf tree top-down left-right, we obtain another Minkowski normal number.

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