ABSTRACT

Title of Dissertation: Ergodic Measure Preserving Transformations with Finite Spectral Multiplicities

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Let $M \geq 2$ be an integer and p a prime such that m divides $p - 1$. There is an isomorphism ϕ of the finite group $\mathbb{Z}/m\mathbb{Z}$ to a subgroup of the multiplicative group $GF(p)^{\times}$ of the finite field of order p. Fo an ergodic measure-preserving transformation $T_0 : X \to X$ and a measurable function $\gamma : X \to \mathbb{Z}/m\mathbb{Z}$ we consider the extension $T : X \times \mathbb{Z}/m\mathbb{Z} \times GF(p) \to X \times \mathbb{Z}/m\mathbb{Z} \times GF(p)$ defined by $T(x,y,z) = T_0x, \gamma(x) + y, \phi(y) + z)$. We show that in general the maximal spectral multiplicity of T is greater than or equal to m and that for a generic set of pairs (T_0, γ) the value m is achieved.

The estimate from below is obtained by a decomposition of $\mathbb{L}_2(X \times \mathbb{Z}/m\mathbb{Z} \times GF(p))$ into $m+1$ invariant subspaces and construction of an operator S cyclicly permuting m of these subspaces such that U_T commutes with S. The estimate from above follows from the method of approximation. We show that if T_0 admist a cyclic approximation by periodic transformations speed $o(1/n)$, then for a generic set of γ, T admits an m-cyclic approximation by periodic transformation speed $o(1/n)$. Using a generalization of this construction, we obtain examples with an arbitrary finite number of values

in the essential range of the multiplicity function. The construction may be made so that the spectrum is continuous. In each case T may be realized as an interval exchange transformation.

A mixing transformation is constructed with nonsimple spectrum of finite multiplicity. This example is based on the Ornstein rank 1 mixing transformation and is constructed by cutting and stacking.

ERGODIC MEASURE PRESERVING TRANSFORMATIONS WITH FINITE SPECTRAL MULTIPLICITIES

by

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Introduction

For an invertible measure-preserving transfor[illegible]ion $T:(X,\mu)\to(X,\mu)$ of a Lebesgue space (X,μ) we con[illegible]der the induced unitary operator $U_T : L_2(X,\mu)\to L_2(X,\mu)$ g[illegible]n by $(U_T f)(x) = f(Tx)$. The simplest concept of spectral [illegible]plicity (cf. [3]) is the maximal spectral multiplicity which we now define. Let $U : H \to H$ be a unitary operator on a Hilbert space H. The maximal spectral multiplicity

$$m(u) = \inf\{m \in \mathbb{Z}_+ \cup \{\infty\} : \exists f_1 \ldots f_m \in H: \text{ linear combinations of } U^i f_j,\ i \in \mathbb{Z},\ j = 1,\ldots,m \text{ are dense in } H\}.$$

When the multiplicity is 1 we say that the spectrum is simple. If for U_T, the eigenvalue 1 is simple and is the only eigenvalue we say T has continuous spectrum.

Questions of spectral multiplicity in ergodic theory have a long and interesting history which we will briefly survey. The problem was first considered by Von Neumann [20] who observed that the spectrum is simple for ergodic flows with purely discrete spectrum and asked what is possible in the continuous spectrum case.

Von Neumann's results on flows were extended to the case of transformations with discrete spectrum by Halmos and Von Neumann in [9] where it was shown that such transformations are equivalent to rotations on compact abelian groups. As in the case of flows, these examples have simple spectrum. In contrast to the discrete spectrum case is another natural class of algebraic examples, the ergodic automorphisms of compact abelian groups. These were shown by Halmos [10] to have continuous spectrum with infinite multiplicity, namely the type of spectrum commonly called countable Lebesgue spectrum. Countable Lebesgue spectrum also occurs in a class of transformations which arise naturally in probability theory, the Bernoulli and Markov shifts. (Cf. e.g. [27] where there is also an interesting general discussion of the spectral multiplicity problem.) Rohlin [26] described the connection between the spectrum and entropy. Any transformation with positive entropy has a countable Lebesgue component in the spectrum.

For many years, no example was found having continuous spectrum of finite multiplicity. The first such example was due to Girsanov, [8] who constructed a transformation with simple continuous spectrum using the theory of Gaussian processes. Soon afterward Veršik proved in [33] and [34] that for any ergodic transformation generated by a Gaussian process, the spectral multiplicity is either 1 or $+\infty$.

Examples with simple continuous spectrum of a geometric rather than probabilistic origin were first constructed by Oseledec [23], and Katok and Stepin [17]. These examples were found among interval exchange transformations and smooth flows on the two-dimensional torus. They are based on the idea of approximation by periodic transformations (cf. [18]). Using these ideas Yuzvinskii ([35] cf. also

[18]) proved that simple continuous spectrum is typical in the weak topology on the space of all transformations.

In [23], Oseledec also proved that for an interval exchange transformation the maximal spectral multiplicity is bounded above by $p - 1$ where p is the number of intervals exchanged. Moreover he constructed the first example of a transformation with continuous spectrum and finite multiplicity greater than 1. Since the example is an exchange of 30 intervals, the maximal spectral multiplicity m satisfies $2 \leq m < 30$.

Variations on the method of approximation by periodic transformations were subsequently introduced and applied to spectral problems by Chacon [4] [5], Schwartzbauer [31], Stepin [32] and Riley [24]. A generalization is also given in [16]. The existence of a simple approximation with multiplicity M is given in [5] as a sufficient condition for an upper bound M on the spectral multiplicity, although no example is given where this bound is achieved for $M > 1$. Baxter [1] noted that the transformations where $M = 1$ are exactly those which may be constructed by cutting and stacking intervals in a single stack, the rank 1 transformations (cf. [14]). Thus rank 1 implies simple spectrum. Del Junco [14] showed that the converse is false. Recently, Thouvenot (unpublished) constructed a transformation with simple spectrum which is not loosely Bernoulli and consequently is not of finite rank. A sufficiently fast m-cycle approximation by periodic transformations (cf. §3) implies a simple approximation with multiplicity m and thus a rank of at most m.

Recently Katok (unpublished) showed that in the Oseledec example $m = 2$ is attainable and is typical within the context of the construction. The upper bound is obtained by using the theory of approximation by periodic transformations. The construction described in Chapter I is a generalization of the Oseledec construction and Katok's upper bound. We show that for every $m > 1$ there exists a measure-preserving transformation T with maximal spectral multiplicity m. We show this to be typical for our construction. Furthermore we show that it is possible to realize such a transformation as an interval exchange.

We can give an equivalent definition of spectral multiplicity in terms of the spectral theorem for unitary operators. The operator U_T is described up to unitary equivalence by a sequence of spectral types [3]

$$\rho_1 < \rho_2 < \rho_3 < \ldots$$

where each ρ_j is an equivalence class of measures on the circle $\mathbb{T}$, and $<$ denotes absolute continuity applied to these classes. The maximal spectral type ρ_{max} is the maximal element of the spectral sequence. The multiplicity function m is defined to be the essential number of ρ_j dominated by ρ_{max}. It is ρ_{max} measurable. Let M_T denote the essential range of the multiplicity function on the subspace of L_2 orthogonal to the constants. $\mathcal{M}_T \subseteq \mathbb{N} \cup \{+\infty\}$, where $\mathbb{N}$ denotes the set of natural numbers. $M_T = \sup \mathcal{M}_T$.

The examples in Chapter I all have $\mathcal{M}_T = \{1,m\}$ for some natural number m. In Chapter III, transformations T are constructed such that

$$\mathcal{M}_T = \{1,p-1,p(p-1),\dots,p^{r-1}(p-1)\}$$

where p is an odd prime and $r > 1$. Transformation T such that $0 < \operatorname{card}(\mathcal{M}_T) < +\infty$ will be said to have highly nonsimple spectrum of finite multiplicity. Katok [16] has recently studied $\mathcal{M}_T(m)$, where $T^{(m)}$ is the m'th cartesian power of a generic transformation T (in the weak topology). He has shown that

$$\mathcal{M}_T(m) \subseteq \{m\} \cup m(m-1)\mathbb{N} \cup \{+\infty\},$$

$$M_T(m) \le m(m-1)\left\lceil \frac{m^m}{m(m-1)} \right\rceil,$$

and that

$$m_T(m) = m$$

where $m_T = \min \mathcal{M}_T$. We note that for all of the examples T constructed here, $1 \in \mathcal{M}_T$ so $m_T = 1$. Katok has conjectured that generically

$$\mathcal{M}_T(m) = \{m,m(m+1),\dots,m!\}$$

so that in the case $m = 2$,

$$\mathcal{M}_T(2) = \{2\}.$$

Transformations T such that $M_T = \{m\}$ are said to have homogeneous spectrum of multiplicity m. There are no known examples of such transformations for $m \neq 1, +\infty$ on Lebesgue probability spaces, although Riley [24] has found an example on an infinite measure space where $M_T = \{2\}$.

In Chapter II we construct a transformation T, which has the mixing property, such that $2 \leq M_T \leq 6$. In order to do this we begin with the Ornstein [22] rank 1 mixing transformation T_0, and repeat a construction similar to the construction in Chapter I. Since the method of approximation by periodic transformations excludes mixing, however, it is replaced by the cutting and stacking method. In this case we obtain $1 \in M_T$; if $k \in M_T$ and $k \neq 1$ then $2|k$; and $M_T \leq 6$. Thus there are five possibilities for M_T.

In the text we will assume all measure-preserving transformations are invertible. We will often neglect to write the measures if no confusion will arise. We will denote by $\mathbb{Z}/m\mathbb{Z}$, the integers mod m, and $GF(p)$ the finite field of order p. Where appropriate we will assume arithmetic is carried out mod m or mod p without so stating. The letter χ will be used both for group characters and characteristic functions.

CHAPTER 1. ARBITRARY FINITE SPECTRAL MULTIPLICITY

In this chapter, examples are constructed showing that every positive integer m can be the maximal spectral multiplicity of an ergodic measure-preserving transformation. In addition, we show that this transformation can always be realized as an interval exchange transformation. The results of this chapter appear in [25], which we essentially reproduce.

§1. Basic Construction

For a fixed prime p let us consider the finite field $GF(p)^{\times}$ and recall that the nonzero elements $GF(p)^{\times}$ form a group under multiplication which is isomorphic to $\mathbb{Z}/(p-1)$. We denote this isomorphism by $\phi_0 : \mathbb{Z}/(p-1) \to GF(p)^{\times} \subseteq GF(p)$.

Lemma 1.1. For any $m > 0$ there exists a finite field $GF(p)$ such that $\mathbb{Z}/m$ is isomorphic to a subgroup of $GF(p)^{\times}$. We denote this isomorphism by ϕ, and the least p satisfying this lemma by $p(m)$.

Proof. By the Dirichlet theorem on primes in an arithmetic progression, the sequence $mk + 1$ contains a prime p. This implies that $m | p - 1$. Consider the subgroup of $\mathbb{Z}/(p-1)$ generated by $m' = (p-1)/m$. This subgroup is isomorphic to $\mathbb{Z}/m$ via $\psi : \mathbb{Z}/m \to \mathbb{Z}/(p-1)$, where $\psi(y) = m'y$. It follows that $\phi = \phi_0 \circ \psi$ is a one to one homomorphism from $\mathbb{Z}/m$ into $GF(p)^{\times}$. □

Let (X, μ_0) be a Lebesgue space, $T_0 : X \to X$ an invertible ergodic measure-preserving transformation and $\gamma : X \to \mathbb{Z}/m$ a measurable function. We say that

$$(1.1) \qquad \begin{cases} T_1 : X \times \mathbb{Z}/m \to X \times \mathbb{Z}/m \\ T_1(x,y) = (T_0 x, \gamma(x)+y) \end{cases}$$

is the $\mathbb{Z}/m$ extension of T_0 corresponding to γ. The natural product measure μ_1 on $X \times \mathbb{Z}/m$, which is defined as the normalized product of μ_0 and the uniform (Haar) measure on $\mathbb{Z}/m$, is clearly preserved by T_1.

All of our examples will be of the following type. Given T_1 as in (1.1) and a prime p which satisfies Lemma 1.1 for m, we construct the $GF(p)$ extension of T_1 corresponding to the homomorphism ϕ.

$$(1.2) \qquad \begin{cases} T : X \times \mathbb{Z}/m \times GF(p) \to X \times \mathbb{Z}/m \times GF(p) \\ T(x,y,z) = (T_0 x, \gamma(x)+y, \phi(y)+z). \end{cases}$$

Note that the T in (1.2) is specified by a choice of T_0 and γ. The natural topology for the set of parts (T_0,γ) is the product of the weak topology for measure-preserving transformations and the L_1-topology for functions. (Cf. §6 for details).

In terms of this topology we can state the first main result of this chapter which is proved in §§2-6:

Theorem 1.1. For a generic set of (T_0,γ), T is ergodic and has continuous spectrum with maximal spectral multiplicity $M_T = m$. Moreover, $\mathcal{M}_T = \{1,m\}$.

The second main result in this chapter, which deals with the realization of a given spectral multiplicity by an interval exchange, is Theorem 7.1 which is formulated and proved in §7.

§2. The Estimate of the Multiplicity from Below

Associated with a finite abelian group extension is a natural orthogonal decomposition of L_2 into U_T invariant subspaces corresponding to the characters of the group. The additive characters of $GF(p)$ are given by $\chi_w(z) = \exp 2\pi izw/p$ where $w \in GF(p)$, so that if T is given by (1.2), we obtain the invariant decomposition

$$L_2(X) \times \mathbb{Z}/m \times GF(p)) = \bigoplus_{w \in GF(p)} H_w$$

where

$$H_w = \{\chi_w(z)f(x,y) : f \in L_2(X \times \mathbb{Z}/m\mathbb{Z})\}.$$

The decomposition is obtained by a discrete Fourier transform with respect to the third variable. Let us define the permutation $\sigma : GF(p) \to GF(p)$ by $\sigma(w) = \phi(1)w$ and for $w \neq 0$, the operator

$$S_w : H_w \to H_{\sigma(w)}$$

by

$$S_w(\chi_w(z)f(x,y)) = \chi_{\sigma(w)}(z)f(x,y+1).$$

Lemma 2.1.

$$U_T\big|_{H_{\sigma(w)}} \circ S_w = S_w \circ U_T\big|_{H_w}$$

Proof

$$U_T\big|_{H_{\sigma(w)}} \circ S_w(\chi_w(z)f(x,y))$$

$$= \chi_{\sigma(w)}(z)\chi_{\sigma(w)}(\phi(y))f(T_0x,\gamma(x)+y+1)$$

$$S_w \circ U_T\Big|_{H_w}(\chi_w(z)f(x,y))$$

$$= \chi_{\sigma(w)}(z)\chi_w(\phi(y+1))f(T_0x,\gamma(x)+y+1).$$

Equality follows from the observation that $\chi_{\sigma(w)}(\phi(y)) = \exp 2\pi i\phi(1)w\phi(y)/p = \chi_w(\phi(y+1))$. □

The following lemma characterizes the action of the permutation σ.

