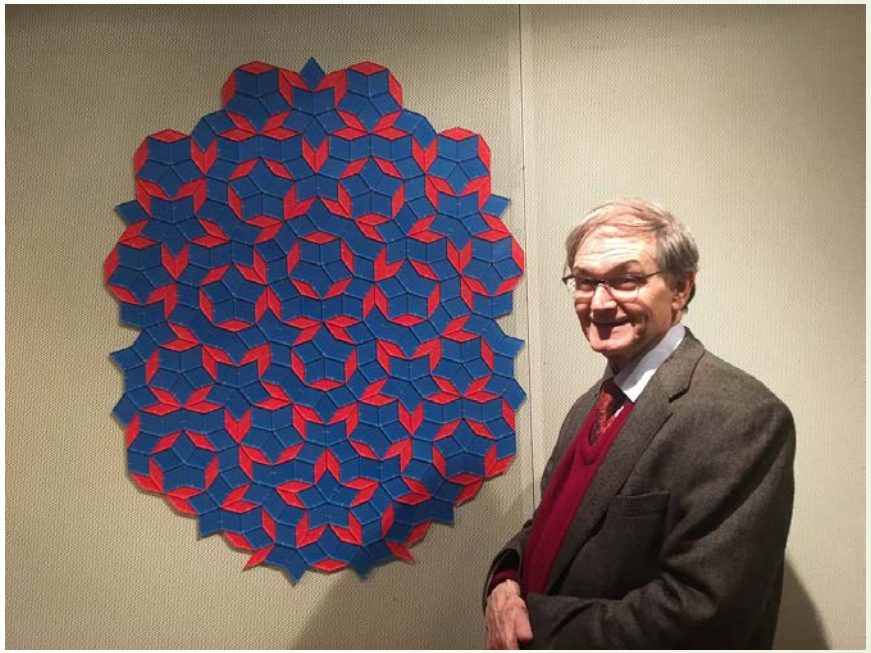
Penrose Tilings I

R. Robinson 1/25/2019

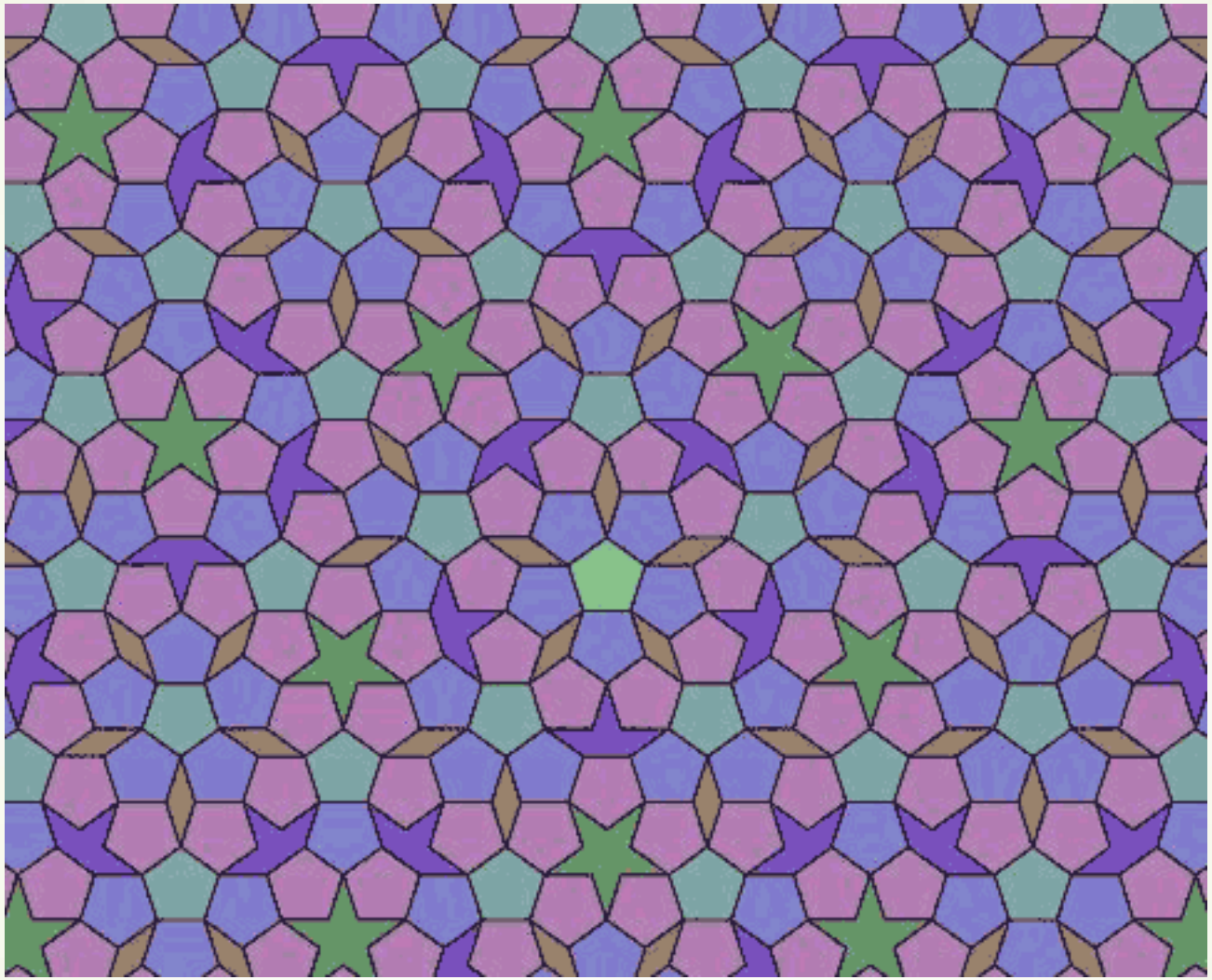
Penrose tilings

Penrose Tiling, mural in Math Department,
Saxe-Patterson Inc., Santa Fe, NM c 1992,
for exhibit at Maryland Science Center

