ON UNIFORM CONVERGENCE IN THE WIENER–WINTNER THEOREM

E. ARTHUR ROBINSON, Jr.

1. *Introduction*

Let $T: X \to X$ be a continuous mapping on a compact metric space X. We say a Borel probability measure μ on X is *T-invariant* if $\mu(T^{-1}E) = \mu(E)$ for all Borel $E \subseteq X$. If μ is the unique T-invariant probability measure, then T is said to be *uniquely ergodic*. A complex Borel function g is called a *measurable eigenfunction* for T if there exists $\lambda \in \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, such that

$$g(Tx) = \lambda g(x) \tag{1}$$

for μ-a.e. $x \in X$. In a convenient abuse of the language, we call λ a '*measurable*' *eigenvalue*, and denote the set of all measurable eigenvalues by M_T. Since T is ergodic, any measurable eigenfunction g satisfies $|g(x)| = \text{const.}$ μ-a.e., and g is unique μ-a.e. up to constant multiples. Let $C(X)$ denote the set of all continuous complex-valued functions on X, and suppose that $g(Tx) = \lambda g(x)$ for some $g \in C(X)$ and for all $x \in X$. In this case, we call g a *continuous eigenfunction* and call λ a '*continuous*' *eigenvalue*. We denote the set of all continuous eigenvalues by C_T. Note that $C_T \subseteq M_T$. For $\lambda \in \mathbb{S}^1$, let us define an operator P_λ on $L^2(X,\mu)$ as follows: if $\lambda \in M_T$, then $P_\lambda f$ is the projection of f to the eigenspace corresponding to λ, and if $\lambda \notin M_T$, then $P_\lambda f = 0$. Since T is uniquely ergodic, it follows, for $\lambda \in M_T$, that $P_\lambda f = \alpha_\lambda g$, where g is a measurable eigenfunction corresponding to λ, and

$$\alpha_\lambda = \|g\|^{-1} \int_X f\bar{g}\, d\mu = \|g\|^{-1} \langle f, g\rangle.$$

Our main result is the following.

THEOREM 1.1. *Let T be a uniquely ergodic mapping on a compact metric space X, with unique T-invariant probability measure μ. Then for all $\lambda \notin M_T \backslash C_T$ and $f \in C(X)$, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(T^k x)\, \lambda^{-k} \tag{2}$$

converges uniformly for $x \in X$ to $P_\lambda f \in C(X)$.

Theorem 1.1 is combination of the Wiener–Wintner Theorem **[9]** and the uniformly convergent argodic theorem of Krylov and Bogolioubov **[5]**). In particular,

---

Received 13 April 1992.

1991 *Mathematics Subject Classification* 28D05.

The author was partially supported by NSF DMS-9007831. This paper was written during the author's Fall 1991 sabbatical in the Department of Mathematics at the University of Maryland, College Park. The author wishes to thank them for their hospitality.

the Wiener–Wintner Theorem says that if T is a measure-preserving transformation of a measure space (X, μ) with $\mu(X) < \infty$, and if $f \in L^1(X, \mu)$, then there exists $X_f \subseteq X$ and $\mu(X_f) = \mu(X)$, such that the limit (2) exists for all λ and all $x \in X_f$; though, in fact, Wiener and Wintner considered only the flow case of this theorem. The uniformly convergent ergodic theorem of Krylov and Bogolioubov is the 'if' part of the following theorem.

THEOREM 1.2 [5]. *If T is a uniquely ergodic mapping on a compact metric space X, with unique T-invariant probability measure μ, then for all $f \in C(X)$ the limit*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} F(T^n x) \tag{3}$$

converges uniformly for $x \in X$ to $\int_X f \, d\mu$.