Lemma 2.2. σ has a fixed point 0 and $m' = (p-1)/m$ cycles of length m. Furthermore, each cycle is represented by exactly one element of the set $\phi_0(\{0,1,\ldots,m'-1\})$.

Proof. Clearly $\sigma(0) = 0$. If $z \neq 0$ then $z = \phi_0(y)$ for some $y \in \mathbb{Z}/(p-1)\mathbb{Z}$ and $\sigma(z) = \phi(1)z = \phi(1)\phi_0(y) = \phi_0(\psi(1))\phi_0(y) = \phi_0(\psi(1)+y) = \phi_0(m'+y)$. It follows that σ on $GF(p)^\times$ is conjugate via ϕ_0 to the permutation $y \to m' + y$ on $\mathbb{Z}/(p-1)$. This permutation has m' cycles of length m represented by $\{0,1,\ldots,m'-1\}$. □

The previous lemma shows how the operators S_w permute the subspaces H_w. For each $j = 0,\ldots,m-1$ we define

$$(2.1) \qquad H^j = H_{\sigma^j(\phi_0(0))} \oplus \cdots \oplus H_{\sigma^j(\phi_0(m'-1))}$$

and also

$$H^* = H_0.$$

Note that by Lemma 2.2 for each $w \neq 0$ $H_w \subseteq H^j$ for some j. We define the linear operator

$$S^j : H^j \to H^{j+1} \tag{2.2}$$

so that if $H_w \subseteq H^j$ then $S^j|_{H_w} = S_w$. Since $S^j|_{H_w} H_w = H_{\sigma(w)} \subseteq H^{j+1}$, the operator S^j is well defined.

Lemma 2.3.

(i) $S^j \circ U_T|_{H^j} = U_T|_{H^{j+1}} \circ S^j$.

(ii) The maximal spectral multiplicity for T in (1.1) is at least m.

Proof. (i) is trivial by Lemmas 2.1 and 2.2. For (ii) we note that U_T is isomorphic on each H^j. Thus each H^j must have the same spectrum $j = 0,\dots,m-1$. □

§3. Approximation by Periodic Transformations

Let (X,μ) be a Lebesgue space and let $\{\xi_n\}$ be a sequence of partitions of X into sets $C_{n,j}$, $j = 1,\ldots,q_n$. We say that $\xi_n \to \xi$ if for any measurable $A \subseteq X$ there exists $\xi_n' \subseteq \xi_n$ and $A_n = \bigcup_{C_n \in \xi_n'} C_n$ such that $\mu(A_n \Delta A) \to 0$ as $n \to \infty$. Let $f(n) \to 0$ as $n \to \infty$. We say that the measure-preserving transformation $T : X \to X$ admits an m-cyclic approximation by periodic transformations with speed $f(n)$, (a.p.t. speed $f(n)$), if there exists $\xi_n \to \varepsilon$ and for every n a measure-preserving permutation σ_n of the collection $\{C_{n,j}\}$ $j = 1,\ldots,q_n$ such that:

(i) $\sum_{j=1}^{q_n} \mu(TC_{n,j} \Delta \sigma_n C_{n,j}) < f(p_n)$

(ii) σ_n has m cycles

where p_n is the length of the longest cycle in σ_n. In the case where $f(n) = o(1/n)$ as $n \to \infty$ we simply say the speed is $o(1/n)$.

Katok and Stepin [18] proved that if T admits a cyclic (1-cyclic) a.p.t. speed $o(1/n)$ then T is ergodic and has simple spectrum. The following result is also due to Katok and Stepin, (unpublished).

Theorem 3.1. If T admits an m-cyclic a.p.t. with speed $o(1/n)$ then the multiplicity of the spectrum of U_T is at most m.

The proof follows directly from a theorem proved in [5].

§4. Combinatorics of Extensions

The next few lemmas discuss the behavior of the extension of an approximation. Let $T_0 : \mathbb{Z}/q \to \mathbb{Z}/q$ be a cyclic permutation. Any function $\gamma : \mathbb{Z}/q \to \mathbb{Z}/m$ may be written as $\gamma = \sum_{y \in \mathbb{Z}/m} y\chi_{A_y}$ where $A_y = \gamma^{-1}(y)$. Letting $a_y = \operatorname{card}(A_y)$ we will define

$$t(y) = \sum_{y \in \mathbb{Z}/m} ya_y.$$

It is easy to see that $t(\gamma_1+\gamma_2) = t(\gamma_1) + t(\gamma_2)$. We will often write t for $t(\gamma)$.

Consider the extension

$$T_1 : \mathbb{Z}/q \times \mathbb{Z}/m \to \mathbb{Z}/q \times \mathbb{Z}/m \tag{4.1}$$

given by

$$T_1(x,y) = (T_0x, \gamma(x)+y).$$

Lemma 4.1. T_1 has m cycles of length q if and only if $t \equiv 0 (\operatorname{mod} m)$. T_1 has one cycle of length m_q if and only if $(t,m) = 1$.

Proof. $T_1^k(x,y) = (T_0^k x, y + \sum_{j=1}^{k-1} \gamma(T_0^j x))$ so that in particular $T_1^q(x,y) = (x, y + \sum_{j=1}^{q-1} \gamma(T_0^j x))$, where q is the least k so that $T_0^k x = x$. Thus it suffices to show that $t(\gamma) = \sum_{j=1}^{q-1} \gamma(T_0^j x)$. This is clear since $\gamma(T_0^j x) = y$ if and only if $T_0^j x \in A_y$. Because T_0 is a cycle this happens a_y times.

If T_1 does not have the behavior described in the last lemma, we can modify γ to obtain it. For the next lemma let

$$\Delta\gamma(x) = \begin{cases} k & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0 \end{cases} \quad \text{and} \quad \tilde{T}_1(x,y) = (T_0x,(\gamma+\Delta\gamma)(x)+y).$$

Lemma 4.2. For arbitrary γ and x_0 there are values of k so that $\tilde{T}_1$ has one cycle and so that it has m cycles.

Proof. If we let $\tilde{a}_y = \text{card}((\gamma+\Delta\gamma)^{-1}(y))$ and $\tilde{t} = \sum_{y=0}^{m} y\tilde{a}_y$ then $\tilde{t} - t = (\gamma+\Delta\gamma)(x_0) - \gamma(x_0) = k$. □

Let us now consider the double extension

$$(4.2) \qquad T : \mathbb{Z}/q \times \mathbb{Z}/m \times GF(p) \to \mathbb{Z}/q \times \mathbb{Z}/m \times GF(p)$$

where

$$T(x,y,z) = (T_0x,\gamma(x)+y,\phi(y)+z).$$

Proposition 4.1. Suppose $T_0 : \mathbb{Z}/q \to \mathbb{Z}/q$ is a cyclic permutation and γ is such that the extension T_1 given by (4.1) has m cycles of length q. Let p and ϕ be chosen according to Lemma 1.1. Then either the extension T given by (4.2) has m cycles of length pq, or for any $\Delta\gamma$ of the form

$$(4.3) \qquad \Delta\gamma = \chi_{\{x_0\}} - \chi_{\{T_0x_0\}}$$

the extension

$$\widetilde{T}(x,y,z) = (T_0x,(\gamma+\Delta\gamma)(x)+y,\phi(y)+z)$$

has m cycles of length pq.

Proof. We adopt the following notation for the cycles of T_1

$$C_\ell = \bigcup_{j=0}^{q-1} T_1^j(0,\ell).$$

Since the extension of T_1 to T respects the decomposition into cycles we may consider the m extensions of $T_1|_{C_\ell}$ to $T|_{C_\ell \times GF(p)}$ separately. To show that T consists of m cycles of length pq it suffices to show that for each ℓ, $T|_{C_\ell \times GF(p)}$ consists of one cycle of length pq. To formulate an equivalent statement we define $a_\ell(z) = \mathrm{card}(C_\ell \cap \phi^{-1}(z))$ and $t_\ell = \sum_{z=0}^{p-1} za_\ell(z)$. It follows from Lemma 4.1 that it suffices to show that $(t_\ell,p) = 1$ for all ℓ.

We first show that if this condition is satisfied for one ℓ then it is satisfied for all ℓ simultaneously. Observe that $(x,y) \in C_\ell$ if and only if $(x,y+k) \in C_{\ell+k}$ and thus

$$a_\ell(z) = a_{\ell+k}(\phi(k)z). \tag{4.4}$$

Making the substitutions $z = \phi(y)$ and $b_\ell(y) = a_\ell(\phi(y))$, we have by (4.4)

$$\begin{aligned} b_{\ell+k}(y) &= a_{\ell+k}(\phi(y)) \\ &= a_{\ell+k}(\phi(k)\phi(y-k)) \\ &= a_{\ell}(\phi(y-k)) \\ &= b_{\ell}(y-k). \end{aligned}$$

Since

$$t_\ell = \sum_{z \in GF(p)} z a_\ell(z) = \sum_{y \in \mathbb{Z}/m} \phi(y) b_\ell(y)$$

and

$$t_{\ell+k} = \sum_{y \in \mathbb{Z}/m} \phi(y) b_{\ell+k}(y) = \sum_{y \in \mathbb{Z}/m} \phi(y+k) b_\ell(y)$$

it follows that

$$\begin{aligned} t_{\ell+k} - t_\ell &= \sum_{y \in \mathbb{Z}/m} (\phi(y+k) - \phi(y)) b_\ell(y) \\ &= (\phi(k)-1) \sum_{y \in \mathbb{Z}/m} \phi(y) b_\ell(y) \\ &= (\phi(k)-1) t_\ell. \end{aligned}$$

We see that $t_{\ell+k} = \phi(k) t_\ell = \sigma^k t$ where σ is the permutation in Lemma 2.2. Since all the t_ℓ are in the same orbit of σ either they are all zero or all nonzero.

We now assume that $T_\ell \equiv 0 (\bmod\ p)$ and consider the perturbation $\tilde{T}$ described in the statement, in particular we take $\Delta\gamma$ as in (4.3) and assume without loss of generality that $x_0 \neq q-1$. Let $\tilde{T}_1(x,y) = (T_0 x, (\gamma+\Delta\gamma)(x)+y)$. It is easy to see that T_1 and $\tilde{T}_1$ are conjugate via the isomorphism

$$R(x,y) = (x,y+\chi_{\{x_0\}}(x))$$

so that these two maps have the same cyclic structure.

Let $\tilde{C}_\ell = \bigcup_{j=0}^{q-1} \tilde{T}_1^j(0,\ell)$ and $\tilde{a}_\ell(z) = \mathrm{card}(\tilde{C}_\ell \cap \phi^{-1}(z))$.

Suppose $x_0 = T_0^k(0)$ and $(x_0,y_0) = \tilde{T}_1^k(0,\ell)$. By the above we also have $(x_0,y_0) = T_1^k(0,\ell)$ and so $y_0 = \ell + \sum_{j=0}^{k-1} \gamma(T_0^j 0)$. Then

$$\tilde{a}_\ell(z) - a_\ell(z) = \begin{cases} 1 & \text{if } z = \phi(y_0+(\gamma+\Delta\gamma)(x_0)) \\ -1 & \text{if } z = \phi(y_0+\gamma(x_0)) \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned} \tilde{t}_\ell - t_\ell &= \sum_{\mathbb{Z}/p} z(\tilde{a}_\ell(z)-a_\ell(z)) \\ &= \phi(y_0+(\gamma+\Delta\gamma)(x_0)) - (y_0+\gamma(x_0)) \\ &= \phi(y_0+\gamma(x_0))(\phi(\Delta\gamma(x_0))-1). \end{aligned}$$

But $\phi(y) \neq 0$ and $\phi(\Delta\gamma(x_0)) = \phi(1) \neq 1$ since ϕ is one to one. It follows that $t_\ell' \neq 0$. $\square$

§5. The Estimate of the Multiplicity from Above

Let L be the set of measurable functions $\gamma : X \to \mathbb{Z}/m$. To describe the topology on L we identify $\mathbb{Z}/m$ with the set $\{0/m, 1/m, \ldots, (m-1)/m\} \subseteq [0,1)$. Let us define

$$\|\gamma_1 - \gamma_2\| = \int_X |\gamma_1(x) - \gamma_2(x)|\,dx.$$

For a finite partition ξ let us define $L(\xi) = \{\gamma \in L : \gamma$ is constant a.e. on each $C \in \xi\}$. Let $\xi' = \xi \times \varepsilon_{\mathbb{Z}/m}$ and $\xi'' = \xi' \times \varepsilon_{GF(p)}$ be the natural extensions of the partition ξ to $X \times \mathbb{Z}/m$ and $X \times \mathbb{Z}/m \times GF(p)$ respectively. If $T_0 : X \to X$ permutes the elements of ξ and $\gamma \in L(\xi)$ then the extension T_1 in (1.1) permutes the elements ξ', and the double extension T in (1.2) permutes the elements of ξ''.

<u>Lemma 5.1</u>. Suppose $T_0 : X \to X$ is a measure preserving transformation which admits an a.p.t. $T_{0,n}$ with speed $f(n)$ such that $\xi_n \to \varepsilon$ is the sequence of partitions permuted by $T_{0,n}$. Suppose $\gamma \in L$, $\gamma_n \in L(\xi_n)$ and $\|\gamma_n - \gamma\| < g(n)$. Then the extension T_1 in (1.1) corresponding to γ has an a.p.t. $T_{1,n}$ with speed $f(n) + g(n)$, where $T_{1,n}$ is given by

$$(5.1) \qquad T_{1,n}(x,y) = (T_{0,n}x, \gamma_n(x) + y).$$

In addition, the double extension T in (1.2) has an a.p.t. $T_{2,n}$ with speed $f(n) + g(n)$, where $T_{2,n}$ is given by

$$(5.2) \qquad T_{2,n}(x,y,z) = (T_{0,n}x, \gamma_n(x)+y, \phi(y)+z).$$

Proof. Let us define

$$T'_{1,n}(x,y) = (T_{0,n}x, \gamma(x)+y)$$

and let μ' be the normalized product measure on $X \times \mathbb{Z}/m$. Then

$$\begin{aligned}
&\sum_{C\in\xi'_n} \mu'(T_1C\Delta T_{1,n}C) \\
&\quad \le \sum_{C\in\xi'_n} \mu'(T_1C\Delta T'_{1,n}C) + \sum_{C\in\xi'_n} \mu'(T'_{1,n}C\Delta T_{1,n}C) \\
&\quad \le \sum_{C\in\xi_n} \mu(T_0C\Delta T_{0,n}C) + \|\gamma_n-\gamma\| \\
&\quad \le f(n) + g(n).
\end{aligned}$$

The second statement follows from the first.