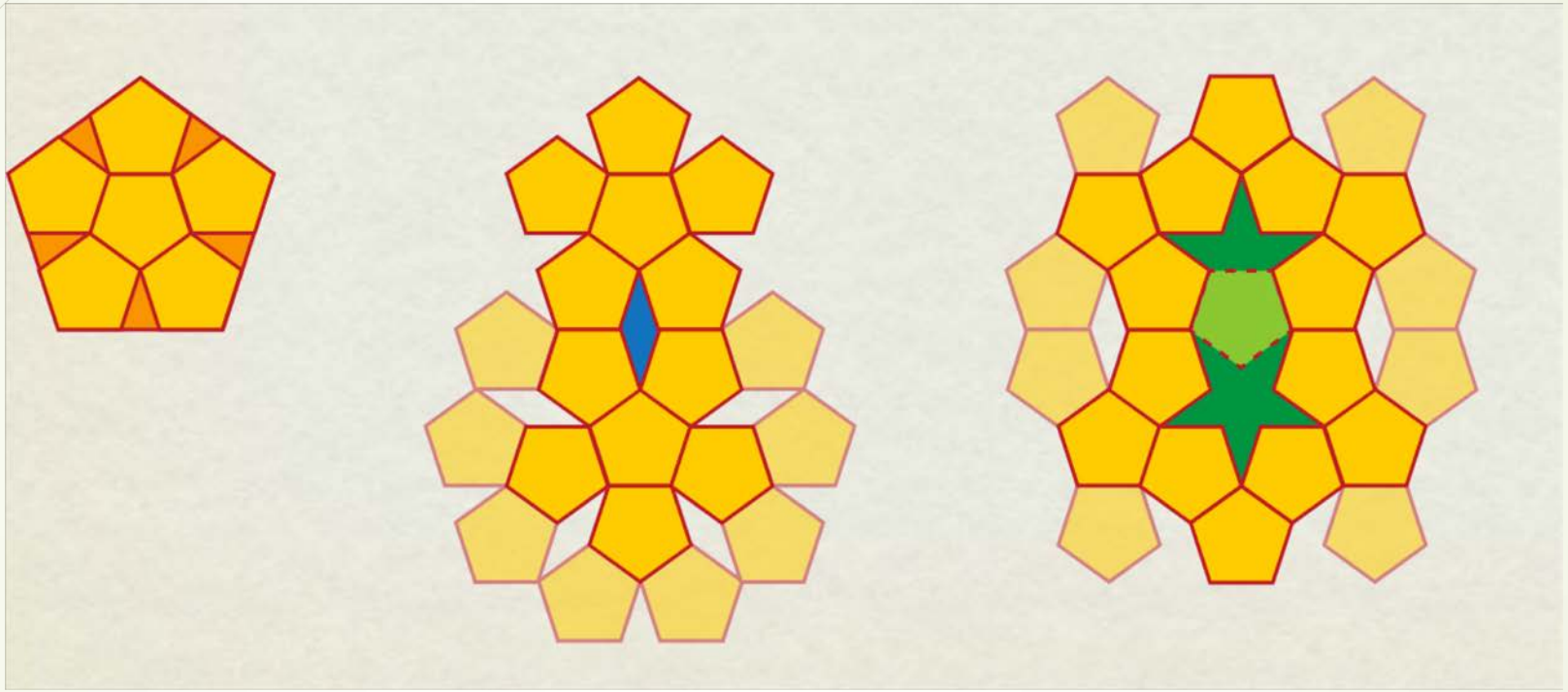
Sir Roger Penrose (1931-)

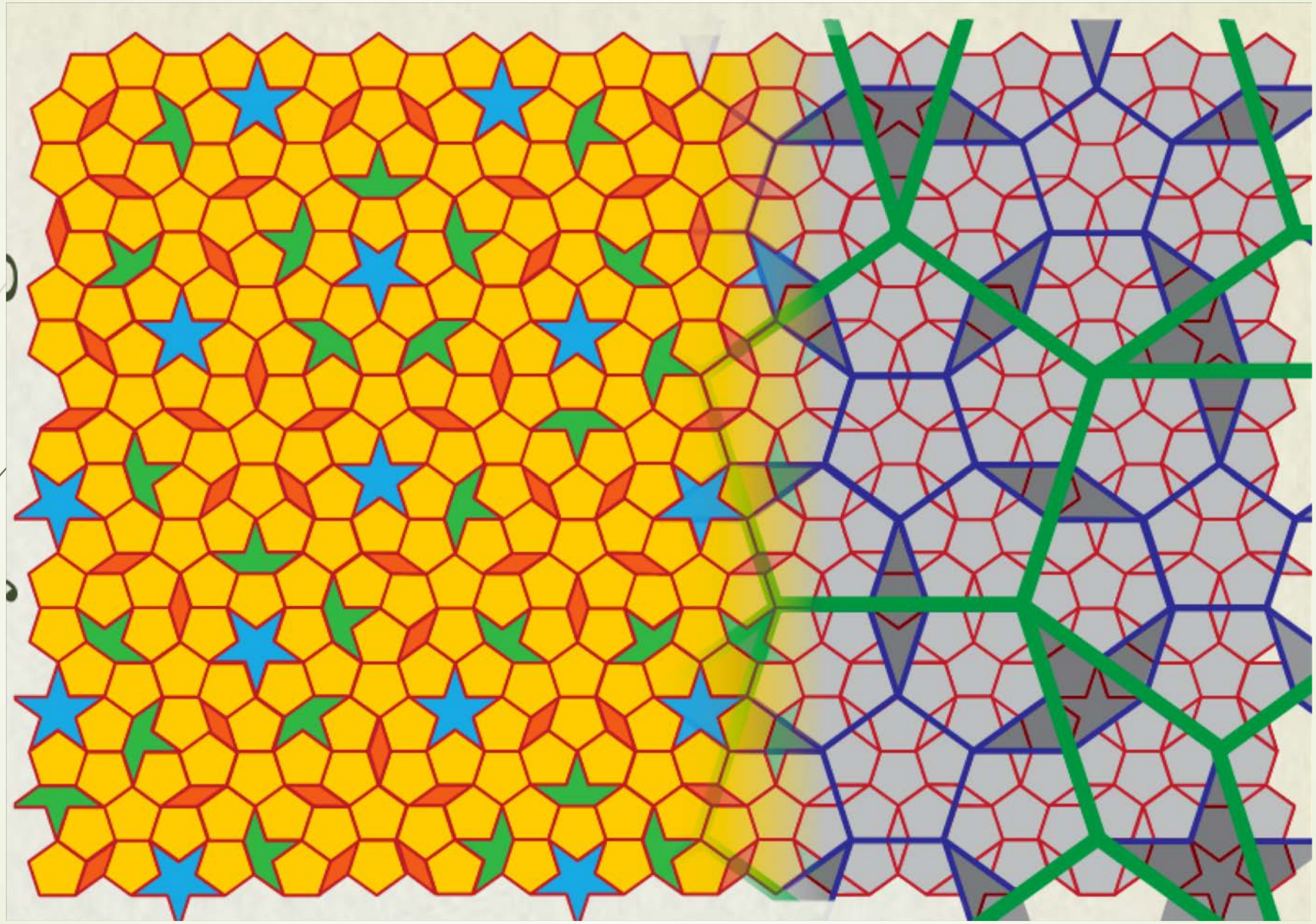


Penrose original “stars” tilings (c1973)