Conversely, T is uniquely ergodic if for every $f \in C(X)$ the limit (3) *converges pointwise on X to a limit which is independent of x.*

We call a mapping T *homogeneous* if it is uniquely ergodic and $M_T = C_T$ (that is, all eigenvalues are continuous). Homogeneous mappings are of interest since it follows from Theorem 1.1 that for such T the limit (2) converges uniformly in x for all $f \in C(X)$ and all $\lambda \in \mathbb{S}^1$. Clearly T is homogeneous if it is (measure theoretically) weakly mixing, and it is well known that any ergodic rotation T on a compact abelian group is homogeneous. A less trivial example is that any substitution dynamical system is homogeneous (cf. Host [4]). More generally, one can show that any invertible ergodic measure-preserving transformation T' on a Lebesgue probability space is (measure theoretically) isomorphic to a homogeneous homeomorphism T on a compact metric space X. This fact may be proven as follows, and I am grateful to B. Weiss for pointing out this argument to me. First, a group rotation is used to provide a homogeneous model for the maximal discrete spectrum factor of T'. Then a homogeneous model for the complementary extension is constructed using the relative Jewitt–Krieger Theorem of Weiss [8]. Contrasting this, Lehrer [6] has shown that any invertible ergodic measure-preserving transformation T' of a Lebesgue probability space is isomorphic to a uniquely ergodic topologically mixing homeomorphism T. This implies that T is topologically weakly mixing, which is equivalent to $M_T = M_T \backslash C_T$. Thus, if T' is not (measure theoretically) weakly mixing then $M_T \backslash C_T$ is nontrivial. In Section 3 we shall explicitly construct a mapping with this latter property.

Recently the author learned that the following result, closely related to Theorem 1.1, was independently obtained by I. Assani [1]. Let $K_T^\perp$ denote the set of all $f \in L^2(X, \mu)$ such that $P_\lambda f = 0$ for all $\lambda \in M_T$.

THEOREM 1.3 [1]. *Let T be a uniquely ergodic mapping on a compact metric space X, with unique T-invariant probability measure μ, and let $f \in C(X) \cap K_T^\perp$. Then the limit* (2) *converges uniformly in $(x, \lambda) \in X \times \mathbb{S}^1$.*

Although Theorems 1.1 and 1.3 overlap, neither result implies the other. For example, suppose that $f \in C(X)$ is such that $P_\nu f \neq 0$ for some $\nu \in M_T$. Then the convergence in (2) cannot be uniform in (x, λ), since the limit function $\tilde{f}(x, \lambda) = P_\lambda f(x)$

cannot be continuous on $X \times \mathbb{S}^1$ (this is because M_T is at most countable). However, Theorem 1.1 still implies that for $\lambda \notin M_T \backslash C_T$, the limit (2) converges uniformly in x for $f \in C(X)$.

2. *Proof of the main theorem*

We begin by considering some special cases of Theorem 1.1. First, let us suppose $\lambda \in C_T$, and let $g \in C(X)$ be an eigenfunction corresponding to λ. Given $f \in C(X)$, define $h = f\bar{g} \in C(X)$. Then by Theorem 1.2 and (1), the limit

$$\int_X f\bar{g}\, d\mu = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)\overline{g(T^n x)} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)\,\lambda^{-n}\overline{g(x)}$$

converges uniformly, proving Theorem 1.1 for this case.

A similar elementary proof can be given if T is weakly mixing (that is, $M_T = \{1\}$) and $\lambda = e^{i\theta}$, where $\theta/2\pi$ is irrational. Let R_θ be the rotation by an angle θ on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ so that $R_\theta t = t+\theta \bmod 2\pi$. Since T is weakly mixing and R_θ is ergodic, $T \times R_\theta$ is ergodic. Moreover, since T and R_θ are disjoint (cf. [3]), and both are uniquely ergodic, it follows that $T \times R_\theta$ is uniquely ergodic. The proof of Theorem 1.1 in this case is completed by applying Proposition 1.2 to the continuous function $h(x,t) = f(x)e^{-it}$ on $X \times \mathbb{T}$. The same argument works for $\lambda = e^{i\theta}$ with $\theta = p/q$ rational, by replacing $\mathbb{T}$ with $\mathbb{Z}/q\mathbb{Z}$ and replacing R_θ with rotation by p on $\mathbb{Z}/q\mathbb{Z}$. Even if T is not weakly mixing, the same line of argument works so long as $\{\lambda^n : n \in \mathbb{Z}\} \cap M_T = \{1\}$ (note that M_T is a group since T is ergodic). However, to get beyond this case we need to use spectral theory.