Let T_0 be a measure-preserving transformation which admits a cyclic a.p.t. $T_{0,n}$ with speed $o(1/n)$.

Definition 5.1. We will say that γ_n is of type 1 if the extension $T_{1,n}$ of $T_{0,n}$ corresponding to γ_n is cyclic. We will say that γ_n is of type 2 if both $T_{1,n}$ and $T_{2,n}$ have m cycles.

Definition 5.2. Let us define $\Gamma(T_0)$ to be the set of all $\gamma \in L$ such that there exists $\gamma_n \to \gamma$ where γ_n is of type 1 when n is even, γ_n is of type 2 when n is odd and $\|\gamma_n-\gamma\| = o(1/n)$.

Definition 5.3. We define $\mathcal{W}$ to be the set of all pairs (T_0,γ) such that:

(i) T_0 has continuous spectrum and admits a cyclic a.p.t. $T_{0,n}$ with speed $o(1/n)$ such that $(q_n,m) = 1$, where $q_n = \text{card}(\xi_n)$ and

(ii) $\gamma \in \Gamma(T_0)$.

The following is a general lemma on the continuity of the spectrum of a finite cyclic group extension:

Lemma 5.2. Suppose T_0 and γ are chosen so that $(T_0,\gamma) \in \mathcal{W}$. Then the extension T_1 of T_0 corresponding to γ, given by (1.1), has continuous spectrum.

The proof depends on the following lemma.

Lemma 5.3. Suppose $(T_0,\gamma) \in \mathcal{W}$. For $k \in \mathbb{Z}/m^{\times}$ let $\tilde{\gamma} = k\gamma - 1$ and let $\tilde{T}_1$ be the $\mathbb{Z}/m$ extension of T_0 corresponding to $\tilde{\gamma}$. Then $\tilde{T}_1$ is ergodic.

Proof. Since $\gamma \in \Gamma(T_0)$ there exists a sequence of type 2 γ_n such that $\|\gamma_n-\gamma\| = o(1/n)$. If we define $\tilde{\gamma}_n = \gamma_n - 1$, then by Lemma 4.1.

$$t(\tilde{\gamma}_n) = kt(\gamma_n) - t(1) = -q_n.$$

We will write $\tilde{T}_{1,n}$ for the $\mathbb{Z}/m$ extension of $T_{0,n}$ corresponding to $\tilde{\gamma}_n$. Since $(q_n,m) = 1$, $\tilde{\gamma}_n$ is type 1 and so $\tilde{T}_{1,n}$ is a cyclic a.p.t. speed $o(1/n)$ for $\tilde{T}_1$. It follows from Theorem 3.1 that $\tilde{T}_1$ is ergodic. □

Proof of Lemma 5.2. We consider the U_{T_1} invariant orthogonal decomposition

$$L_2(X \times \mathbb{Z}/m\mathbb{Z}) = \bigoplus_{k \in \mathbb{Z}/m} H_k$$

where

$$H_k = \{\chi_k(y)f(x) : f \in L_2(X)\}.$$

Each eigenvalue ζ corresponds to an eigenfunction of the form $\chi_k(y)f(x) \in H_k$ for some k. Since

$$\zeta\chi_k(y)f(x) = U_{T_1}\chi_k(y)f(x) = \chi_k(\gamma(x)+y)f(T_0x)$$

and since $\chi_k^m \equiv 1$,

$$f^m(T_0x) = \zeta^m f^m(x).$$

The continuity of the spectrum of T_0 implies $\zeta^m = 1$.

We next establish the ergodicity of T_1. Since $\gamma \in \Gamma(T_0)$ there exists a sequence of type 1 γ_n such that $\|\gamma_n-\gamma\| = o(1/n)$. By Lemma 5.1 T_1 admits a cyclic a.p.t. speed $o(1/n)$ and it follows from Theorem 3.1 that T_1 is ergodic.

Ergodicity implies that the eigenvalues ζ_j of U_{T_1} form a subgroup of the m'th roots of unity and thus

$$\zeta_j = \exp 2\pi i j \ell_1/m, \quad j = 0,\ldots,m_1 - 1$$

where $m_1 | m$ and $\ell_1 = m/m_1$. Since each eigenvalue ζ_j is simple, the associated eigenfunction $F_j \in H_k$ for some

$k \in \mathbb{Z}/m$. The relation between j and k is that $j = \psi(k)$ for some function $\psi : \mathbb{Z}/m_1 \to \mathbb{Z}/m$. We may assume without loss of generality [11] that $F_j F_{j'} = F_{j+j'}$, from which it follows that $\psi(j+j') = \psi(j) + \psi(j')$. In fact, ψ is one to one since otherwise there are two eigenvalues ζ and ζ' corresponding to different eigenfunctions f and f' in H_0, contradicting the continuity of the spectrum of T_0. Thus $\psi(j) = k\ell_1 j$ for some $k \in \mathbb{Z}/m^\times$. It follows that

$$\chi_{\psi(j)}(\gamma(x))f(T_0 x) = \zeta_1 f(x)$$

and so

$$\chi_{\ell_1 j}(k\gamma(x)-1)f(T_0 x) = f(x). \tag{5.3}$$

We note however that (5.3) implies that the function $\chi_{\ell_1 j}(y)f(x)$ is an invariant function for the transformation $\tilde{T}_1$ constructed in Lemma 5.2. This contradicts the ergodicity of $\tilde{T}_1$. □

<u>Proposition 5.1</u>. Let $(T_0, \gamma) \in \mathcal{W}$. Then the double extension T in (1.2) has continuous spectrum with maximal spectral multiplicity m. Moreover the spectrum in $\bigoplus_{\omega \in GF(p)^\times} H_w$ is homogeneous with multiplicity m.

<u>Proof</u>. Since $\gamma \in \Gamma(t_0)$ there exists $\gamma_n \to \gamma$ such that γ_n is of type 2, and by Lemma 5.1 an m-cyclic a.p.t. speed $o(1/n)$, $T_{2,n}$ of T. By Theorem 3.1 the maximal spectral multiplicity of T is at least m and by Lemma

2.3 it is at least m.

By Lemma 2.3 the spectra in each H^j of the decomposition (2.1) are identical. It follows that the spectra in each H^j are simple, since if not the multiplicity would be at least $2m$ contradicting the upper bound.

Now suppose that f is an eigenfunction for $U_T|_{H^0 \oplus \dots \oplus H^{m-1}}$. Then at least one projection f_j into H^j is nonzero. Let $f_0, \dots, f_{m-1}$ be the images under the operators S^k in (2.2) of f_j' in the spaces $H^0, \dots, H^{m-1}$. Then $f_0, \dots, f_{m-1}$ are a set of m with the same eigenvalue. If we let $f_j' = f_j/f_0$ then the functions f_j' are a set of invariant functions, which for $j > 0$ are not constant.

We now show that for $j > 0$, $f_j' \perp H^*$ where H^* is given by (2.1). It is clear that $U_T|_{H^*}$ is equivalent to U_{T_1}. By Lemma 5.2 T_1 is ergodic and thus has no invariant functions besides constants.

For some $j > 0$ let $f_0'', \dots, f_{m-1}''$ be the images of f_j' under the operators S^k with $f_k'' \in H^k$. The functions $f_0'', \dots, f_{m-1}''$ are a set of m non-constant invariant functions. Together with the constants this implies that the multiplicity of 1 in the spectrum is $m+1$ contradicting the established upper bound. It follows that the spectra in each H^j are continuous. This fact combined with the continuity of the spectrum of T_1 implies T has continuous spectrum. It is clear that the spectrum in H^* is disjoint from the spectra in the H^j. Otherwise the multiplicity at some point would be $m+1$ contradicting the upper bound. $\square$

§6. Genericity

To complete the proof of Theorem 1.1 we must show that $(T_0,\gamma) \in \mathcal{W}$ generically. In §5 we defined a topology on the set of $\gamma \in L$. We now recall the definition of the weak topology on the set $\mathcal{U}$ of invertible measure-preserving transformations of X. For $T \in \mathcal{U}$ a subbase for the neighborhoods of T is given by sets of the form

$$N(T,\xi,\varepsilon) = \{S : \sum_{C\in\xi} \mu(TC\Delta SC) < \varepsilon\}$$

where ξ is an arbitrary finite partition and $\varepsilon > 0$. We give the set of pairs $(T_0,\gamma) \in \mathcal{U} \times L$ the product topology.

Proposition 6.1. The set $\mathcal{W}$ has a subset which is an everywhere dense G_δ subset of $\mathcal{U} \times L$ in the product topology.

Proof. Let $\xi_n \to \varepsilon$ be a fixed sequence of partitions and $f(n) = o(1/n)$ a fixed speed. Define Z_n the set of cyclic measure-preserving permutations of the elements of ξ_n. For $\sigma \in Z_n$ we define

$$U_\sigma(\xi_n) = N(\sigma,\xi_n,f(n))$$

and

$$\mathcal{U}_1 = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{\sigma\in Z_n} U_\sigma(\xi_n)$$

Let $\mathcal{U}_2$ be the set of those elements of $\mathcal{U}_1$ which have continuous spectrum. In [5] it is proved that $\mathcal{U}_1$ contains a subset which is everywhere dense and G_δ in $\mathcal{U}$. It follows from the fact that the T_0 with continuous spectra are an everywhere dense G_δ set [11], that $\mathcal{U}_2$ has an everywhere dense G_δ subset in $\mathcal{U}$.

We now make some definitions: Let

$$B(\gamma,f(n)) = \{\gamma' \in L : \|\gamma-\gamma'\| < f(n)\}$$

and for $j = 1,2$

$$L_j(\sigma,\xi_n) = \{\gamma \in L(\xi_n) : \gamma \text{ is type } j \text{ for } \sigma\}.$$

We also define

$$V_\sigma(\xi_n) = \bigcup_{\sigma \in L_j(\sigma,\xi_n)} B(\gamma,f(n))$$

where

$$j = \begin{cases} 1 & \text{if } 2 \nmid n \\ 2 & \text{if } 2 \mid n. \end{cases}$$

Let

$$\mathcal{W}_1 = \bigcap_{n=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{\sigma \in Z_n} (U_\sigma(\xi_n) \times V_\sigma(\xi_n))$$

and $\mathcal{W}_2$ be the set of those $(T_0,\gamma) \in \mathcal{W}_1$ such that T_0 has continuous spectrum. If $(T_0,\gamma) \in \mathcal{W}_2$ then $T_0 \in \mathcal{U}_2$ and $\gamma \in \Gamma(T_0)$. Thus $\mathcal{W}_2 \subseteq \mathcal{W}_1 \subseteq \mathcal{W}$. Since $\mathcal{W}_2$ is G_δ it remains to show that $\mathcal{W}_2$ is everywhere dense.

By Lemma 4.2 and Proposition 4.1, for any $\gamma \in L(\xi_n)$ there exists $\gamma' \in L_j(\sigma,\xi_n)$ such that $\|\gamma-\gamma'\| \le \frac{m+1}{mq_n}$. Since for $\gamma_0 \in L$ there is a $\gamma \in L(\xi_n)$ such that $\|\gamma_0-\gamma'\| \le \frac{1}{mq_n}$, we can find $\gamma' \in L_j(\sigma,\xi_n)$ such that $\|\gamma_0-\gamma'\| \le \frac{m+2}{mq_n}$, for

$j = 1,2.$

Let $T_{0,n}$ be a cyclic a.p.t. speed $f(n)$ of T_0 on a subsequence ξ_{n_k}. Refine this subsequence further so that

(6.1) $$\frac{m+2}{mq_{n_{k+1}}} < f(q_{n_k}).$$

Let us define

$$\mathcal{V}(T_0) = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} V_{T_{0,n_k}}(\xi_{n_k})$$

and

$$\mathcal{W}_3 = \{(T_0,\gamma) : T_0 \in \mathcal{U}_2, \gamma \in \mathcal{V}(T_0)\}.$$

Clearly $\mathcal{W}_3 \subseteq \mathcal{W}_2$.

To complete the proof we will show that $\mathcal{V}(T_0)$ is dense in L. Given $\gamma_0 \in L$ and $\varepsilon > 0$ we choose n_1 large enough so that $D_1 \neq \emptyset$ where

$$D_1 = L_1(T_{0,n_1},\xi_{n_1}) \cap B(\gamma_0,\varepsilon).$$

Given $\gamma_k \in D_k$ let us define D_{k+1} inductively by

$$D_{k+1} = L_{k+1}(T_{0,n_{k+1}},\xi_{n_{k+1}}) \cap B(\gamma_k,f(q_{n_k})).$$

It follows from (6.1) that $D_{k+1} \neq \emptyset$. Furthermore $\gamma_k \to \gamma \in \mathcal{V}(T_0)$, and $\|\gamma_0-\gamma\| < \varepsilon$. $\square$

§7. Interval Exchange Transformation

In this section we show that our construction may be realized as an interval exchange transformation. Recall that an invertible transformation $T : [0,1) \to [0,1)$ is called an interval exchange if it is piecewise continuous, Lebesgue measure preserving and orientation preserving. The simplest nontrivial case is when three intervals are exchanged according to the permutation $(1,2,3) \to (3,2,1)$. Katok and Stepin [ ] have shown that for almost every pair (α,β) with $\alpha < \beta$ the transitive three interval exchange with discontinuities α and β has simple continuous spectrum.

The proof relies on the following fact. $T^{(\alpha,\beta)}$ is equivalent to the mapping induced on the interval $[0,B)$ by the rotation $T^{(A)} : [0,1) \to [0,1)$ where $T^A x = (x+A) = (\text{mod } 1)$. The relation between α, β, A and B is given by

$$(7.1) \qquad A = \frac{1-\alpha}{1+\beta-\alpha} \qquad 1 - B = \frac{\beta-\alpha}{1+\beta-\alpha} .$$

Note that a finite cyclic group extension of an interval exchange is an interval exchange provided γ is piecewise constant. The following theorem is the second main result of this paper.

<u>Theorem 7.1</u>. For any $m > 1$ there exists an interval exchange transformation T which is ergodic and has continuous spectrum of multiplicity m. In fact T has

the form (1.2) where T_0 an exchange of three intervals and γ is piecewise constant with three points of discontinuity.

<u>Proof</u>. We choose (α,β) such that $T_0 = T^{(\alpha,\beta)}$ has simple continuous spectrum and such that the following conditions hold

(7.2) There exists a sequence $q_n \to \infty$ such that for some p_n and r_n

$$|A - p_n/q_n| = o(1/q_n^2) \quad \text{and} \quad |B - r_n/q_n| = o(1/q_n).$$

(7.3) $(q_n,m) = 1$

(7.4) $\alpha > 1/2$.