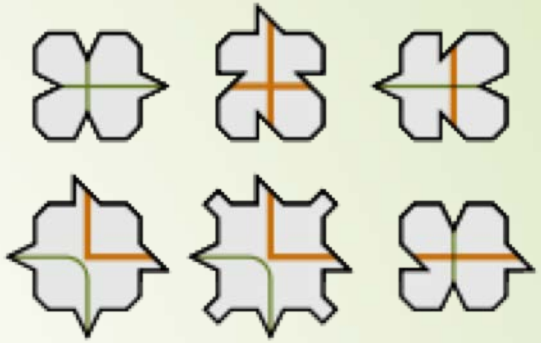


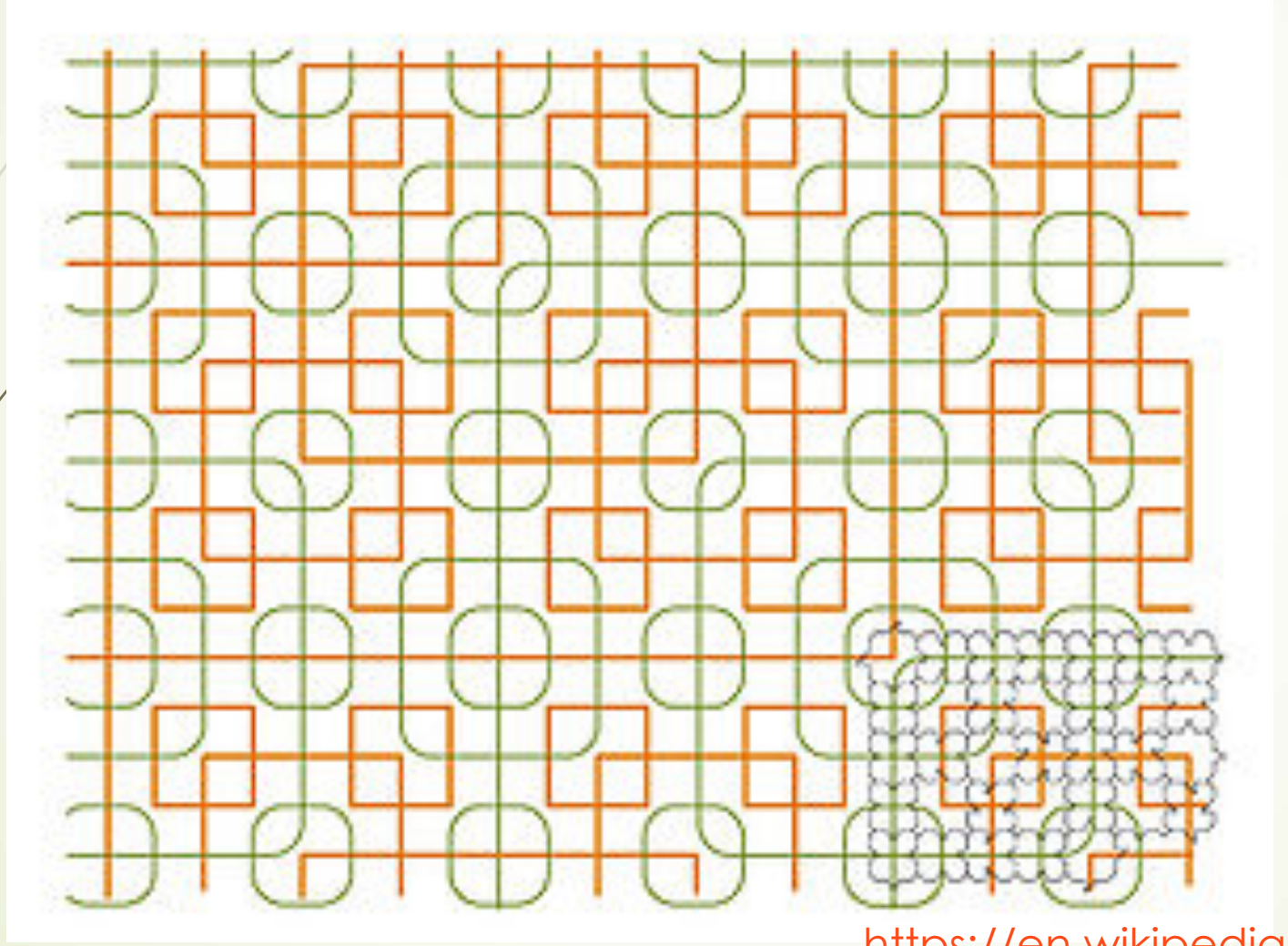
http://www.ams.org/publicoutreach/feature-column/fcarc-penrose





Raphael Robinson (c 1971)