Suppose that T is a continuous mapping of a compact metric space X and μ is a T-invariant probability measure – we do not necessarily assume that T is uniquely ergodic, or even ergodic. Let us extend the definition of M_T to this case by defining it to be the set of all λ such that for some $g \in L^2(X,\mu)$, equation (1) holds for μ-a.e. x. For $f \in L^2(X,\mu)$ and $n \geq 0$, let

$$\hat{\sigma}_{f,T,\mu}(n) = \int_X f(T^n x)\overline{f(x)}\, d\mu(x). \tag{4}$$

For $n < 0$, define $\hat{\sigma}_{f,T,\mu}(n) = \overline{\hat{\sigma}_{f,T,\mu}(-n)}$. It is well known that the sequence $\hat{\sigma}_{f,T,\mu}(n)$ is positive definite (cf. Queffelec [7]), so that by the Bochner–Herglotz Theorem there exists a finite Borel measure $\sigma_{f,T,\mu}$ on $\mathbb{T}$ such that

$$\hat{\sigma}_{f,T,\mu}(n) = \int_{\mathbb{T}} e^{-in\theta}\, d\sigma_{f,T,\mu}(\theta)$$

for all $n \in \mathbb{Z}$. The measure $\sigma_{f,T,\mu}$ on $\mathbb{T}$, is called the *spectral measure* for f. Let U_T denote the induced isometry on $L^2(X,\mu)$, defined by $U_T f(x) = f(Tx)$. From the Spectral Theorem applied to U_T, it follows that the atoms of the measures $\sigma_{f,T,\mu}$ for $f \in L^2(X,\mu)$, correspond to M_T (cf. Queffelec, [7]). In particular, if $\lambda = e^{-i\theta} \in M_T$ then

$$\sigma_{f,T,\mu}(\{\theta\}) = \|P_\lambda f\|^2, \tag{5}$$

and if $\lambda = e^{-i\theta} \notin M_T$, then

$$\sigma_{f,T,\mu}(\{\theta\}) = 0, \tag{6}$$

where P_λ now denotes projection to the (possibly multi-dimensional) eigenspace corresponding to λ. The next lemma, which is the main ingredient of our proof of Theorem 1.1, is also of some independent interest.

LEMMA 2.1. *Suppose that T is a uniquely ergodic mapping on a compact metric space X, with unique T-invariant probability measure μ. Let $\{x_N\}$ be a sequence in X. Then for all $f \in C(X)$, we have that*

$$\sigma_{f,T,\mu}(\{\theta\})^{1/2} \geqslant \limsup_{N\to\infty} \frac{1}{N}\left|\sum_{n=0}^{N-1} f(T^n x_N)\,\lambda^{-n}\right|. \tag{7}$$

Proof. Choose $N_j \to \infty$ such that

$$\lim_{j\to\infty} \frac{1}{N_j}\left|\sum_{n=0}^{N_j-1} f(T^n x_{N_j})\,\lambda^{-n}\right| = \limsup_{N\to\infty} \frac{1}{N}\left|\sum_{n=0}^{N-1} f(T^n x_N)\,\lambda^{-n}\right|. \tag{8}$$

Let $\lambda = e^{-i\theta}$ and consider the homeomorphism $\tilde{T} = T \times R_\theta$ of $\tilde{X} = X \times \mathbb{T}$ so that

$$\tilde{T}^n(x,t) = (T^n x, t+n\theta).$$

For $N \in \mathbb{N}$, define a Borel measure

$$\eta_N = \frac{1}{N}\sum_{n=0}^{N-1} \delta_{\tilde{T}^n(x_N,0)},$$

where $\delta_{(x,t)}$ denotes unit point mass at $(x,t) \in \tilde{X} = X \times \mathbb{T}$. Then

$$\int_{X\times\mathbb{T}} f(y)\,e^{it}\,d\eta_N(y,t) = \frac{1}{N}\sum_{n=0}^{N-1} f(T^n x_N)\,\lambda^{-n}. \tag{9}$$