These conditions clearly hold for a set of (α,β) of positive measure. Condition (7.4) is equivalent to the condition $1 - A > B/2$.

It suffices to show that there is a piecewise constant $\gamma \in \Gamma(T_0)$. We write $T_0^* = T^{(A)}$ and consider first an extension of T_0^* by a piecewise constant. $\gamma \in \Gamma(T_0^*)$ such that $\operatorname{supp}\gamma \subseteq [0,B)$, where α, β, A and B are related as in (7.1). Let $\xi_n^* = \{[k/q_n, k+1/q_n) : k = 0,\dots,q_n-1\}$ and $T_{0,n}^* x = (x+p_n/q_n)(\text{mod } 1)$. Let ξ_n be the restriction of ξ_n^* to the interval $[0,r_n/q_n)$ and $T_{0,n}$ the mapping induced by $T_{0,n}^*$ on $[0,r_n/q_n)$. It is clear from (7.2) and (7.3) that $T_{0,n}^*$ and $T_{0,n}$ determine cyclic a.p.t.'s speed $o(1/n)$ of T_0^* and T_0 respectively.

We will define $\gamma_n \to \gamma$ inductively such that at each step

$$\gamma_n = \chi_{Q_{1,n} \cup Q_{2,n}}$$

where

$$Q_{1,n} = [g_n, h_n)$$

and

$$Q_{2,n} = [h_n + p_n, r_n).$$

Also define γ_n' corresponding to g_n' and h_n'. Choose a subsequence of $q_n \to \infty$ such that $q_{n+1} > q_n^2$. Assume that q_1 is large enough that for some arbitrary $0 < g_1 < h_1$ the inequality

$$(7.5) \qquad 0 < g_n < h_n < h_n + p_n < q_n - p_n < r_n < q_n$$

holds for $n = 1$. This determines γ_1.

We now show how to construct γ_{n+1}' given γ_n. Let us define

$$g_{n+1}'' = \min_{j \in \mathbb{N}} \left| g_n/q_n - j/q_{n+1} \right|$$

$$h_{n+1}'' = \min_{j \in \mathbb{N}} \left| h_n/q_n - j/q_{n+1} \right|.$$

The construction breaks into two cases depending on whether γ_n is of type 1 or type 2. If we let $s_n = r_n - p_n - q_n$ then $t(\gamma_n) = s_n (\mathrm{mod}\ m)$. By Lemma 4.1, if γ_n is of type 1 then $(s_n,m) = 1$ and if γ_n is of type 2 then $s_n \equiv 0 (\mathrm{mod}\ m)$. Let us define $s''_{n+1} = r_{n+1} - p_{n+1} - q''_{n+1}$.

Case 1. γ_n is of type 2.

We wish to construct γ'_{n+1} of type 1. If $(s''_{n+1},m) = 1$ then let $g'_{n+1} = g''_{n+1}$ and $h'_{n+1} = h''_{n+1}$. It follows that γ'_{n+1} is of type 1. If $(s''_{n+1},m) \neq 1$ then there exists $k < m$ such that $(s''_{n+1}+k,m) = 1$. Let $q'_{n+1} = q''_{n+1} + k$ and $h'_{n+1} = h''_{n+1}$. γ'_{n+1} is of type 1.

Case 2. γ_n is of type 1.

In this case we do not require that γ'_{n+1} be of type 2 but only that $t(\gamma'_{n+1}) = 0$. To do this we repeat the procedure of Case 1, making $g'_{n+1} = g''_{n+1} + k$ for some $k < m$ if necessary to insure that $t(\gamma'_{n+1}) = r_{n+1} - p_{n+1} - q'_{n+1} \equiv 0(\mathrm{mod}\ m)$.

We note the following fact. If $T_{1,n+1}(x,y) = (T_{0,n+1}x, \gamma'_{n+1}(x)+y)$ then the cyclic structure of $T_{1,n}$ depends only on the type of γ_n. The corresponding fact for $T^*_{0,n+1}$ is the consequence of the construction of γ'_{n+1}. The statement for $T_{0,n+1}$ follows from the remark that $T_{0,n+1}$ is equivalent to the transformation induced on the interval $[0,r_n+1/q_{n+1})$ by $T^*_{0,n+1}$, and that $\mathrm{supp}(\gamma'_{n+1}) \subseteq [0,r_{n+1}/q_{n+1})$. Consequently we need only consider extensions of the three interval exchange T_0.

The final step is to show that we can construct γ_{n+1} from γ'_{n+1} such that $\gamma_n \to \gamma$ where $\gamma \in \Gamma(T_0)$. In the case where γ_n is type 2, γ'_{n+1} is type 1 and so we define $\gamma_{n+1} = \gamma'_{n+1}$ making $g_{n+1} = g'_{n+1}$ and $h_{n+1} = h'_{n+1}$. In the case where γ_n is a type 1 we must insure that γ_{n+1} is of type 2. We consider the double extension

$$T_{2,n+1}(x,y,z) = (T_{0,n+1}, \gamma'_{n+1}(x)+y, \phi(y)+z).$$

If $T_{2,n+1}$ has m cycles then we again define $\gamma_{n+1} = \gamma'_{n+1}$. If not then we apply Proposition 4.1 with

$$\Delta\gamma_{n+1} = \chi_{R^1_{n+1}} - \chi_{R^1_{n+1}}$$

where

$$R^1_{n+1} = [h'_{n+1}/q_{n+1}, (h'_{n+1} + 1)/q_{n+1})$$

$$R^{-1}_{n+1} = [(h'_{n+1} + p_{n+1})/q_{n+1}, (h'_{n+1} + p_{n+1} + 1)/q_{n+1}).$$

It follows from condition (7.4) that $\Delta\gamma_{n+1}$ is of the form (4.3). Thus we define $\gamma_{n+1} = \gamma'_{n+1} + \Delta\gamma_{n+1}$, making $g_{n+1} = g'_{n+1}$ and $h_{n+1} = h'_{n+1} + 1$.

Inequality (7.5) follows from the inequality $q_{n+1} > q_n^2$. In addition we have the following

$$|g_{n+1}/q_{n+1} - g_n/q_n| < m/q_{n+1}$$

$$|h_{n+1}/q_{n+1} - h_n/q_n| < 2/q_{n+1}$$

$$|p_{n+1}/q_{n+1} - p_n/q_n| = o(1/q_n^2)$$

$$|r_{n+1}/q_{n+1} - r_n/q_n| = o(1/q_n)$$

and so

$$\|\gamma_{n+1}-\gamma_n\| < \frac{m+2}{q_{n+1}} + o(1/q_n^2) + o(1/q_n) = o(1/q_n).$$

It follows that

$$\|\gamma_n-\gamma\| = o(1/q_n).$$

Letting $G = \lim_{n\to\infty} g_n$ and $H = \lim_{n\to\infty} h_n$ we have

$$\gamma = \chi_{[G,H)\cup[H+A,B)}.$$

Thus $\gamma \in \Gamma(T_0)$ and γ is piecewise constant. □

We note that the minimum number of intervals exchanged in this example for $M_T = m$ is $6m\,p(m)$, where $p(m)$ is the prime satisfying Lemma 1.1. Thus, in these examples M_T is always much less than the theoretical upper bound in terms of the number of intervals exchanged, given in the introduction.

CHAPTER II. A MIXING TRANSFORMATION WITH NONSIMPLE SPECTRUM OF FINITE MULTIPLICITY

At the beginning of Chapter I a general construction was presented, such that starting with an arbitrary measure preserving transformation T_0 and any given natural number $m \geq 2$, a new transformation T could be obtained satisfying $M_T \geq m$. The bulk of the chapter considered specific choices of T_0 and the other parameters of the construction so that T would satisfy $M_T = m$. In order to do this we required that both T and T_0 admit good approximations by periodic transformation, a property implying the absence of mixing. (cf. [18]).

There is a somewhat weaker approximation property which does not necessarily exclude mixing, namely that T admits a cutting and stacking construction of some type. There are also conditions on cutting and stacking constructions which imply upper bounds on the spectral multiplicity in a manner analogous to the estimates used in Chapter I for transformations admitting good approximations. An example of a transformation satisfying all of these conditions is Ornstein's rank 1 mixing transformation, which has simple spectrum (cf. §2). In this chapter, starting with this transformation as T_0 and applying the general construction from Chapter I, we obtain a mixing transformation T such that $2 \leq M_T \leq 6$. Using generalizations of the methods in this chapter, it is possible to produce, for any $m \leq 2$, a mixing transformation T satisfying $m \leq M_T \leq mP(m)$, where $P(m)$ satisfies Lemma 1.1 of Chapter I.

§1. Group Extensions

We begin by recalling the general construction from Chapter I. Let $T_0 : (X_0, \mu_0) \to (X_0, \mu_0)$ be a measure preserving transformation of a Lebesgue probability space and let $\gamma : X_0 \to \mathbb{Z}/m$ be a measurable function. A new Lebesgue space $(X_1, \mu_1) = (X_0 \times \mathbb{Z}/m, \mu_0 \times \delta_{\mathbb{Z}/m})$ is constructed, where $\delta_{\mathbb{Z}/m}$ is the normalized Haar measure on $\mathbb{Z}/m$. We define the $\mathbb{Z}/m$ extension T_1 of T_0 corresponding to ϕ by

$$
\begin{aligned}
&T_1 : (X_1, \mu_1) \to (X_1, \mu_1), \\
&T_1(x,y) \;=\; (T_0 x, \gamma(x)+y).
\end{aligned}
\tag{1.1}
$$

The function γ is called the cocycle of the extension.

Let us consider the extension (1.1) in the case $m = 2$. Let $\phi : \mathbb{Z}/2 \to \mathbb{Z}/3$ be the function

$$
\phi(y) = \begin{cases} 1 & \text{if } y = 0 \\ 2 & \text{if } y = 1 \end{cases}
\tag{1.2}
$$

and let $(X, \mu) = (X_1, \times\, \mathbb{Z}/3, \mu_1 \times \delta_{\mathbb{Z}/3})$. We will consider the $\mathbb{Z}/3$ extension T of T_1 corresponding to the cocycle ϕ:

$$
\begin{aligned}
&T : (X, \mu) \to (X, \mu) \\
&T(x,y,z) \;=\; (T_0 x, \gamma(x)+y, \phi(y)+z)
\end{aligned}
$$

which we will call the second extension. It follows from Chapter I, Lemma 2.3 that $M_T \geq 2$. This estimate depends in a fundamental way on the choice (1.2) of the cocycle ϕ.

The algebraic interpretation of ϕ given in Chapter I will be studied in greater detail in Chapter III, §1.

We will now state the main result of this chapter.

Theorem 1.1. There exist T_0 and γ such that the transformation T of (1.3) is mixing and satisfies $2 \leq M_T \leq 6$.

We note that we do not explicitly determine $\mathcal{M}_T$ or even M_T for this example. In fact we know of no way to construct any mixing transformation T, with $1 < M_T < +\infty$, where a better estimate of M_T is possible. The remainder of this chapter will be primarily devoted to the proof of Theorem 1.2. This section will be concluded with a compilation of some basic facts about group extensions which will be needed for this proof.

Lemma 1.1. Suppose T_1 and S are measure-preserving transformations of (X_1, μ_1) such that:

(i) T and S commute.

(ii) S is periodic and has least period m for almost every $x \in X$. Then there exists a measure-preserving transformation T_0 of a Lebesgue space (X_0, μ_0) and measurable $\gamma : X_0 \to \mathbb{Z}/m$ such that T_1 is isomorphic to the $\mathbb{Z}/m$ extension (1.1) of T_0.

Proof. There exists a partition

$$\zeta = \{A_j : j = 0, \ldots, m-1\}$$

of X_1 such that $\mu(A_j) = 1/m$ for all j, and

$$(1.3) \qquad \begin{aligned} SA_j &= A_{j+1}, \ j < m-1 \\ SA_{m-1} &= A_0 \end{aligned}$$

([ ] p. 70). In other words A_0 is a fundamental domain for the action of S. We define $X_0 = A_0$ and $\mu_0 = (1/m)\mu_1$, and we identify A_j with $X_0 \times \{j\}$, $j \in \mathbb{Z}/m$. The transformation

$$T_0 : (X_0,\mu_0) \to (X_0,\mu_0)$$

is given by

$$(1.4) \qquad T_0 x = S^{-\ell(Tx)}(Tx)$$

where $\ell : X_1 \to \mathbb{Z}/m$ is defined by $\ell(x) = j$ if $x \in A_j$. The cocycle γ is the function

$$\gamma(x) = \ell(Tx)$$

for $x \in X_0$. □

Given a measure-preserving transformation T_0 on (X_0,μ_0), two cocycles $\gamma,\gamma' : X_0 \to \mathbb{Z}/m$ are called cohomologous (cf. [18]) if there exists a measurable $\psi : X_0 \to \mathbb{Z}/m$ such that

$$(1.5) \qquad \gamma(x) = \gamma'(x) + \psi(Tx) - \psi(x).$$

A cocycle γ is called a coboundary if it is a trivial cocycle, i.e. if

$$(1.6) \qquad \gamma(x) = \psi(Tx) - \psi(x).$$

The function ψ in (1.5) and (1.6) has been called the transfer function by Kočergin [19]. If T_0 is ergodic, then if ψ exists it is unique up to a constant. The next fundamental lemma is well known.

Lemma 1.2. Let T_0 be a measure-preserving transformation of (X_0,μ_0) and let γ and γ' be cohomologous cocycles. Then the extensions (1.1) corresponding to γ and γ' are isomorphic.

Proof. The argument is almost trivial. Let ψ be the transfer function (1.5) and define the transformation S^ψ of (X_1,μ_1) by

$$(1.7) \qquad S^\psi(x,y) = (x,\psi(x)+y).$$

Denoting by T_1 and T_1' the extensions corresponding respectively to γ and γ' it follows from (1.7) and (1.5) that

$$\begin{aligned} T_1 \circ S^\psi(x,y) &= (T_0x,\gamma(x)+\psi(x)+y) \\ &= (T_0x,\gamma'(x)+\psi(Tx)+y) \\ &= S^\psi \circ T_1'(x,y). \end{aligned}$$
□

An immediate corollary of Lemma 1.2 is

Corollary 1.1. If T_1 is ergodic then T_0 is ergodic and γ is not a coboundary.

Proof. If γ is a coboundary then T_1 is isomorphic to $T_0 \times \mathrm{id}$ on $X_0 \times \mathbb{Z}/m$ so there are m invariant sets of positive measure. If T_0 is not ergodic then there is a nonconstant invariant function f for T_0. Extending f trivially to X_1, we obtain a nonconstant invariant function for T_1. □

For the sake of completeness, we state a converse to the last lemma. Let $R(\gamma)$ denote the essential range of γ in $\mathbb{Z}/m$, and for $Z \subseteq \mathbb{Z}/m$, let $\langle Z \rangle$ denote the subgroup of $\mathbb{Z}/m$ generated by Z. To each cocycle γ we assign a subgroup G_γ of $\mathbb{Z}/m$ defined by

$$G_\gamma = \bigcap_\psi \langle R(\gamma + \psi \circ T_0 - \psi) \rangle .$$

Proposition 1.1. Suppose T_0 is ergodic. Then T_1 is ergodic if and only if $G_\gamma = \mathbb{Z}/m$.