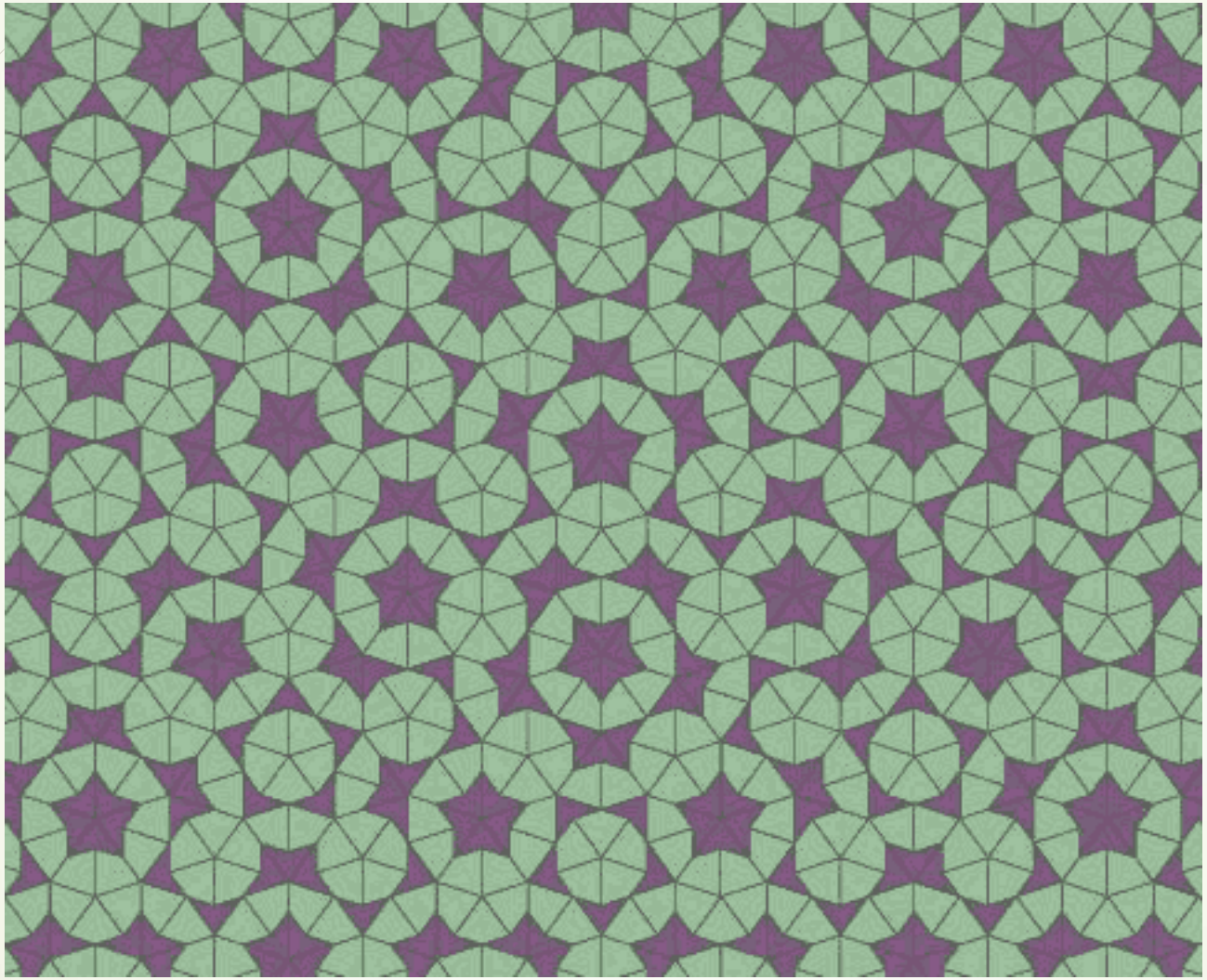




Raphael Robinson, 1911–1995

https://en.wikipedia.org/wiki/Aperiodic_tiling

Penrose Kites and Darts



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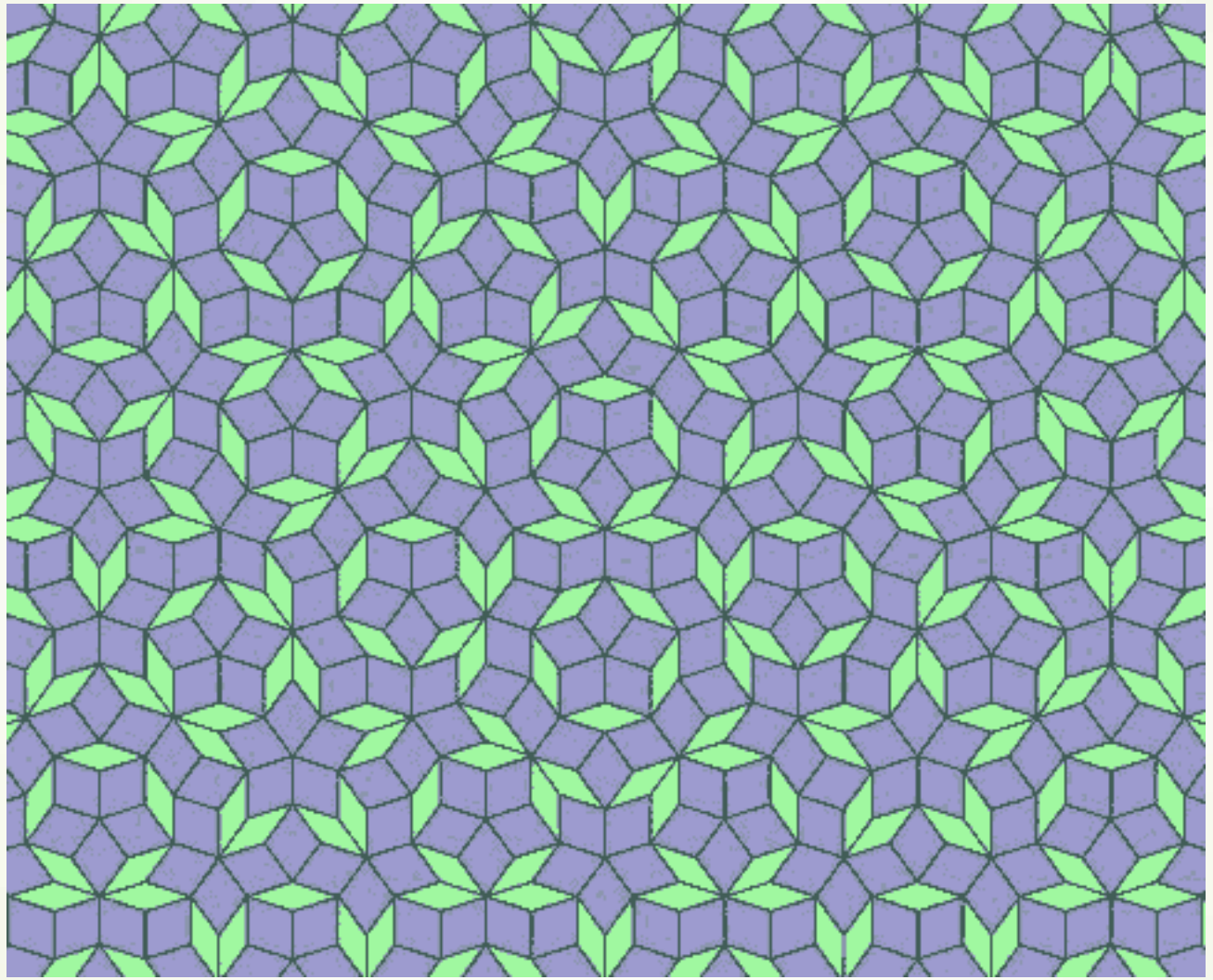
Penrose Rhombs