Let $h(y,t) = f(y)\,e^{it}$. Let ρ be a weak-* limit point of the set of the measures $\{\eta_{N_j} : j \in \mathbb{N}\}$. Note that ρ exists by the Banach–Alaoglu Theorem. By (9), and the fact that $h \in C(X \times T)$, we have (passing to a subsequence if necessary)

$$\left|\int_{X\times\mathbb{T}} f(y)\,e^{it}\,d\rho(y,t)\right| = \lim_{j\to\infty} \frac{1}{N_j}\left|\sum_{n=0}^{N_j-1} f(T^n x_{N_j})\,\lambda^{-n}\right|. \tag{10}$$

By its construction, the measure ρ is $\tilde{T}$-invariant on $\tilde{X}$. Define the *X-marginal* $\rho|_X$ of ρ to be the Borel measure on X satisfying $\rho|_X(E) = \rho(E \times \mathbb{T})$ for all Borel $E \subseteq X$. Since the σ-algebra of sets of the form $E \times \mathbb{T}$ is $(T \times R_\theta)$-invariant, it follows that $\rho|_X$ is a T-invariant on X. Thus, the unique ergodicity of T implies that

$$\rho|_X = \mu. \tag{11}$$

Using (4) and (11), it follows that

$$\hat{\sigma}_{h,\tilde{T},\rho}(n) = \int_{\tilde{X}} h(\tilde{T}^n(y,t))\,\overline{h(y,t)}\,d\rho(y,t) = \int_{X\times\mathbb{T}} f(T^n y)\overline{f(y)}\,\lambda^{-n}\,d\rho(y,t)$$

$$= \lambda^{-n}\int_X f(T^n y)\overline{f(y)}\,d\mu(y) = \lambda^{-n}\hat{\sigma}_{f,T,\mu}(n).$$

Now for $n \geqslant 0$,

$$(\sigma_{f,T,\mu}\circ R_\theta^{-1})^\wedge(n) = \int_{\mathbb{T}} e^{-int}\,d(\sigma_{f,T,\mu}\circ R_\theta^{-1})(t) = \int_{\mathbb{T}} e^{-in(t+\theta)}\,d\sigma_{f,T,\mu}(t) = \lambda^{-n}\hat{\sigma}_{f,T,\mu}(n),$$

so that, by the Fourier Uniqueness Theorem, it follows that $\sigma_{h,\tilde{T},\rho}\circ R_\theta^{-1} = \sigma_{f,T,\mu}$.

In particular,

$$\sigma_{h,\tilde{T},\rho}(\{0\}) = \sigma_{f,T,\mu}(\{\theta\}). \tag{12}$$

By (5)

$$\sigma_{h,\tilde{T},\rho}(\{0\}) = \|P_1 h\|^2 \geqslant \|P_{\text{const.}} h\|^2, \tag{13}$$

where $P_{\text{const.}}$ denotes projection to the constant functions. The inequality in (13) reflects the fact that $\tilde{T}$ may not be ergodic for ρ. Now

$$\|P_{\text{const.}} h\|_2 = \left| \int_{\tilde{X}} h(y,t)\, d\rho(y,t) \right| = \left| \int_{X\times\mathbb{T}} f(y)\, e^{-it}\, d\rho(y,t) \right| \tag{14}$$

and a combination of (8), (10) and (14), yields the equation

$$\limsup_{N\to\infty} \frac{1}{N} \left| \sum_{n=0}^{N-1} f(T^n x_N)\, \lambda^{-n} \right| = \|P_{\text{const.}} h\|. \tag{15}$$

The proof is completed by combining (12), (13) and (15).

COMMENT. This lemma generalizes a similar result for correlation measures in [7].