Proof. If $G_\gamma \neq \mathbb{Z}/m$ then there exists a cocycle γ' cohomologous to γ such that $\langle R(\gamma') \rangle = G_\gamma$. Let T be the G_γ extension of T_0 corresponding to γ'. T_1 is then isomorphic to $(\mathbb{Z}/m)/G_\gamma$ extension of T corresponding to a coboundary γ''. It follows from Corollary 1.1 that T_1 is not ergodic.

Conversely, suppose T_1 is not ergodic. Let E be an invariant set such that $0 < \mu(E) < 1$. By the Fubini Theorem, for almost every $x \in X_0$ there exists $E_x \subseteq \mathbb{Z}/m$ such that

$$E = \bigcup_{x \in X_0} \{x\} \times E_x, \; E_x \leq \mathbb{Z}/m.$$

Let $S_0 : \mathbb{Z}/m \to \mathbb{Z}/m$ be defined $S_0 y = y + 1$. By the invariance of E,

$$E_{T_0 x} = T_1 E_x = S_0^{\phi(x)} E_x,$$

for μ_0 almost every x. The ergodicity of T_0 implies that these sets E_x are all translates of one another in $\mathbb{Z}/m$, so that there exists $\tilde{E}_0 \subsetneq \mathbb{Z}/m$, $0 \in \tilde{E}_0$, and measurable ψ such that $\tilde{E}_0 = S_0^{\psi(x)} E_x$. Since $0 < \mu_1(E) < 1$, $\delta_{\mathbb{Z}/m}(\tilde{E}_0) = k/m$ for some $0 < k < m$. Let $\gamma'(x) = -\gamma(x) + \psi(T_0 x) - \psi(x)$ and let T_1' be the $\mathbb{Z}/m$ extension of T_0 corresponding to γ'. By Lemma 1.2, T_1 and T_1' are isomorphic and the invariant set E for T_1 corresponds to the invariant set $\tilde{E} = X_0 \times \tilde{E}_0$ for T_1'. Assuming without loss of generality that $\tilde{E}$ is an ergodic component for T_1', it is straightforward that $\tilde{E}_0$ is a nontrivial proper subgroup of $\mathbb{Z}/m$. □

<u>Definition 1.1</u>. Let T_0 be a measure-preserving transformation of (X_0, μ_0) and let γ_0 be a $\mathbb{Z}/m$ cocycle. Define a new family of cocycles $\gamma_k = \gamma_0 - k$, $k \in \mathbb{Z}/m$, and corresponding extensions $T_{1,k}$. The m different extensions $T_{1,k}$ will be called complementary extensions to the extension $T_1 = T_{1,0}$ (1.1).

<u>Lemma 1.3</u>. Let T_0 be weak mixing. The extension T_1 is mixing if and only if all the complementary extensions $T_{1,k}$, $k \in \mathbb{Z}/m$ are ergodic.

The proof of this lemma is similar to the proof of Chapter I, Lemma 5.2.

In the next theorem, (i)is due to Thouvenot (unpublished) (ii) is due to D. Rudolph (unpublished).

<u>Theorem 1.2</u>. (i) If T_0 is mixing and T_1 is weak mixing then T_1 is mixing.

(ii) If T_0 is mixing or order k and T is weak mixing then T_1 is mixing of order k.

This theorem is also true for the K-property and [29] the Bernoulli property, and is true for extensions much more general than the sort considered here.

§2. Cutting and Stacking

We begin by reviewing the well known cutting and stacking method for constructing a Lebesgue measure-preserving transformation of the unit interval $I = [0,1)$ (cf. Friedman [6]). A tower $\mathcal{T}$ of height $h \geq 1$ is an ordered collection of h disjoint intervals of equal length, called levels, together with a transformation $T_{\mathcal{T}}$ which carries the first $h-1$ levels linearly into their successors. $T_{\mathcal{T}}$ is undefined on level h. The first level is called the tower's base; the last its top. The width of a tower is the length of its base.

Two operations are defined on towers:

(i) A tower $\mathcal{T}$ of width w may be cut into p towers $\mathcal{T}_j$, $1 \leq j \leq p$ of widths w_j, $w = \sum_{j=1}^{p} w_j$, as follows: the base of $\mathcal{T}$ is partitioned into p subintervals having lengths w_j, $1 \leq j \leq p$; $\mathcal{T}_j$ is defined as the h consecutive $T_{\mathcal{T}}$ images of the jth base interval; and $T_{\mathcal{T}_j} = T_{\mathcal{T}}\big|_{\mathcal{T}_j}$. The cutting operation will be denoted by

$$\mathcal{T} = |\mathcal{T}_1|\mathcal{T}_2|\cdots|\mathcal{T}_p|$$

(ii) p disjoint towers $\mathcal{T}_j$, $1 \leq j \leq p$, having equal widths may be stacked to form a new tower $\mathcal{T}$, denoted

$$\mathcal{T} = \mathcal{T}_1 \circ \mathcal{T}_2 \circ \cdots \circ \mathcal{T}_p,$$

in the following way: the levels of $\mathcal{T}$ are the levels of the $\mathcal{T}_j$, $1 \leq j \leq p$. $T_{\mathcal{T}}$ is defined to be some $T_{\mathcal{T}_j}$ wherever they are defined, and extended to carry the top of each $\mathcal{T}_j$

$1 \leq j \leq p$ linearly into the base of $\mathcal{T}_{j+1}$.

A Lebesgue measure-preserving transformation may be constructed inductively by cutting and stacking as follows:

Step 1. The unit interval $[0,1)$ is cut into m_1 subintervals $\mathcal{T}_{1,1},\ldots,\mathcal{T}_{1,m_1}$, called 1-towers. Since each 1-tower has height 1, no transformation is yet defined.

Assuming that Step n has been completed, suppose that m_n n-towers $\mathcal{T}_{n,1},\ldots,\mathcal{T}_{n,m_n}$ have been defined.

Step n+1. The cutting and stacking operations (i) and (ii) are each applied to the n-towers a finite number of times. The resulting towers: $\mathcal{T}_{n+1,1},\ldots,\mathcal{T}_{n+1,m_{n+1}}$ are the (n+1)-towers.

Let W_n denote the sum of the widths of all the towers at the end of Step n. To insure that the construction converges, we will assume that $W_n \to 0$ as $n \to \infty$. We can then define

$$(2.1) \qquad Tx = T_{\mathcal{T}_{n,i}} x$$

whenever

$$(2.2) \qquad x \in \text{Domain } (T_{\mathcal{T}_{n,i}}).$$

It is clear that T is defined for almost every x, since for almost every x there is an n large enough that (2.2) holds for some i. Thus (2.1) defines a Lebesgue measure-preserving transformation T of $[0,1)$, called the limit transformation of the cutting and stacking construction.

Let r_n denote the number of n-towers of height $h > 1$. It is customary to refer to towers of height 1 as spaces.

Definition 1.1. A cutting and stacking construction is said to be rank r if $r_n = r$ for all n. The rank r of a transformation T is defined as the least positive integer r such that there is a rank r cutting and stacking construction for a transformation metrically isomorphic to T. If there is no such construction, then T is said to have infinite rank.

We note that if a transformation is constructed by a rank r construction, its rank may actually be less than r. The definition of the rank of a transformation which we have given here is equivalent to various other well known definitions, although the term "rank" has not always been used (cf. [22],[5],[21],[14]).

We will now state two theorems which are important to the constructions which will be given in the next section.

Theorem 2.1. (i) (Baxter [1]) A rank 1 transformation T has simple spectrum.

(ii) (Chacon [5]) A rank r transformation has maximal spectral multiplicity $M_T \leq r$.

Theorem 2.2. (i) (Ornstein [22]) There exists a rank 1 mixing transformation.

(ii) (D. Rudolph [30]) There exists a rank 1 transformation which is mixing of all orders.

Kalikow [15] has shown that if T is rank 1 mixing then it is mixing of order 2.

We will now prove a criterion for the ergodicity of a cutting and stacking construction which is fairly general. As before, let $T_{n,1},\ldots,T_{n,m_n}$ be the n-towers of a cutting and stacking construction (including the spaces). We will assume that these towers are ordered so that there heights $h_{n,j}$, $1 \le j \le m_n$, satisfy

$$h_{n,1} \le h_{n,2} \le \cdots \le h_{n,m_n}.$$

Let $w_{n,j}$ be the widths of these towers and

$$W_{n,j} = \sum_{i=1}^{j} w_{n,i}, \quad 1 \le j \le m_n.$$

We will define the nth height function

$$H_n : [0, W_{n,m_n}) \to [0,\infty)$$

by

$$H_n(x) = \begin{cases} h_{n,1} & \text{if } x < W_{n,1} \\ \\ h_{n,j} & \text{if } W_{n,j-1} \le x < W_{n,j},\ 1 < j \le m_n. \end{cases}$$

We will also define step functions

$$G_n^{p,q} = p\chi_{[W_{n,m_n}-q,\ W_{n,m_n})}.$$

where $p \geq 0$, $0 \leq q \leq W_{n,m_n}$. A pair (p_n, q_n) will be called admissible if $p_n = h_{n,k}$ and $q_n = W_{n,k-1}$ for some $1 \leq k \leq m_n$ and $G_n^{p_n q_n} \leq H_n$.

Definition 2.2. We will say that the construction satisfies the uniform growth condition if there exists an admissible sequence of pairs (p_n, q_n) such that $\lim_{n\to\infty} p_n q_n = 1$. The n-towers $\mathcal{T}_{n,j}$ for $j \geq k$, will be called substantial and $\bar{h}_n = h_{n,k}$ will be called the minimum substantial height.

Let $\mathcal{T}_{n,j}$, $1 \leq j \leq \ell_n$, be the substantial n-towers (renumbered), let $B_{n,j}$ be the bases of these towers, and let $B_n = \bigcup_{j=1}^{\ell_n} B_{n,j}$. μ_n will denote Lebesgue measure, normalized so that $\mu_n(B_n) = 1$, and ζ_n will denote the partition of B_n into the sets $B_{n,j}$. We will say that $A \in \mathcal{A}(\zeta_n)$ if A is the union of elements of ζ_n, and $\tilde{A}$ will denote the complement of A in B_n.

Since T preserves Lebesgue measure, almost every point $x \in B_n$ is recurrent and it is possible to define the induced map

$$T_{B_n} : (B_n, \mu_n) \to (B_n, \mu_n)$$

as follows: For $x \in B_n$, let

$$t_n(x) = \{\min t \geq 1 : T^t x \in B_n\}.$$

Then $T_{B_n} x = T^{t_n(x)} x$.

Proposition 2.1. Let T be the limit transformation of a cutting and stacking construction satisfying the uniform growth condition. Then T is ergodic if

$$\limsup_{n\to\infty} \min_{\substack{A\in\mathcal{A}(\zeta_n)\\ 0<\mu_n(A)<1}} \frac{\mu_n(T_{B_n} A \cap \tilde{A})}{\mu_n(A)\mu_n(\tilde{A})} > 0.$$

Proof. Let $\alpha > 0$ be the upper limit in (2.3). Suppose E is an invariant set, $0 < \mu(E) < 1$. Choose

$$0 < \delta < \frac{\alpha}{64} \mu(E)\mu(\tilde{E})$$

where $\tilde{E}$ is the complement of E in $[0,1)$. Let ξ_n denote the partition of $[0,1)$ into levels of the n-towers. Clearly $\xi_n \to \varepsilon$, so that for some large n_0 and all $n \geq n_0$ there exist E' and $\tilde{E}'$, both unions of elements of ξ_n, such that $E' \cup \tilde{E}' = [0,1)$, $E' \cap \tilde{E}' = \emptyset$, and

$$\mu(E\Delta E') < \delta \min(\mu(E),\mu(\tilde{E})),$$

$$\mu(\tilde{E}\Delta\tilde{E}') < \delta \min(\mu(E),\mu(\tilde{E})).$$

Now let us consider a level $C \in \xi_n$. $C = (C \cap E) \cup (C \cap \tilde{E})$, and the invariance of E and $\tilde{E}$ imply that

$$\mu(T^k C \cap E) = \mu(C \cap E)$$

(2.4) and

$$\mu(T^k C \cap \tilde{E}) = \mu(C \cap \tilde{E})$$

whenever $T^k C$ is another level in the same tower as C. Thus, we can assume that E' and $\tilde{E}'$ consist of entire towers with the error distributed equally among the levels of each tower.

By making n_0 larger if necessary, we can insure that for all $n \geq n_0$

$$(2.5) \qquad 1 - p_n q_n < \delta \min(\mu(E), \mu(\tilde{E})).$$

Finally, by (2.3), there exists $n \geq n_0$ so that

$$(2.6) \qquad \min_{\substack{A \in \mathcal{A}(\zeta_n) \\ 0<\mu_n(A)<1}} \frac{\mu_n(T_{B_n} A \cap \tilde{A})}{\mu_n(A)\mu_n(\tilde{A})} > \frac{\alpha}{2}.$$

Beginning with the substantial n-towers $\mathcal{T}_{n,1}, \ldots, \mathcal{T}_{n,\ell_n}$, we remove the top few levels to obtain new towers $\mathcal{T}'_{n,1}, \ldots, \mathcal{T}'_{n,\ell_n}$, each having height $\bar{h}_n$. E'' and $\tilde{E}''$ are defined to be the unions of all levels of those truncated towers corresponding, respectively, to the n-towers belonging to E' and $\tilde{E}'$.