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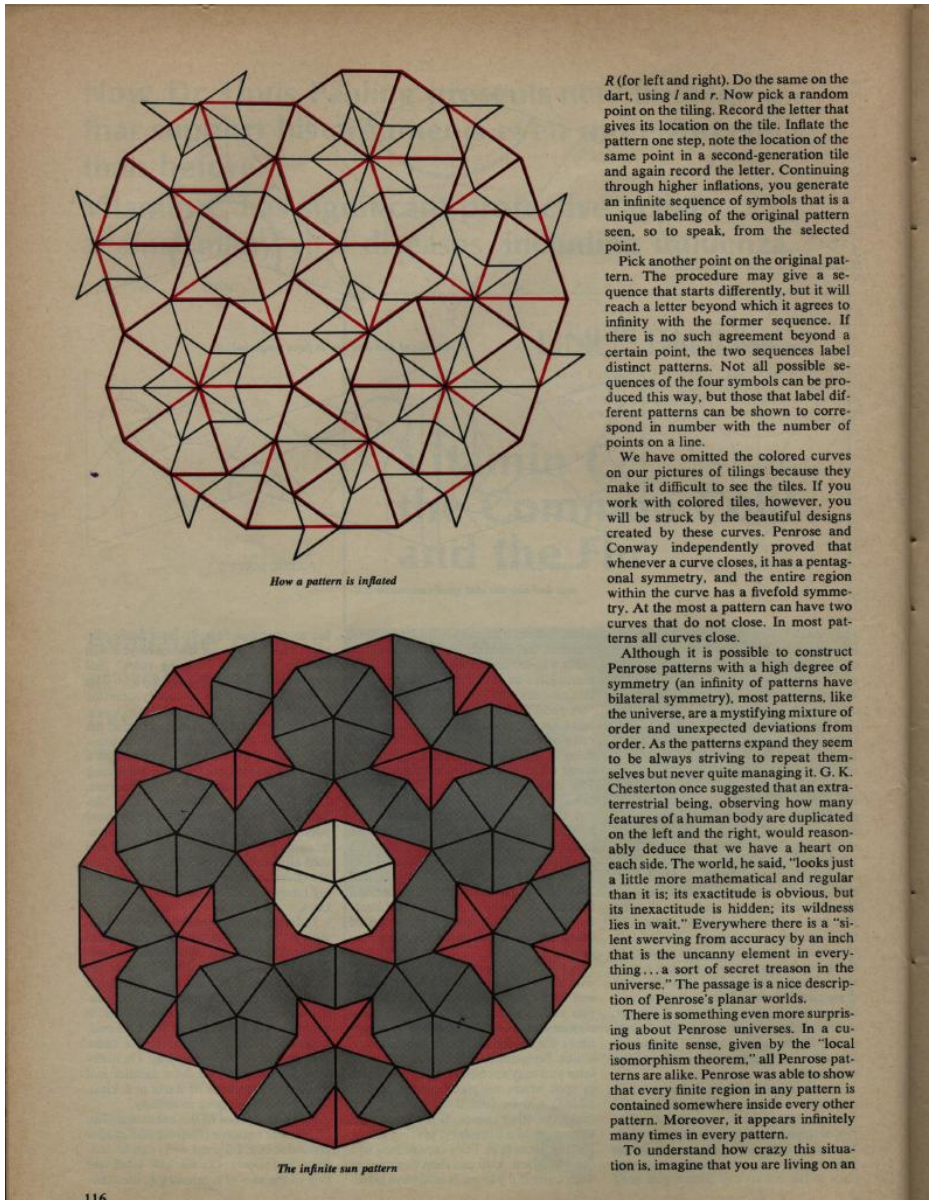
Gardner and Conway



R (for left and right). Do the same on the dart, using *l* and *r*. Now pick a random point on the tiling. Record the letter that gives its location on the tile. Inflate the pattern one step, note the location of the same point in a second-generation tile and again record the letter. Continuing through higher inflations, you generate an infinite sequence of symbols that is a unique labeling of the original pattern seen, so to speak, from the selected point.

Pick another point on the original pattern. The procedure may give a sequence that starts differently, but it will reach a letter beyond which it agrees to infinity with the former sequence. If there is no such agreement beyond a certain point, the two sequences label distinct patterns. Not all possible sequences of the four symbols can be produced this way, but those that label different patterns can be shown to correspond in number with the number of points on a line.

We have omitted the colored curves on our pictures of tilings because they make it difficult to see the tiles. If you work with colored tiles, however, you will be struck by the beautiful designs created by these curves. Penrose and Conway independently proved that whenever a curve closes, it has a pentagonal symmetry, and the entire region within the curve has a fivefold symmetry. At the most a pattern can have two curves that do not close. In most patterns all curves close.

How a pattern is inflated

Although it is possible to construct Penrose patterns with a high degree of symmetry (an infinity of patterns have bilateral symmetry), most patterns, like the universe, are a mystifying mixture of order and unexpected deviations from order. As the patterns expand they seem to be always striving to repeat themselves but never quite managing it. G. K. Chesterton once suggested that an extraterrestrial being, observing how many features of a human body are duplicated on the left and the right, would reasonably deduce that we have a heart on each side. The world, he said, "looks just a little more mathematical and regular than it is; its exactitude is obvious, but its inexactitude is hidden; its wildness lies in wait." Everywhere there is a "silent swerving from accuracy by an inch that is the uncanny element in everything . . . a sort of secret treason in the universe." The passage is a nice description of Penrose's planar worlds.

There is something even more surprising about Penrose universes. In a curious finite sense, given by the "local isomorphism theorem," all Penrose patterns are alike. Penrose was able to show that every finite region in any pattern is contained somewhere inside every other pattern. Moreover, it appears infinitely many times in every pattern.

To understand how crazy this situation is, imagine that you are living on an

The infinite sun pattern

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Scientific American, vol 236 (1977), 111-119

MATHEMATICS

Proceedings A 84 (1), March 20, 1981

of the Koninklijke Nederlandse Akademie van Wetenschappen

Sequences of zeros and ones generated by special production rules

by N.G. de Bruijn

Department of Mathematics, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, the Netherlands

Communicated at the meeting of September 27, 1980

1. INTRODUCTION

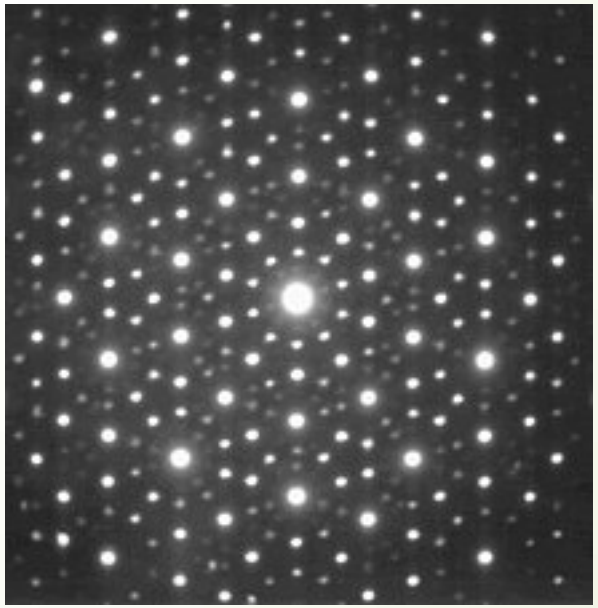
Consider the following rule for the production of (finite or infinite) sequences of 0's and 1's: Take any sequence, and replace every 0 by 10 and every 1 by 100. Example: the finite sequence 10010 transforms into 100101010010. The new sequence is called a *successor* of the old one, and the old one is called a *predecessor* of the new one. The process of replacing 0 by 10 and 1 by 100 will be called *deflation*. The inverse action, to be called *inflation*, can be applied only to sequences that can be subdivided into groups 10 and 100. (The names inflation and deflation are not very suggestive in this context. They have been chosen because of the correspondence with operations in the Penrose tilings mentioned below.)