Proof of Theorem 1.1. By the discussion following the statement of the theorem, we may assume that $\lambda \notin M_T$. It suffices to show that for $f \in C(X)$,

$$\lim_{N\to\infty} \frac{1}{N} \left\| \sum_{n=0}^{N-1} f(T^n x)\, \lambda^{-n} \right\|_\infty = 0, \tag{16}$$

where $\|\cdot\|_\infty$ denotes the uniform norm on $C(X)$. Now if (16) does not hold, there exists $\varepsilon > 0$, and a sequence $N_j \to \infty$ and a sequence of points $y_j \in X$ such that

$$\frac{1}{N_j} \left| \sum_{n=0}^{N_j-1} f(T^n y_j)\, \lambda^{-n} \right| \geqslant \varepsilon.$$

Thus for any sequence $x_N \in X$ with $x_{N_j} = y_j$, it follows that

$$\limsup_{N\to\infty} \frac{1}{N} \left| \sum_{n=0}^{N-1} f(T^n x_N)\, \lambda^{-n} \right| \geqslant \varepsilon.$$

Using (7), this implies that $\sigma_{f,T,\mu}(\{0\}) > 0$, which by (6) implies $\lambda = e^{-i\theta} \in M_T$.

3. *Essentially discontinuous eigenfunctions and divergence*

The purpose of this section is to show that, in general, the condition $\lambda \notin M_T \backslash C_T$ is necessary for Theorem 1.1. For a pair (X, μ) consisting of a compact metric space X together with a Borel probability measure μ on X, we refer to a complex Borel function g on X as *essentially discontinuous* if g is not equal μ-a.e. to a continuous function. Note that in order to have $\lambda \in M_T \backslash C_T$, an eigenfunction g corresponding to λ must be essentially discontinuous.

Let $\phi: \mathbb{T} \to \mathbb{T}$ be continuous and let $\theta/2\pi$ be irrational. Define the Lebesgue measure preserving homeomorphism T of $\mathbb{T}^2 = \mathbb{T}\times\mathbb{T}$ (called an *Anzai skew product*) by

$$T(s,t) = (R_\theta s, \phi(s) + t). \tag{17}$$

Furstenberg [2] showed that such a transformation T is ergodic if and only if it is uniquely ergodic with respect to Lebesgue measure, and that this is equivalent to the condition that for each $k \in \mathbb{Z}, k \neq 0$, there is no Borel function $\psi: \mathbb{T} \to \mathbb{T}$ such that

$$k\phi(s) = \psi(R_\theta s) - \psi(s) \tag{18}$$

for μ-a.e. s (the arithmetic is understood to be mod 2π). The equation (18) is called a *cohomological equation*, and the function ψ is called a solution to (18). Note that if

ψ is continuous, then (18) holds for all s. Recall that a homeomorphism T of a compact metric space X is called *minimal* if there are no proper closed T-invariant subsets of X. An Anzai skew product (17) is minimal if and only if the cohomological equation (18) has no continuous solutions for any nonzero $k \in \mathbb{Z}$. In particular, uniquely ergodic Anzai skew products are always minimal. A homeomorphism which is both minimal and uniquely ergodic is called *strictly ergodic* (this terminology is now standard, but it conflicts with [**2**]). Furstenberg [**2**] constructed an example of an Anzai skew product T which is minimal but not uniquely ergodic, and showed that there exist points $x = (s, t)$ for which the limit (3) fails to exist for such T. The following proposition can be viewed as the Wiener–Wintner version of Furstenberg's result.

PROPOSITION 3.1. *There exists a strictly ergodic real analytic Anzai skew product T of $\mathbb{T}^2$ which has an essentially discontinuous eigenfunction (that is, $M_T \backslash C_T \neq \phi$). Moreover, for some $\lambda \in M_T \backslash C_T$, and for some $f \in C(\mathbb{T}^2)$, there exists $(s, t) \in \mathbb{T}^2$ such that the limit*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(s, t)) \lambda^{-n}$$

fails to exist.

The proof is based on the next two lemmas of Furstenberg [**2**].

LEMMA 3.2 [**2**]. *There exists an irrational number $\theta/2\pi$ and real analytic function $\gamma\colon \mathbb{T} \to \mathbb{T}$ such that for $k = 1$, the cohomological equation* (18) *has an essentially discontinuous solution ψ.*

Note that since R_θ is ergodic, the solutions to (18) are unique a.e. up to an additive constant.