Let A and $\tilde{A}$ be unions of the bases of those towers contained, respectively, in E'' and $\tilde{E}''$. Clearly $A, \tilde{A} \in \mathcal{A}(\zeta_n)$. By (2.4), and because the towers $\mathcal{T}'_{n,j}$ all have height $\bar{h}_n$,

$$\mu_n(\tilde{E} \cap A) = \frac{\mu(\tilde{E} \cap E'')}{\mu(E'' \cup \tilde{E}'')} = \frac{\mu(\tilde{E} \cap E'')}{p_n q_n}$$

and since by (2.5), $p_n q_n > 1-\delta$, it follows that

$$\mu_n(\tilde{E} \cap A) < \frac{1}{1-\delta}\,\mu(\tilde{E} \cap E'')$$

$$< \frac{1}{1-\delta}\,\mu(E \,\Delta\, E'')$$

$$\leq \frac{2\delta}{1-\delta}\,\mu(E) < 4\delta u(E),$$

for sufficiently small δ. Similarly,

$$\mu(E) < \mu(E'') + (1-p_n q_n) + \mu(E \,\Delta E')$$

$$< \mu(E'') + \delta\mu(E),$$

so that

$$\mu(E) < \frac{1}{1-2\delta}\,\mu(E'') < 2\mu(E''),$$

and

$$\begin{aligned} \mu_n(\tilde{E} \cap A) &< 8\delta\mu(E'') \\ &= 8\delta p_n q_n \mu_n(A) \qquad (2.7) \\ &\leq 8\delta\mu_n(A). \end{aligned}$$

Reasoning in the same way, we obtain

$$\mu_n(E \cap \tilde{A}) < 8\delta\mu_n(\tilde{A}). \qquad (2.8)$$

It follows from the choice of δ that

$$\delta < \frac{\alpha}{64}\,\mu(E)\mu(\tilde{E})$$

$$< \frac{\alpha}{16}\,\mu_n(A)\mu_n(\tilde{A})$$

$$= \frac{\alpha\mu_n(A)\ {}_n(\tilde{A})}{16(\mu_n(A)+\mu_n(\tilde{A}))},$$

so that

$$(2.9) \qquad \frac{\alpha}{2}\,\mu_n(A)\mu_n(\tilde{A}) - 8\delta\mu_n(A) > 8\delta\mu_n(\tilde{A}).$$

Then, by (2.6), (2.7) and (2.8)

$$\begin{aligned}
\mu_n(E \cap \tilde{A}) &\geq \mu_n(T_{B_n}(E \cap A) \cap \tilde{A}) \\
&\geq \mu_n(T_{B_n} A \cap \tilde{A}) - \mu_n(\tilde{E} \cap A) \\
&\geq \frac{\alpha}{2}\,\mu_n(A)\mu_n(\tilde{A}) - 8\delta\mu_n(A) \\
&> 8\delta\mu_n(\tilde{A})
\end{aligned}$$

contradicting (2.8). It follows that T is ergodic. □

A particularly convenient corollary to Proposition 2.1 appears if we impose two additional conditions on the cutting and stacking construction.

Definition 2.2. Let $W_n = \mu(B_n)$ and $w_n = \min_{1\leq j\leq \ell_n} \mu(B_{n,j})$. A cutting and stacking construction will be said to have a growth coefficient $\rho > 0$ if

$$(2.10) \qquad W_{n+1} < \frac{1}{\rho}\, w_n, \quad \text{for all } n.$$

We note that it is always possible to make ρ as large as we want by combining several construction steps.

Definition 2.3. Let B_n be the union of the bases of the substantial n-towers. A cutting and stacking construction such that $B_{n+1} \subseteq B_n$ will be said to satisfy the descending base condition.

We now define some useful sets. The set $R_n = T^{-1}B_n$ will be called the set of roofs of the n-towers. To assign roofs to individual n-towers, let

$$s_n(x) = \min\{s \geq 1 : T^s x \in R_n\}$$

for $x \in B_n$. We define the roof of the j^{th} n-tower as

$$R_{n,j} = \{x \in R_n : T^{-s_n(x)} x \in B_{n,j}\}.$$

Of particular interest here, is that part of the roof constructed during Step $n+1$. Let T_n denote the partial transformation defined on the substantial n-towers at the end of Step n, and let D_n be the domain of T_n. We define

$$\overline{D}_n = D_{n+1} \setminus D_n$$

$$\overline{R}_n = R_n \cap \overline{D}_n$$

$$\overline{R}_{n,j} = R_{n,j} \cap \overline{D}_n$$

and also

$$\overline{B}_n = \{x \in B_n : T^{s_n(x)} x \in \overline{R}_n\}$$

$$\overline{\overline{B}}_n = T\overline{R}_n.$$

It is easy to see that $B_n = \overline{B}_n \cup \overline{\overline{B}}_n$ and $\mu(\overline{B}_n) = \mu(\overline{\overline{B}}_n)$. For any $A \in \mathcal{A}(\zeta_n)$ we will denote $A' = A \cap \overline{B}_n$ and $A'' = A \cap \overline{\overline{B}}_n$.

Corollary 2.1. Suppose that the cutting and stacking construction for T satisfies the uniform growth condition, the descending base condition and has growth coefficien $\rho > 1$, then T is ergodic if

$$(2.11) \qquad \limsup_{n\to\infty} \min_{\substack{A\in A(\zeta_n)\\ 0<\mu_n(A)<1}} \frac{\mu_n(T_{B_n}A'\cap \widetilde{A}'')}{\mu_n(A')\mu_n(\widetilde{A}'')} > 0.$$

Proof. If x is the upper limit in (2.11) then there are infinitely many n such that for any $A \in A(\zeta_n), 0 < \mu_n(A) < 1$,

$$(2.12) \qquad \mu_n(T_{B_n}A \cap \widetilde{A}) \geq \mu_n(T_{B_n}A'\cap \widetilde{A}'')$$

$$> \frac{\alpha}{2}\mu_n(A')\mu_n(\widetilde{A}'')$$

$$= \frac{\alpha}{2}(\mu_n(A)-\mu_n(B_n\setminus\overline{B}_n))(\mu_n(\widetilde{A})-\mu_n(B_n\setminus\overline{\overline{B}}_n))$$

$$= \frac{\alpha}{2}(\mu_n(A)\mu_n(\widetilde{A})-\mu_n(B_n\setminus\overline{B}_n)(\mu_n(A)+\mu_n(A))+\mu_n(B_n\setminus\overline{B}_n)^2)$$

$$= \frac{\alpha}{2}(\mu_n(A)\mu_n(\widetilde{A}) - (\mu_n(B_n\setminus\overline{B}_n) - \mu_n(B_n\setminus\overline{B}_n)^2)).$$

It follows from the descending base condition that

$$(2.13) \qquad \mu_n(B_n\setminus\overline{B}_n) - \mu_n(B_n\setminus\overline{B}_n)^2$$

$$\leq \frac{W_{n+1}}{W_n} - \frac{W_{n+1}^2}{W_n^2}$$

$$< \frac{1}{\rho}\left(\frac{w_n}{W_n} - \frac{w_n^2}{W_n^2}\right)$$

$$\leq \frac{1}{\rho}\mu_n(A)\mu_n(\widetilde{A}).$$

Combining (2.12) and (2.13) with the fact that $\rho > 1$,

$$\mu_n(T_{B_n} A \cap \widetilde{A}) > \frac{\alpha}{2}(1 - \frac{1}{\rho}) > 0.$$

It follows from Proposition 2.1 that T is ergodic. $\square$

§3. Constructions

In this section we first describe simultaneously, three cutting and stacking constructions which we will call Construction 1, Construction 2, and Construction 3. Respectively, the ranks of these constructions will be 1, 2 and 6, and the limit transformations will be called T_0, T_1 and T_2. T_0 will be an arbitrary rank 1 transformation. T_2 will be the $\mathbb{Z}/2$ extension (1.1) of T_0 for a cocycle γ which will depend on two parameters $\nu, \omega \in \{0,1\}^{\mathbb{N}}$. The transformation T will be the $\mathbb{Z}/3$ extension (1.3) of T_1. Following the constructions, we will show that T satisfies Theorem 1.1.

All three constructions will share the following parameters:

i) a sequence of positive integers $p_n \geq 2$, $n \geq 2$;

ii) for each $n \geq 2$, a finite sequence $t_{n,j}$ of non-negative integers, $1 \leq j \leq p_n$.

The choice of i) and ii) will be subject to the following constraint: If

$$(3.1) \qquad s_{n,k} = \sum_{j=1}^{k} t_{n,j}, \quad n \leq 2 \quad \text{and} \quad 1 \leq k \leq p_n,$$

and

$$\rho_n = \prod_{k=2}^{n} p_k, \qquad n \geq 2,$$

then

$$(3.2) \qquad u = \sum_{n=2}^{\infty} s_{n,p_n}/\rho_n < \infty.$$

We will index the towers of Construction i, $i = 1,2,3$, by $\theta \in \Theta_i$, where $\Theta_1 = \{0\}$, $\Theta_2 = \mathbb{Z}/2$ and $\Theta_3 = \mathbb{Z}/2 \times \mathbb{Z}/3$. As well as depending on the parameters above, each construction will also depend on two sequences of permutations of Θ_i:

$$\alpha(i,n,j,k;\cdot) \quad \text{and} \quad \beta(i,n,j;\cdot),$$

where $n \geq 2$, $1 \leq j \leq p_n$ and $1 \leq k \leq t_{n,j}$. In the case $i = 1$, these permutations are obviously trivial. For $i = 2$ and $i = 3$, these permutations will be defined below. We are now ready to describe the constructions.

Step 1. The unit interval $[0,1)$ is cut into card (Θ_i) subintervals I^θ, $\theta \in \Theta_i$, of equal measure. These, in turn, are each cut into a 1-tower T_1^θ of measure $\mu(I^\theta)/(u+1)$ and 1-spacer S_1^θ of measure $\mu(I\theta)u/(u+1)$.

We suppose now that Step n has been completed so that for each $\theta \in \Theta_i$, there is an n-tower T_n^θ and n-spacer S_n^θ.

Step n+1. For each $\theta \in \Theta_i$ we cut

$$T_n^\theta = |\hat{T}_{n,1}^\theta|\hat{T}_{n,2}^\theta|\cdots|\hat{T}_{n,p_{n+1}}^\theta|$$

into p_{n+1} subtowers of equal width. We also cut

$$S_n^\theta = \underbrace{|\hat{S}_n^\theta|\cdots|\hat{S}_n^\theta|}_{s_{n+1},p_{n+1} \text{ copies}}S_{n+1}^\theta|$$

into $s_{n+1,p_{n+1}}$ subspacers of the same width as $\hat{T}_{u,j}$, leaving an interval S^{θ}_{n+1} which will be the $(n+1)$-spacer. The inequality (3.12) guarantees that this will always be possible.

Stacking proceeds in three stages. First, we inductively stack the spacers:

$$\overline{S}^{\theta}_{n,j}(1) = \hat{S}_n^{\alpha(i,n+1,j,1;\theta)}$$

and

$$\overline{S}^{\theta}_{n,j}(k) = \overline{S}^{\theta}_{n,j}(i-1) \circ \hat{S}_n^{\alpha(i,n+1,j,k;\theta)}$$

for $1 < k \leq t_{n+1,j}$, $1 \leq j \leq p_{n+1}$.

(3.1) guarantees that there will be enough spacers. These piles of spacers are then stacked:

$$\tilde{T}^{\theta}_{n,j} = \hat{T}^{\theta}_{n,j} \circ \overline{S}^{\theta}_{n,j}(t_{n+1,j}).$$

Finally, the subtowers $\tilde{T}^{\theta}_{n,j}$, $1 \leq j \leq p_{n+1}$ are stacked by induction to obtain T^{θ}_{n+1}:

$$\overline{T}^{\theta}_{n+1}(1) = \tilde{T}^{\theta}_{n,1},$$

$$\overline{T}^{\theta}_{n+1}(j) = \overline{T}^{\theta}_{n+1}(j-1) \circ \tilde{T}^{\beta(i,n,j;\theta)}_{n,j}, \quad 1 < j \leq p_{n+1},$$

and

$$T^{\theta}_{n+1} = \overline{T}^{\theta}_{n+1}(p_{n+1}).$$

This completes Step $(n+1)$ of the construction.

It is obvious that Construction 1 is an arbitrary rank 1 construction depending only on the parameter i) and ii).

We will often drop the superscript θ when referring to this construction. We will also identify the interval $[0,1)$ with the Lebesgue space (X_0,μ_0), where μ_0 will correspond to Lebesgue measure. By doing this, we can consider the limit transformation T_0 a measure-preserving transformation of (X_0,μ_0).

<u>Lemma 3.1</u>. Let

$$K_n = \operatorname{card}\{j : t_{n,j} \neq 0,\ j = 1,\ldots,p_n\}.$$

If

$$\limsup_{n\to\infty} \frac{K_n}{p_n} = 0 \tag{3.3}$$

then T_0 is not mixing.

<u>Proof</u>. Let h_n be the height of the n-tower. It is easy to see that for a measurable set E, $0 < \mu_0(E) < 1$, (3.3) implies

$$\mu_0(T_0^{h_n}E\Delta E) \to \mu_0(E)$$

as $n \to \infty$. Thus T_0 is rigid (cf. [7]), and therefore cannot be mixing. $\square$

We now consider Construction 2, first describing the permutations α and β needed for the construction. α will depend on a parameter $\nu \in \{0,1\}^{\mathbb{N}}$. We will denote

$$\nu = (\nu^{(2)},\nu^{(3)},\ldots).$$

and

$$\nu^{(n)} = (\ldots \nu_{j,k}^{(n)} \ldots) \in \{0,1\}^{s_{n},p_{n}}$$

where $1 \leq j \leq p_n$, $1 \leq k \leq t_{n,j}$.

$$\alpha(2,n,j,k;\theta) = \begin{cases} \theta & \text{if } \nu_{j,k}^{(n)} = 0 \\ \theta+1 & \text{if } \nu_{j,k}^{(n)} = 1. \end{cases}$$

The permutation β will depend on a second parameter

$$\omega = (\omega^{(2)}, \omega^{(3)}, \ldots) \in \{0,1\}^{\mathbb{N}}$$

where

$$\omega^{(n)} = (\omega_1^{(n)}, \omega_2^{(n)}, \ldots, \omega_{p_{n-1}}^{(n)}) \in \{0,1\}^{p_{n-1}}.$$

The definition of β is similar to the definition of α, namely

$$\beta(2,n,j;\theta) = \begin{cases} \theta & \text{if } \omega_j^{(n)} = 0 \\ \theta+1 & \text{if } \omega_j^{(n)} = 1. \end{cases}$$

The parameter $\nu^{(n)}$ determines which spacers to use when building the (n-1) subtowers $\tilde{T}_{n=1,j}^{\theta}$. The parameter $\omega^{(n)}$ determines whether or not we exchange the $\theta = 0$ and $\theta = 1$ (n-1)-subtowers in stacking the n-towers.

<u>Lemma 3.2.</u> If

$$\limsup_{n\to\infty} \frac{1}{p_n - 1} \sum_{j=1}^{p_n - 1} \omega_j^{(n)} > 0$$

then T_1 is ergodic.