We have to be a bit more careful about the definition of deflation in the case of doubly-infinite sequences, i.e. sequences indexed by the elements of Z, since

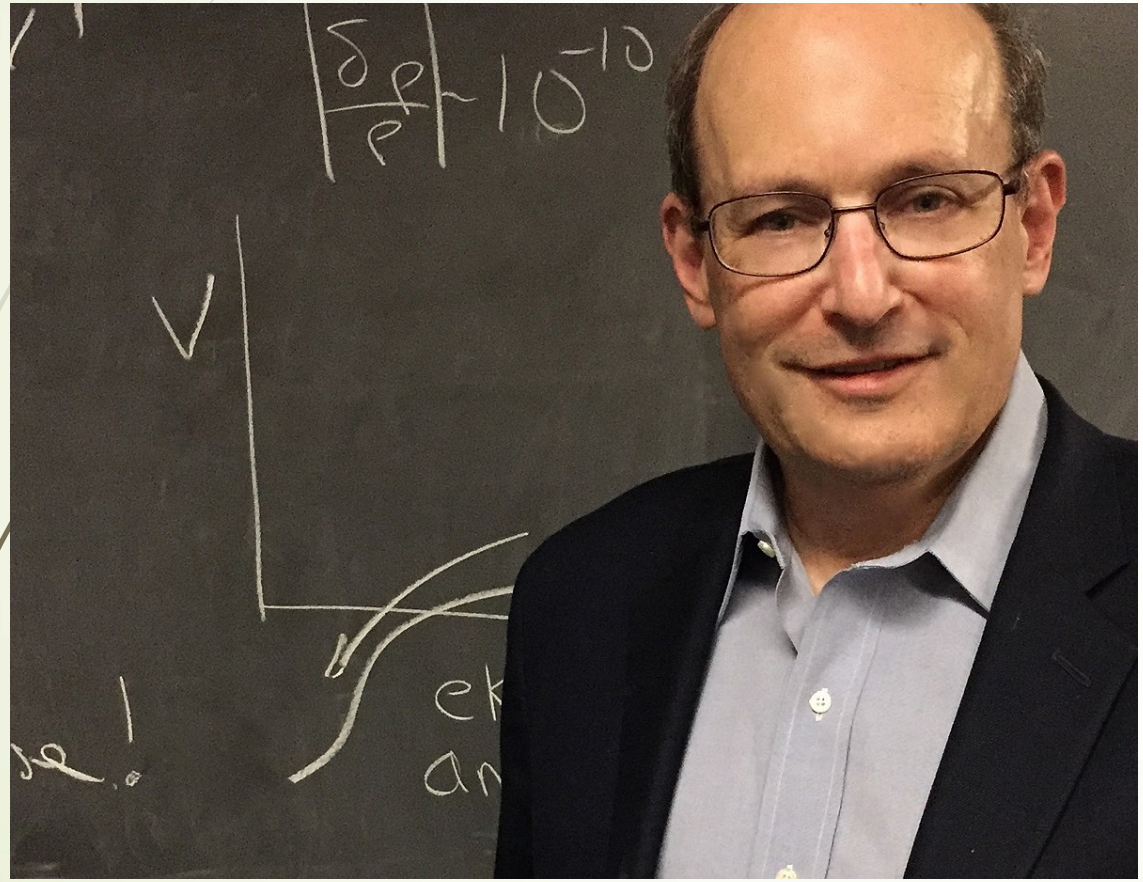
Quasicrystals

Dan Schectman

- Israeli Material Scientist working at NIST (then NBS).
- Discovered Quasicrystals in 1982
- Nobel Prize, Chemistry, 2011