LEMMA 3.3 [**2**]. *Suppose that T is an Anzai skew product* (17) *with $\theta/2\pi$ irrational. If for $k = 1$ there exists an essentially discontinuous solution ψ to* (18), *then there exists $(s, t) \in \mathbb{T}^2$ such that the limit*

$$\lim_{N \to \infty} \sum_{n=0}^{N-1} f(T^n(s, t))$$

fails to exist for the continuous function $f(s, t) = e^{i(\phi(s)+t)}$.

Proof of Proposition 3.1. Using Lemma 3.2, choose θ and $\gamma\colon \mathbb{T} \to \mathbb{T}$ so that the cohomological equation $\gamma(s) = \psi(R_\theta s) - \psi(s)$ has an essentially discontinuous solution ψ. Let $\lambda = e^{i\nu}$ be such that $\lambda^n \notin M_{R_\theta} = \{e^{ik\theta}\colon k \in \mathbb{Z}\}$ for all $n \in \mathbb{Z}$. Define $\phi(s) = \nu + \gamma(s)$ and note that

$$\phi(s) - \nu = \psi(R_\theta s) - \psi(s), \tag{19}$$

for Lebesgue a.e. s.

Let T be defined by (17). First we show that T is uniquely ergodic. As noted above, since T is an Anzai skew product, it suffices to show that T is ergodic. This is accomplished by showing that T is isomorphic to $R_\theta \times R_\nu$, which is ergodic by the choice of ν. In particular, if $S(s, t) = (s, t - \psi(s))$, then by (19),

$$S \circ T(s, t) = (R_\theta s, -\psi(R_\theta s) + \phi(s) + t) = (R_\theta s, -\psi(s) + \nu + t) = (R_\theta \times R_\nu) \circ S(s, t).$$

Note that the isomorphism S is essentially discontinuous.

Next, define $g(s,t) = e^{i(\psi(s)+t)}$, and observe that g is essentially discontinuous. By (19), it follows that

$$g(T(s,t)) = e^{i(\psi(R_\theta s)-\phi(s)-t)} = e^{i(\psi(s)-\nu-t)} = \lambda g(s,t).$$

Thus $\lambda \in M_T \setminus C_T$.

To complete the proof, let us define $f(s,t) = e^{i(\gamma(s)+t)}$, and note that f is real analytic, so that in particular, it is continuous. Define a new Anzai skew product $T_1(s,t) = (R_\theta s, \gamma(s)+t)$. Then

$$T_1^n(s,t) = (R_\theta s, \gamma(R_\theta^{n-1}s)+\ldots+\gamma(R_\theta s)+\gamma(s)+t),$$

and by the definition of ϕ,

$$T^n(s,t) = (R_\theta s, \gamma(R_\theta^{n-1}s)+\ldots+\gamma(R_\theta s)+\gamma(s)+t+n\nu).$$

This implies that

$$f(T^n(s,t)) = e^{i(\gamma(R_\theta^n s)+\gamma(R_\theta^{n-1}s)+\ldots+\gamma(s)+t+n\nu)} = \lambda^n e^{i(\gamma(R_\theta^n s)+\gamma(R_\theta^{n-1}s)+\ldots+\gamma(s)+t)} = \lambda^n f(T_1^n(s,t)) \tag{20}$$

for all $n \geq 0$. It now follows from Lemma 3.3 there exists $(s,t) \in \mathbb{T}^2$ such that the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} f(T^n(s,t))\,\lambda^{-n} = \lim_{N\to\infty} \sum_{n=0}^{N-1} f(T_1^n(s,t))$$

does not exist.

REMARK. We note that equation (20) still holds if γ is replaced with an arbitrary function $\omega\colon \mathbb{T} \to \mathbb{T}$ in the definition of f.

4. *The case of* $\mathbb{Z}^d$ *and* $\mathbb{R}^d$

In this section we show how to generalize Theorem 1.1 to the cases of uniquely ergodic actions of $\mathbb{Z}^d$ and $\mathbb{R}^d$. Although the proofs in these two cases are essentially identical to the proof of Theorem 1.1, the statements have a different appearance. This difference is a bit more than superficial, since in the homeomorphism case (that is, the $\mathbb{Z}^d$ case with $d = 1$) we obtain a slightly different formulation (Corollary 4.2) of Theorem 1.1.