Proof. Let $p = \limsup_{n\to\infty} p_n$. Clearly the construction has growth coefficient ρ for any $\rho < p/4$. By combining several steps if necessary, we can make $p > 4$ and take $\rho > 1$. Since there are only two towers, any $A \in A(\zeta_n)$ with $0 < \mu_{1,n}(A) < 1$ is the base of one of the towers. Let $N_n = \sum_{j=1}^{p_n-1} \omega_j^{(n)}$. It follows from the construction that for n sufficiently large

$$(3.4) \qquad \frac{\mu_{1,n}((T_1)_{B_n} A' \cap \widetilde{A}'')}{\mu_{1,n}(A')\mu_{1,n}(\widetilde{A}'')} = \frac{4N_n(p_n-1)}{(p_n-3)^2}$$

$$\geq \frac{4}{p_n-1} \sum_{j=1}^{p_n-1} \omega_n^{(j)} .$$

The statement follows from Corollary 2.1 by taking the upper limit of (3.4). □

We recall that $T_0 : (X_0,\mu_0) \to (X_0,\mu_0)$ where X_0 is the unit interval $[0,1)$. Let $(X_1,\mu_1) = (X_0 \times \mathbb{Z}/2, \mu_0 \times \delta_{\mathbb{Z}/2})$, where $\delta_{\mathbb{Z}/2}$ is Haar measure on $\mathbb{Z}/2$, so that X_1 will consist of two intervals $X_0 \times \{0\}$ and $X_0 \times \{1\}$, each having measure $1/2$. These two intervals will be identified with the intervals I^0 and I^1 of Construction 2, so that the limit transformation T_1 can be considered a measure-preserving transformation of the Lebesgue space (X_1,μ_1).

Lemma 3.3. T_1 is the $\mathbb{Z}/2$ extension of (1.1) of T_0.

Proof. Let $S:(X_1,\mu_1) \to (X_1,\mu_1)$ be the map which interchanges the intervals I^0 and I^1. By the symmetry of Construction 2, this map interchanges the n-towers: T_n^0 and T_n^1, and the n-spacers: S_n^0 and S_n^1, for all n. Therefore S commutes with T_1. It follows from Lemma 1.1 that T_1 is a $\mathbb{Z}/2$ extension. Taking I^0 as a fundamental domain for S, and identifying it with X_0, it can be seen that T_1 modulo S is equal to T_0. □

Let us now analyze the cocycle γ in Construction 2. Since $\gamma: X_0 \to \mathbb{Z}/2$, we can express $\gamma = \chi_E$, thereby identifying γ with its support E. We will also consider the complementary cocycle $\gamma' = \gamma - 1$ and its support $\tilde{E} = X_0 \setminus E$.

As in §2, let R_n denote the roof of the n-tower (for Construction 1). During Step n+1 of the construction, the n-tower is cut into p_{n+1} subtowers, inducing a cutting of R_n into p_{n+1} subroofs R_n^j, $j = 1,\ldots,p_{n+1}$. It is easy to see that

$$R_{n+1} = R_n^{p_{n+1}}$$

and

$$\overline{R}_n = \bigcup_{j=1}^{p_{n+1}-1} R_n^j .$$

We define a partition η of R_1 by

$$\eta = \{R_n^j : n \geq 2,\ 1 \leq j \leq p_{n+1}-1\}.$$

For a semi-partition ξ, $|\xi|$ will denote the union of the elements in ξ. Beginning with the semi-partition ξ_n of X_0 into levels of the single n-tower T_n, we construct a semi-partition ξ_n' by removing the top level and another semi-partition ξ_n'' by removing the elements of ξ_n' contained in elements of ξ_{n-1}'. Clearly

$$|\xi_n''| \cap |\xi_k''| = \phi \quad \text{if} \quad n \neq k.$$

Let us define

$$\xi_* = \bigcup_{n=3}^{\infty} \xi_n'',$$

and note that $\mu_0(|\xi_*|) = 1$ since $\xi_n \to \varepsilon$, so that ξ_* is a partition of X_0.

Lemma 3.4. i) E and $\tilde{E}$ are measurable with respect to ξ_*.

ii) There is a surjective mapping B taking the collection of sets F measurable with respect to ξ_* onto the collection of subsets of R_1 measurable with respect to η in such a way that there exists a $\mathbb{Z}/2$ valued measurable function ψ_F such that

$$(3.5) \qquad \chi_{B(F)}(x) = \chi_F(x) + \psi_F(T_0x) - \psi_F(x).$$

ii) There is a one to one correspondence between sets $G \subseteq R_1$ which are measurable with respect to η and choices of the parameters $\omega, \nu \in \{0,1\}^{\mathbb{N}}$ for Construction 2 such that $\nu_{j,k}^{(n)} = 0$ for all n, j, k. More specifically, $R_n^j \subseteq G$ if and only if $\omega_j^{(n)} = 1$.

Proof. i) We consider the tower levels L^1 of the n-tower T_n^1 of Construction 2 such that $L^1 \subseteq I^\theta$ and $TL^1 \subseteq I^{\theta+1}$. For the corresponding level L^2 of T_n^2, $L^2 \subseteq I^{\theta+1}$ and $TL^2 \subseteq I^\theta$. By Lemma 1.1, the union of all such levels, modulo S, is the support E of Υ. E is clearly ξ_* measurable. It follows that $\tilde{E}$ is ξ_* measurable.

ii) We first consider the special case $F \in \xi_*$. Let t be the least non-negative integer such that $T^tF \subseteq R_1$. We define $B(F) = T^tF$ and note that $B(F) \in \eta$. If $t = 0$, $\psi_F = 0$ satisfies (3.5). Otherwise, let

$$H = \bigcup_{j=0}^{t-1} T^tF$$

and define $\psi_F = \chi_H$. It is straightforward that (3.5) is satisfied.

For a particular element $F' \in \eta$ there are only finitely many $F \in \xi_*$ such that $B(F) = \eta$. For a finite union $\hat{F} = F_1 \cup F_2 \cup \ldots \cup F_k$ of such sets, let $B(\hat{F}) = F'$ if k is odd and $B(\hat{F}) = \phi$ if k is even. Then (3.5) is satisfied by $\psi_{\hat{F}} = \sum_{j=1}^{k} \chi_{H_j}$.

The general case is handled by expressing F as a countable disjoint union of sets $\hat{F}$, taking $B(F)$ to be the union of the sets $\mathcal{B}(\hat{F})$ and letting ψ_F be the sum of the functions $\psi_{\hat{F}}$. Since the sets $\hat{F}$ form a countable disjoint collection, the sum converges to a measurable function. $\mathcal{B}$ is surjective since $\eta \subsetneq \xi_*$.

iii) This statement follows easily from the proof of i) and the construction. □

Considering now the parameters $t_{n,j}$, $n \geq 2$, $1 \leq j \leq p_n$ from Construction 1, let

$$(3.6) \qquad q_{n,j} = \begin{cases} t_{2,j} & \text{if } n = 2 \\ t_{n,j} + \sum_{k=2}^{n-1} t_{k,p_k} & \text{if } n \geq 2 \end{cases}$$

for $1 \leq j \leq p_{n-1}$. Let $\overline{\omega}_j^{(n)}$ be the residue of $q_{n,j} - \omega_j^{(n)} - \sum_{k=1}^{t_{n,j}} \nu_{j,k}^{(n)}$ modulo 2. We will write

$$\overline{\omega}^{(n)} = (\overline{\omega}_1^{(n)}, \ldots, \overline{\omega}_{p_n-1}^{(n)}) \in \{0,1\}^{p_n-1}$$

$$\overline{\omega} = (\omega^{(2)}, \omega^{(3)}, \ldots) \in \{0,1\}^{\mathbb{N}}.$$

Let T_1'' be the transformation constructed via Construction 2 using the parameters $\omega = \overline{\omega}$ and $\nu = 0$. □

<u>Lemma 3.5</u>. i) T_1'' is isomorphic to the complementary extension T_1' to T_1.

ii) Suppose T_0 is weak mixing.

If

$$(3.7) \qquad \limsup_{n\to\infty} \frac{1}{p_n-1} \sum_{j=1}^{p_n-1} \omega_j^{(n)} > 0,$$

and

$$(3.8) \qquad \limsup_{n\to\infty} \frac{1}{p_n-1} \sum_{j=1}^{p_n-1} \bar{\omega}_j^{(n)} > 0,$$

then T_1 is weak mixing.

Proof. i) since the cocycles $\gamma' = \chi_{\tilde{E}}$ and $\gamma'' \stackrel{\text{def}}{=} \chi_{B(\tilde{E})}$ are cohomologous, Lemma 1.2 implies that the corresponding extensions of T_0 are isomorphic. The extension corresponding to γ'' is, by Lemma 3.4 and (3.5), T_1''.

ii) By Lemma 3.2, the inequality (3.8) implies the ergodicity of T_1'', which by i) is isomorphic to T_1'. The inequality (3.7) implies the ergodicity of T_1. It follows from Lemma 1.3 that T_1 is weak mixing. □

We will now define the permutations α and β of $\Theta_3 = \mathbb{Z}/2 \times \mathbb{Z}/3$ for Construction 3. Let $\hat{\tau}(n,j;\theta) = \theta'$, where θ' is such that the top of $\hat{T}^{\theta}_{n,j}$ is a subset of $I^{\theta'}$. Denoting $\theta' = (\hat{\tau}_1(n,j;\theta), \hat{\tau}_2(n,j;\theta))$ and $\theta = (\theta_1,\theta_2)$, we define

$$(3.9) \quad \alpha(3,n,j,k;\theta) = (\alpha(2,n,j,k;\theta), \hat{\tau}_2(n,j;\theta)+k\phi(\hat{\tau}_1(n,j;\theta)))$$

where $\phi : \mathbb{Z}/2 \to \mathbb{Z}/3$ is given by (1.2).

To define β, let $\tilde{\tau}$ be defined in the same way as $\hat{\tau}$, using the tower $\tilde{T}^{\theta}_{n,j}$ instead of $\hat{T}^{\theta}_{n,j}$. Then let

(3.10) $\beta(3,n,j;\theta) = (\beta(2,n,j;\theta_1), \phi(\tilde{\tau}_1(n,j;\theta) + \tilde{\tau}_2(n,j;\theta)).$

Since the Lebesgue space (X_1, μ_1) has been identified with the intervals I^0 and I^1 of Construction 2, the Lebesgue space $(X,\mu) = (X \times \mathbb{Z}/3,\ \mu_1 \times \delta_{\mathbb{Z}/3}) = (X_0 \times \mathbb{Z}/2 \times \mathbb{Z}/3,\ \mu_1 \times \delta_{\mathbb{Z}/2} \times \delta_{\mathbb{Z}/3})$ can be identified with $[0,1)$ by identifying each of the intervals $I^{(\theta_1,\theta_2)}$ in Construction 3 with $X_0 \times \{\theta_1\} \times \{\theta_2\}$. In this way we make the limit transformation T of Construction 3 a measure-preserving transformation of (X,μ).

Lemma 3.6. T is the $\mathbb{Z}/3$ extension of T_1 corresponding to ϕ.

Proof. Let R be the transformation exchanging the intervals $I^{(\theta_1,\theta_2)}$ of Construction 3 according to the scheme

$$RI^{(\theta_1,\theta_2)} = I^{(\theta_1,\theta_2+1)}.$$

We will show that R commutes with T along the n-towers for any n. We prove this by induction. It is trivially true for the 1-towers. We now suppose it is true for the $(n-1)$ towers. Fixing j, consider the subtowers $T^{\theta}_{n,j}$ for $\theta \in \Theta_3$ and note that T commutes with R along these towers. It follows from (3.9) that T commutes with R along the spacers stacked on top of these towers to obtain $\tilde{T}^{\theta}_{n,j}$. Finally, we note that in stacking the towers $T^{\theta}_{n,j}$ on $\bar{T}^{\theta}_{n+1}(j-1)$ to obtain $\bar{T}^{\theta}_{n+1}(j)$, (3.10) implies that T, from the tops of $\mathcal{T}_{n+1}(j-1)$ to the bases of $\tilde{\mathcal{T}}_{n,j}$, commutes with R.

By considering (3.9) and (3.10), we obtain that T modulo R is equivalent to T_1 and that the cocycle is ϕ. □

Let us consider the group of permutations of $\Theta_3 = \mathbb{Z}/2 \times \mathbb{Z}/3$ generated by

$$a(\theta_1,\theta_2) = (\theta_1, \phi(\theta_1) + \theta_2)$$

and

$$b(\theta_1,\theta_2) = (\theta_1 + 1, \theta_2).$$

It is easy to see that $a^3 = b^2 = abab = 1$, so that this group is isomorphic to S_3, the group of symmetries of a triangle. One can also see by considering (3.9) and (3.10), that the permutations α and β used in Construction 3 belong to this group.

There is a dual notion of a permutation as an arrangement of objects. Let us assign the identity element $1 \in S_n$ to an initial arrangement of the six intervals $I^\theta \subseteq [0,1)$, $\theta \in \Theta_3$. For any n, this arrangement is inherited by the bases of the six n-towers T_n^θ, $\theta \in \Theta_3$. Let $\rho(n,\ell;\theta)$ be the element of S_3 corresponding to the arrangement of the ℓ'th levels of the n-towers, in the sense that if $L_{n,\ell}^\theta$ denotes the ℓ'th level of T_n^θ, then $L_{n,\ell}^\theta \subseteq I^{\rho(n,\ell;\theta)}$. Let $\delta(n,\ell;\theta) = \rho(n,\ell+1;\theta)\rho(n,\ell;\theta)^{-1}$. By (3.9) and (3.10), $\delta(n,\ell;\theta) = a$ or $\delta(n,\ell;\theta) = ba$.

The $(n+1)$-towers T_{n+1}^θ are made up of stacks of p_{n+1}

n-subtowers $\tilde{T}^{\theta}_{n,1},\ldots,\tilde{T}^{\theta}_{n,\rho_{n+1}}$. Let $\ell_1,\ldots,\ell_{\rho_{n+1}-1}$ be the levels of the towers T^{θ}_{n+1} corresponding to the tops of all except the last of these subtowers, and let $\pi(n,j;\theta) = \rho(n,\ell;+1;\theta)$. The permutations $\pi(n,j;\theta)$ enable us to keep track of the images of the roofs $\bar{R}_n$ of the towers T^{θ}_n under the mapping T. This information will be useful in establishing the ergodicity of T.