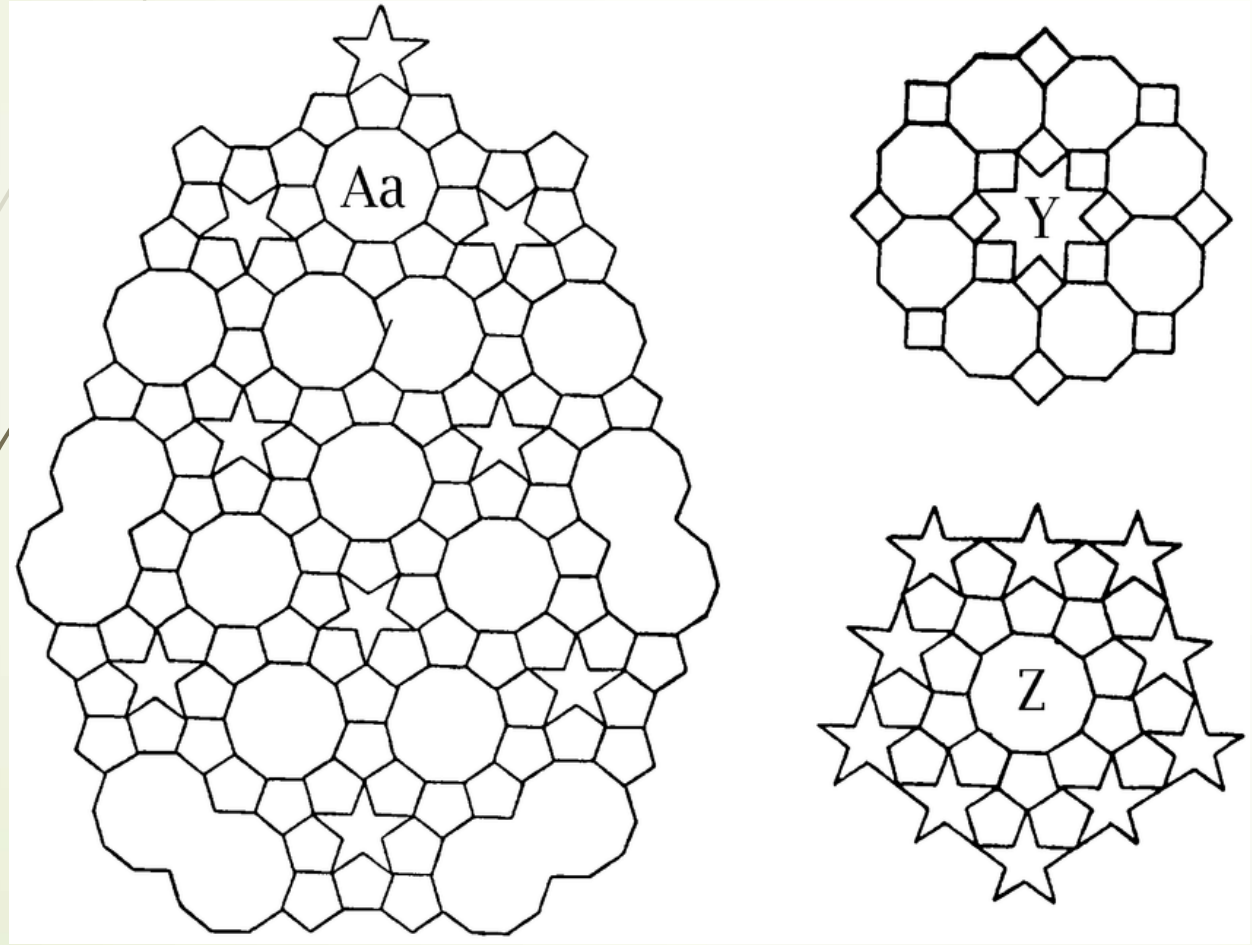
Paul Steinhardt



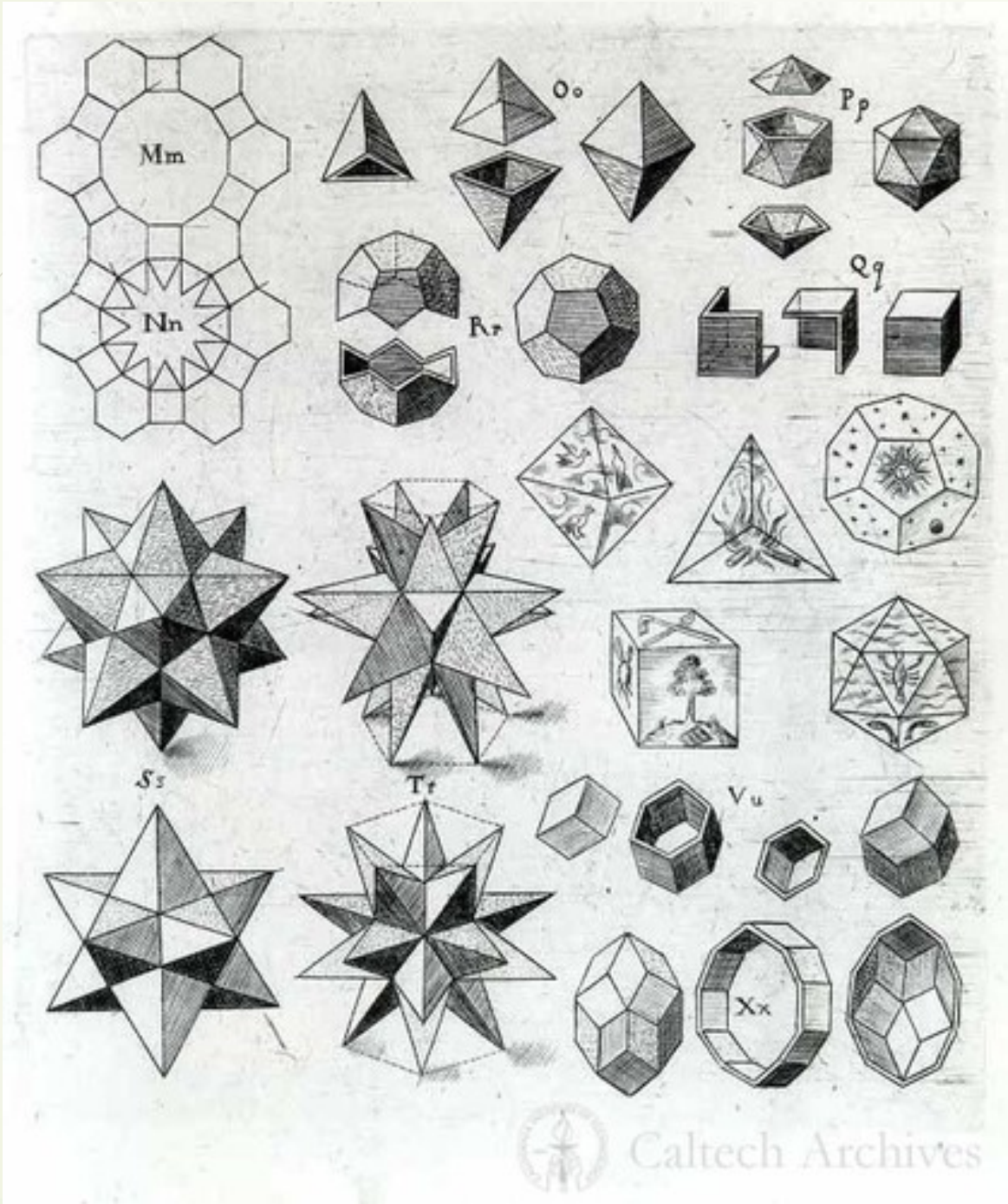
Levine & Steinhardt, *Quasicrystals: A New Class of Ordered Structures,* Physical Review Letters 53 (1984)

Earlier examples

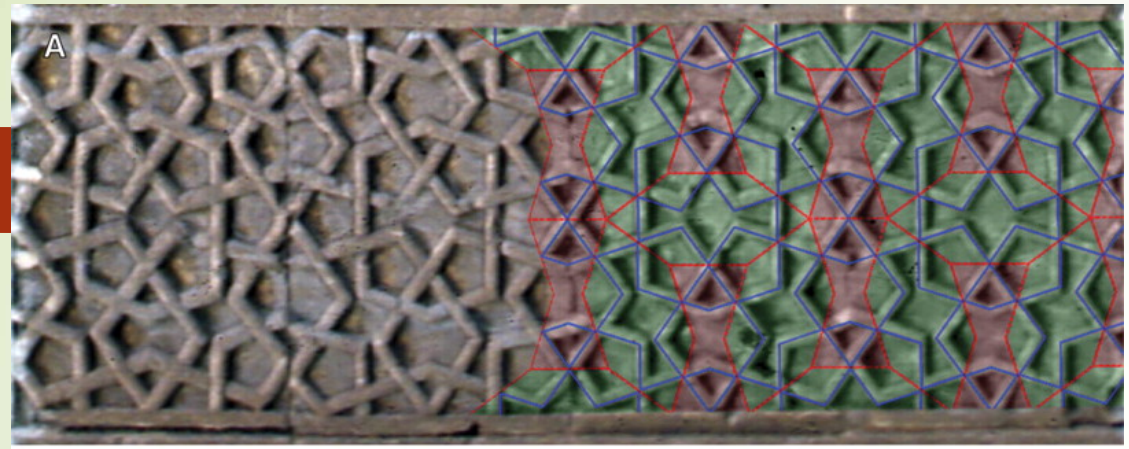
Johannes Kepler
Harmoicus Mundi (1619)