Suppose T is a continuous uniquely ergodic action of $\mathbb{Z}^d$ for $d \geq 1$, on a compact metric space X, with unique T-invariant measure μ. We denote the action of $\mathbf{n} \in \mathbb{Z}^n$ on $x \in X$ by $T^{\mathbf{n}}x$. Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. We say $\mathbf{w} \in \mathbb{T}^d$ is an *eigenvalue* for T if there exists a complex Borel function g on X such that

$$g(T^{\mathbf{n}}x) = e^{2\pi i\langle \mathbf{n}, \mathbf{w}\rangle} g(x), \tag{21}$$

holds for μ-a.e. $x \in X$ (note that the inner product $\langle \mathbf{n}, \mathbf{w}\rangle$ in (21) depends only on $\mathbf{w} \in \mathbb{R}^d \bmod \mathbb{Z}^d$). As in the homeomorphism case, we say $\mathbf{w}$ is a *continuous eigenvalue*, denoted $\mathbf{w} \in C_T$, if (21) has a continuous solution g, and we say that $\mathbf{w}$ is a *measurable eigenvalue*, denoted $\mathbf{w} \in M_T$, if (21) has only essentially discontinuous solutions. If

$\mathbf{w} \in M_T$, then $P_{\mathbf{w}}$ will denote the projection to the eigenspace corresponding to $\mathbf{w}$, and otherwise $P_{\mathbf{w}} f = 0$. For $N \geqslant 1$, define $Q_N \subseteq \mathbb{Z}^d$ by $Q_N = \{(t_1, \ldots, t_n) \colon |t_i| \leqslant N \text{ for all } i\}$.

THEOREM 4.1. *Let T be a continuous uniquely ergodic $\mathbb{Z}^d$ action on a compact metric space X with unique T-invariant probability measure μ. Then for all $\mathbf{w} \notin M_T \backslash C_T$ and all $f \in C(X)$, the limit*

$$\lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{\mathbf{n}\in Q_N} f(T^{\mathbf{n}} x)\, e^{-2\pi i\langle \mathbf{t}, \mathbf{w}\rangle}$$

converges uniformly for $x \in X$ to $P_{\mathbf{w}} f$.

COROLLARY 4.2. *Let T be a uniquely ergodic homeomorphism of a compact metric space X with unique T-invariant probability measure μ. Then for all $\lambda \notin M_T \backslash C_T$, the limit*

$$\lim_{N\to\infty} \frac{1}{(2N+1)} \sum_{k=-N}^{N} f(T^k x)\, \lambda^{-k}$$

converges uniformly for $x \in X$ to $P_\lambda f$.

Now suppose that F is a continuous uniquely ergodic $\mathbb{R}^n$ action on X, with unique F-invariant measure μ. We denote the action of $\mathbf{t} \in \mathbb{R}^d$ on $x \in X$ by $F^{\mathbf{t}} x$. In this case we write the eigenvalue equation

$$g(F^{\mathbf{t}} x) = e^{2\pi i\langle \mathbf{t}, \mathbf{w}\rangle} g(x),$$

where now $\mathbf{w} \in \mathbb{R}^d$. We define M_T, C_T and $P_{\mathbf{w}}$ in analogy to the $\mathbb{Z}^d$ case. For $R > 0$, we define $Q_R \subseteq \mathbb{R}^d$ by $Q_R = \{(t_1, \ldots, t_n) \colon |t_i| \leqslant R \text{ for all } i\}$.

THEOREM 4.3. *Let F be a continuous uniquely ergodic $\mathbb{R}^d$ action on a compact metric space X with unique invariant probability measure μ. Then for all $\mathbf{w} \notin M_T \backslash C_T$ and all $f \in C(X)$, the limit*

$$\lim_{R\to\infty} \frac{1}{(2R)^d} \int_{Q_R} f(F^{\mathbf{t}} x)\, e^{-2\pi i\langle \mathbf{t}, \mathbf{w}\rangle}\, dt$$

converges uniformly for $x \in X$ to $P_{\mathbf{w}} f$.