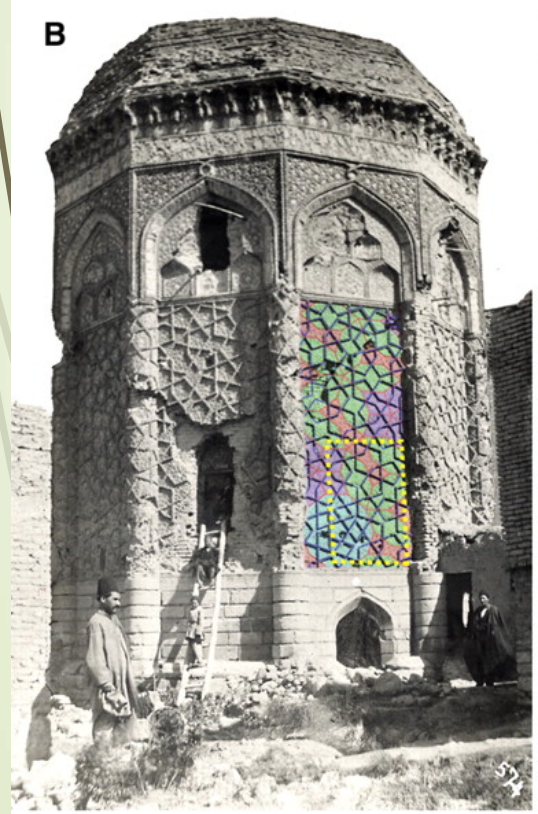


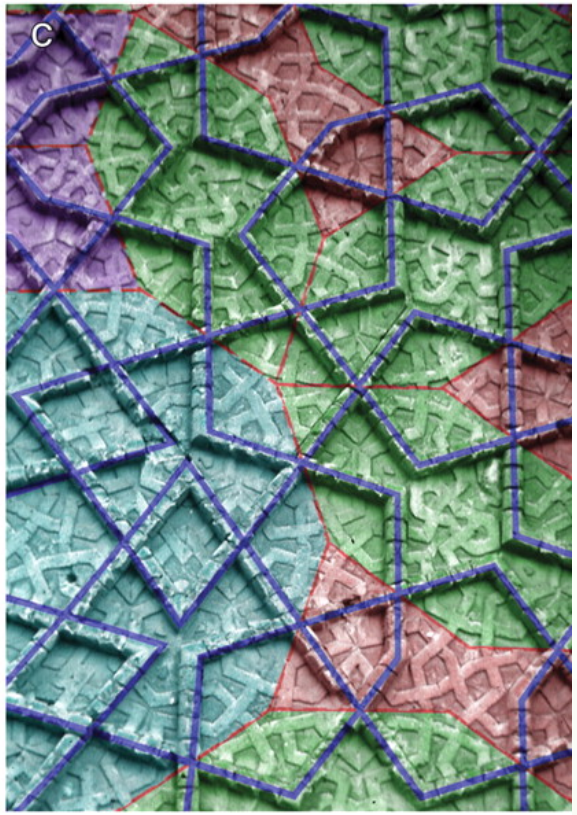
Johannes Kepler
1571-1630

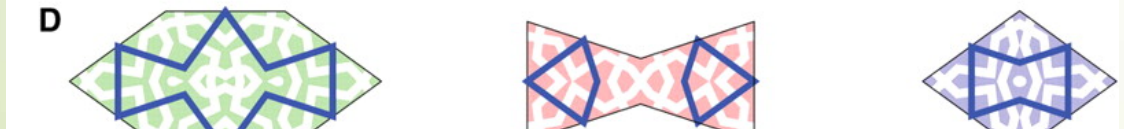


More pictures from Kepler's *Harmoicus Mundi*.









Peter J. Lu, Paul J. Steinhardt, ***Decagonal and Quasi-Crystalline Tilings in Medieval Islamic Architecture***, Science **315**, 2007.

(**A**) Seljuk Mama Hatun Mausoleum in Tercan, Turkey (~1200 C.E.), (**B**) Photograph by A. Sevruguin (~1870s) octagonal Gunbad-i Kabud tomb tower in Maragha, Iran (1197 C.E.), (**C**) Close-up of marked area in (B). (**D**) Hexagon, bowtie, and rhombus girih tiles.

http://science.sciencemag.org/content/315/5815/1106

My paper…

TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 348, Number 11, November 1996

THE DYNAMICAL PROPERTIES OF PENROSE TILINGS

E. ARTHUR ROBINSON, JR.

ABSTRACT. The set of Penrose tilings, when provided with a natural compact metric topology, becomes a strictly ergodic dynamical system under the action of $\mathbf{R}^2$ by translation. We show that this action is an almost 1:1 extension of a minimal $\mathbf{R}^2$ action by rotations on $\mathbf{T}^4$, i.e., it is an $\mathbf{R}^2$ generalization of a *Sturmian dynamical system*. We also show that the inflation mapping is an almost 1:1 extension of a hyperbolic automorphism on $\mathbf{T}^4$. The local topological structure of the set of Penrose tilings is described, and some generalizations are discussed.

1. INTRODUCTION

Penrose tilings are a remarkable family of aperiodic tilings of the plane, first described in the mid 1970s by Roger Penrose [11]. They consist of the edge-to-edge tilings of $\mathbf{R}^2$ by copies of the two marked *Penrose tiles* (Figure 1, on the next page), subject to the *matching rule* that the arrows on adjacent edges must match (Figure 2).

While it is not completely trivial to show that *any* Penrose tilings exist, there are in fact uncountably many incongruent Penrose tilings [7], all of which are aperiodic. It has often been noted that many of the basic properties of Penrose tilings are similar to properties well known in dynamical systems theory. For example, the matching rule resembles a subshift of finite type (the tiles play the role of the symbols and the arrows play the role of the adjacency graph). Similarly, the Penrose