For $f \in L^2(X, \mu)$ let $\sigma_{f,T,\mu}$ and $\sigma_{f,F,\mu}$ be the finite Borel measures on $\mathbb{T}^d$ and $\mathbb{R}^d$ respectively, satisfying

$$\int_{\mathbb{T}^d} e^{-2\pi i\langle \mathbf{n}, \mathbf{w}\rangle}\, d\sigma_{f,T,\mu}(\mathbf{w}) = \int_X f(T^{\mathbf{n}} x)\overline{f(x)}\, d\mu(x),$$

for all $\mathbf{n} \in \mathbb{Z}^d$, and

$$\int_{\mathbb{R}^d} e^{-2\pi i\langle \mathbf{t}, \mathbf{w}\rangle}\, d\sigma_{f,F,\mu}(\mathbf{w}) = \int_X f(F^{\mathbf{t}} x)\overline{f(x)}\, d\mu(x),$$

for all $\mathbf{t} \in \mathbb{R}^d$. The following lemma plays the same role in the proofs of Theorems 4.1 and 4.3 that Lemma 2.1 plays in the proof of Theorem 1.1.

LEMMA 4.4. *If T is a uniquely ergodic $\mathbb{Z}^n$ action on a compact metric space X with unique T-invariant probability measure μ, then for any sequence $x_N \in X$, and all $f \in C(X)$,*

$$\sigma_{f,T,\mu}(\{\mathbf{w}\})^{1/2} \geqslant \limsup_{N\to\infty} \frac{1}{(2N+1)^d}\left|\sum_{\mathbf{n}\in Q_N} f(T^{\mathbf{n}}x_N)\, e^{-2\pi i\langle \mathbf{n},\mathbf{w}\rangle}\right|.$$

Similarly, if F is a uniquely ergodic $\mathbb{R}^n$ action on X with unique F-invariant probability measure μ, then for any function $R \mapsto x_R : \{r\in\mathbb{R} : r \geqslant 0\} \to X$, and all $f\in C(X)$,

$$\sigma_{f,F,\mu}(\{\mathbf{w}\})^{1/2} \geqslant \limsup_{R\to\infty} \frac{1}{(2R)^d}\left|\int_{Q_R} f(F^{\mathbf{t}}x_R)\, e^{-2\pi i\langle \mathbf{t},\mathbf{w}\rangle}\, d\mathbf{t}\right|.$$

Acknowledgement. B. Weiss has informed the author that he has obtained results similar to some of those in this paper, but has not published them.

References

1. I. ASSANI, 'Uniform Wiener–Wintner theorems for weakly mixing dynamical systems', preprint, 1992.
2. H. FURSTENBERG, 'Strict ergodicity and transformations of the torus', *Amer. J. Math.* 83 (1961) 573–601.
3. H. FURSTENBERG, 'Disjointness in ergodic theory and a problem in Diophantine analysis', *Math. Sys. Th.* 1 (1967) 1–79.
4. B. HOST, 'Valeurs propres des systèmes dynamiques définis par des substitutions de longueur variable', *Ergod. Th. and Dynam. Sys.* 6 (1986) 529–540.
5. N. KRYLOV and N. BOGOLIOUBOV, 'La théorie générale de la mesure dans son application à l'étude des systèmes dynamiques de la mécanique non-linéaire', *Ann. of Math.* 38 (1937) 65–113.
6. E. LEHRER, 'Topological mixing and uniquely ergodic systems', *Israel J. Math.* 57 (1987) 239–255.
7. M. QUEFFELEC, *Substitution dynamical systems – spectral analysis*, Lecture Notes in Mathmematics 1294 (Springer, 1987).
8. B. WEISS, 'Strictly ergodic models for dynamical systems', *Bull. Amer. Math. Soc.* 13 (1985) 143–146.
9. N. WIENER and A. WINTNER, 'On the ergodic dynamics of almost periodic systems', *Amer. J. Math.* 63 (1941) 794–824.

Department of Mathematics
The George Washington University
Washington
District of Columbia 20052
USA