The permutations of Θ_3 in S_3 can be represented by a group of 6×6 zero-one matrices, generated by the matrices $G(a)$ and $G(b)$ corresponding to a and b. Putting the dictionary order on Θ_3, we have:

$$G(a) = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & 0 & \\ 1 & 0 & 0 & & & \\ \hline & & & 0 & 0 & 1 \\ & 0 & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \end{array}\right]$$

$$G(b) = \left[\begin{array}{c|c} 0 & Id_{3\times 3} \\ \hline Id_{3\times 3} & 0 \end{array}\right]$$

Let G be the set of all 6×6 matrices G with entries $g_{ij} \geq 0$. Clearly $G(a), G(b) \in G$. Let us denote by P the collection of all subsets $A \subseteq \{1,\ldots,6\}$. $|A|$ will denote the cardinality of A and $\tilde{A}$ the complement of A in P. For $G \in G$ and $A \in P$ we define

$$(3.11) \qquad G^A = \sum_{\substack{i \in A \\ j \in \tilde{A}}} g_{ij}$$

and

$$(3.12) \qquad \|G\| = \min_{A \in P} G^A.$$

Each set $A \in P$ can be identified with a set $A \in A(\zeta_n)$, (cf. §2), namely, the union of the bases of the towers indexed, in dictionary order, by the elements of $A \in P$. Combining (3.11) and (3.12), we have

$$(3.13) \qquad \min_{\substack{A \in A(\zeta_n) \\ 0<\mu_n(A)<1}} \frac{\mu_n(T_{B_n} A' \cap \tilde{A}'')}{\mu_n(A')\mu_n(\tilde{A}'')}$$

$$= \min_{A \in P} \frac{\dfrac{1}{6p_{n+1}} \displaystyle\sum_{j=1}^{p_{n+1}-1} G^A(\pi(n,j;\cdot))}{\dfrac{(p_{n+1}-1)^2}{p_{n+1}^2} |A| \cdot |\tilde{A}|}$$

$$\geq \frac{1}{54p_{n+1}} \left\| \sum_{j=1}^{p_{n+1}-1} G(\pi(n,j;\cdot) \right\|,$$

where $G(\pi(n,j;\cdot))$ denotes the representation of the permutation $\pi(n,j;\cdot)$ as a matrix in G.

<u>Lemma 3.7</u>. Let

$$M_n^1 = \text{card}\{j : 1 \leq j < p_{n+1},\ \pi(n,j;\cdot) = a \ \text{ or } \ a^2\}$$

$$M_n^2 = \text{card}\{j : 1 \leq j < p_{n+1},\ \pi(n,j;\cdot) = ba\}$$

$$M_n^3 = \text{card}\{j : 1 \leq j < p_{n+1},\ \pi(n,j;\cdot) = ba^2\}.$$

Then

$$(3.14) \quad \left\| \sum_{j=1}^{p_{n+1}-1} G(\pi(n,j;\cdot)) \right\| \geq \min(M_n^1, \max(M_n^2, M_n^3)).$$

The proof of this lemma follows straightforwardly from a case by case analysis, after observing that the matrices in G correspond naturally to directed graphs on six vertices with positively weighted edges. □

Lemma 3.8. There exists a choice of the parameters $\nu^{(n)}$ and $\omega^{(n)}$ such that

$$(3.15) \qquad M_n^1 \geq \tfrac{1}{4}(K_n-1) \quad \text{and} \quad (M_n^2 + M_n^3) \geq \tfrac{1}{4}(K_n-1).$$

Proof. We first show that it is possible to make $M_n^1 + M_n^2 + M_n^3 = K_n-1$. Let us consider the first K_n-1 subtowers of the six n-towers where spacers are used. Suppose that the jth subtowers $\tilde{T}_{n,j}^{\theta}$, $\theta \in \Theta_3$ are such subtowers and that for this j, $\pi(n,j;\ ) = 1$ or b.

For each θ,

$$(3.16) \qquad \pi(n,j;\theta) = \delta(n,\ell_j;\theta)\rho(n,\ell_j;\theta).$$

By changing $\nu_{j,t_{n,j}}^{(n)}$, we obtain instead of (3.16),

$$(3.16a) \qquad \pi(n,j;\theta) = \delta(n,\ell_j;\theta)b\rho(n,\ell_j\theta).$$

One can check that any pair of elements $c, d \in S_3$, such that $c, d \neq 1$ or b and $cd = 1$ or b, have the property that $cbd \neq 1$ or b. Thus after changing ν in this way we have $\pi(n,j;\theta) \neq 1$ or b.

In a similar way, the effect of changing $\omega_j^{(n)}$ is that (3.16) is replaced by

$$\text{(3.16b)} \qquad \pi(n,j;\theta) = b\,\delta(n,\ell_j;\theta)\,\rho(n,\ell_j;\theta).$$

This enables us to modify M_n^1, M_n^2 and M_n^3 enough to obtain (3.15). □

Proof of Theorem 1.1. Let T_0 be a rank 1 mixing transformation. By Lemma 3.1,

$$\beta = \limsup_{n\to\infty} K_n/p_n > 0.$$

Let $\alpha = \beta/1000$. Choose $\nu, \omega \in \{0,1\}^{\mathbb{N}}$ such that (3.7) and (3.8) are satisfied, and so that for infinitely many n, (3.15) is satisfied.

Combining (3.7) and (3.8) with Lemma 3.2, we find that T_1 is weak mixing, so that by Theorem 1.2, T_1 is mixing. Furthermore, (3.13),(3.14) and (3.15), together with Corollary 2.1 imply that T is ergodic. By Theorem 1.2 it suffices to show that T is weak mixing.

Recall from Chapter I, Lemma 2.3, that

$$L_2(X,\mu) = H_0 \oplus H_1 \oplus H_2,$$

where (3.17) is a U_T invariant orthogonal decomposition, and that there are no eigenfunctions in $H_0\setminus\{\text{constant}\}$ because T_1 is weak mixing. Furthermore, there is an isomorphism $V : H_1 \to H_2$ which is U_T equivariant.

Suppose there is an eigenfunction f for U_T. Then f has a nontrivial projection f' in one g of the subspaces H_1 or H_2. Vf' is another eigenfunction with the same eigenvalue, and f'/Vf' is a nonconstant invariant function. This contradicts the ergodicity of T. $\square$

CHAPTER III. HIGHLY NONSIMPLE SPECTRA OF FINITE MULTIPLICITY

In the introduction, the spectral invariant M_T was discussed. The main result of this section provides examples of transformations T where $\mathrm{Card}(M_T) > 2$. The transformations T can always be realized as interval exchange transformations. These results are contained in Theorem 2.1 and Corollary 3.1.

§1 does not pursue this goal directly, but instead consists of an algebraic analysis of the symmetry responsible for all the examples in Chapters I, II and III. In this sense it represents a generalization of the original Oseledec construction. §§2 and 3, as well as providing the proofs of Theorem 2.1 and Corollary 3.1, can be considered as a step toward extending the construction providing upper estimates in Chapter I to the more general situation in §1. In particular, Proposition 2.1 is likely to be true in greater generality than we prove here.

§1. A - B Extensions

In this section we will construct a class of double finite group extensions which will generalize the algebraic construction at the beginning of Chapter I. These extensions, which we will call A - B extensions, have the property that L_2 splits into an orthogonal sum of U_T invariant subspaces with U_T equivariant isomorphisms between various factors. This, in analogy to Chapter I, will imply lower bounds on the spectral multiplicity, and in particular, estimates on M_T.

Let A be a finite abelian group. We will apply the structure theorem for finite abelian groups, (cf. [ ]), in order to obtain a representation for A:

$$(1.1) \qquad \bigoplus_{j=1}^{k} \mathbb{Z}/\alpha_j$$

where $\{\alpha_j\}$ is a finite sequence of positive integers, subject to the conditions i) - iii) below, which determines A up to isomorphism. The conditions are:

i) $\alpha_j = p_j^{n_j}$, p_j prime, $n_j \geq 1$,

ii) $p_{j+1} \geq p_j$,

iii) if $p_{j+1} = p_j$ then $n_{j+1} \geq n_j$.

A character of A is a function $\chi : A \to \mathbb{C}$ such that $\chi(a+b) = \chi(a)\chi(b)$ and $|\chi| = 1$. The set $\hat{A}$ of all characters of A forms a group under the operation of pointwise multiplication called the dual group of A. It is well known that the groups A and $\hat{A}$ are isomorphic. In terms of (1.1), we can exhibit this isomorphism. Let $z,w \in A$ with $z = (z_1,\ldots,z_k)$ and $w = (w_1,\ldots,w_k)$ such that $w_j, z_j \in \mathbb{Z}/\alpha_j$. For $w \in A$ we define $\chi_w \in \hat{A}$ by

$$(1.2) \qquad \chi_w(z) = \exp 2\pi i \sum_{j=1}^{k} z_j w_j/\alpha_j .$$

Now let us consider the groups $A_L(A)$ and $A_R(A)$ of abelian group automorphisms acting, respectively, on left and right. We note that since A and $\hat{A}$ are isomorphic, so are their isomorphism groups. We will show that these correspond naturally to $A_L(A)$ and $A_R(A)$. For $\sigma \in A_L(A)$, we define $\sigma^* \in A_R(A)$ by

$$(1.3) \qquad \chi_{w\sigma^*}(z) = \chi_w(\sigma z)$$

for all $w, z \in A$.

<u>Lemma 1.1</u>. The mapping $\sigma \to \sigma^*$ from $A_L(A)$ to $A_R(A)$ is an isomorphism.

<u>Proof</u>. The mapping $*$ is well defined, and σ^* is an automorphism acting on the right since $(\gamma\delta)^* = \delta^*\gamma^*$. The symmetry of the construction implies that $*$ is invertible. □

Lemma 1.1 may be interpreted as showing what symmetries are present in the table of character values for A, (i.e. the matrix with entries $\chi_w(z)$ for $w, z \in A$).

We write $z \in A$ as a column vector $[z_j]$ of length k having jth entry $z_j \in \mathbb{Z}/\alpha_j$. In [2] it is shown that any $\sigma \in A_L(A)$ may be represented as a $k \times k$ matrix $[\sigma_{ij}]$ with integer entries σ_{ij}, subject to certain divisibility conditions involving the numbers α_ℓ in (1.1) which do not concern us. The action of σ on z is represented by the matrix multiplication

$$[(\sigma z)_i] = [\sigma_{ij}][z_j].$$

It is clear from (1.2) and (1.3) that the action of σ^* on $Z \in A$ is represented by the transpose of $[\sigma_{ij}]$:

$$[(z\sigma^*)_i]^T = [z_j]^T[\sigma_{ij}]^T = [z_j]^T[\sigma_{ji}].$$

This observation is summarized in the following lemma:

<u>Lemma 1.2.</u> Suppose $\mathcal{B} \subseteq A_L(A)$ and

$$\mathcal{B}^* = \{\sigma^* \in A_R(A) : \sigma \in \mathcal{B}\}.$$

Then the left action of $\mathcal{B}$ and the right action of $\mathcal{B}^*$ have the same orbits. In fact, $\sigma a = a\sigma^*$ for all $a \in A$, $\sigma \in \mathcal{B}$.

The following notation will be used for the orbits of the actions of $\mathcal{B}$ and $\mathcal{B}^*$ on A. Elements $a_1, \ldots, a_n \in A$ can be chosen so that

$$\mathcal{B}a_j \stackrel{\text{def}}{=} \{\sigma a_j : \sigma \in \mathcal{B}\} \stackrel{\text{def}}{=} \{a_j = a_j^0, \ldots, a_j^{s_j - 1}\},$$

and

$$a_j\mathcal{B}^* \stackrel{\text{def}}{=} \{a_j\sigma^* : \sigma \in \mathcal{B}\},$$

satisfying the following conditions:

i) $a_1 = 0$,

ii) $\mathcal{B}a_j = a_j\mathcal{B}^*$ (by Lemma 1.2),

iii) $\mathcal{B}a_j \cap \mathcal{B}a_{j'} = \emptyset$ if $j \neq j'$,

iv) $s_j \stackrel{\text{def}}{=} \text{card}(\mathcal{B}a_j) \leq \text{card}(\mathcal{B}a_{j+1})$.

The n-tuple $(s_1,\ldots,s_n)$ will be called the multiplicity of the action of $\mathcal{B}$ on A. Let $(m_1,\ldots,m_\ell)$ denote the ℓ-tuple of distinct values of s_j, $j = 1,\ldots,n$, written so that $m_j < m_{j+1}$. The ℓ-tuple $(m_1,\ldots,m_\ell)$ will be called the reduced multiplicity. The actions $\mathcal{B}$ and $\mathcal{B}^*$ have the same multiplicity and reduced multiplicity. We also note that $m_1 = s_1 = 1$ since any automorphism fixes $0 \in A$.

Definition 1.1. Let A and B be groups such that A is abelian and $A_L(A)$ has a subgroup $\mathcal{B}$ isomorphic to B. We will call A a B-group. In the case $B = \mathbb{Z}/m$, we will call A an m-group.

For $b \in B$, we denote the corresponding element in $\mathcal{B}$ by σ_b. It follows that for $b,c \in B$, $\sigma_{bc} = \sigma_b \circ \sigma_c$. For $a \in A$ and $b \in B$ define

$$(1.4) \qquad \overline{\phi}(a,b) = \sigma_b a \in A.$$

Then

$$\overline{\phi}(a,bc) = \sigma_{bc}a = (\sigma_b \circ \sigma_c)a$$
$$= \sigma_b\overline{\phi}(a,c)$$

and

$$(1.5) \quad \chi_w(\overline{\phi}(a,bc)) = \chi_w(\sigma_b\overline{\phi}(a,b)) = \chi_{w\sigma_b^*}(\overline{\phi}(a,c)).$$

We are now ready to describe the construction of A - B extensions. Let (X_0,μ_0) be a Lebesgue probability space and let T_0 be a measure-preserving transformation of (X_0,μ_0). Let A be a B group. For a measurable function

$\gamma : X_0 \to B$, we define a measure-preserving transformation T_1 on the space

$$(X_1, \mu_1) \;=\; (X_0 \times B,\ \mu_0 \times \delta_B),$$

where δ_B denotes normalized counting measure on B, by

$$T_1(x,y) \;=\; (T_0 x, \gamma(x)y).$$

The transformation T_1 is called the B extension of T_0 corresponding to γ. Since the function γ is unspecified, this construction is of a very general nature.

Let $\eta : X_0 \to A$ be a measurable function. Define $\phi : X_1 \to A$ by

$$(1.6) \qquad \phi(x,y) \;=\; \overline{\phi}(\eta(x),y) \;=\; \sigma_y \eta(x).$$

We repeat the previous construction starting with the transformation T_1, the group A, and the cocycle ϕ, obtaining the measure-preserving T of the space

$$(X,\mu) \;=\; (X_0 \times B \times A, \mu_0 \times \delta_B \times \delta_A),$$

defined

$$T(x,y,z) \;=\; (T_0 x, \gamma(x)y, \phi(x,y) + z).$$

We will call T the $A - B$ extension of T_0 corresponding to γ and η. In contrast to T_1, the transformation T has special properties which we now study.

Let us denote $H = L_2(X,\mu)$ and $H^* = L_2(X_1,\mu_1)$. We recall the definition of the induced unitary operator U_T

corresponding to T, namely

$$(1.7) \qquad U_T f(x,y,z) = f(T_0 x, \gamma(x)y, \phi(x,y)+z).$$

As in Chapter I, we will analyze the U_T invariant subspaces of H. For $a \in A$, let

$$(1.8) \qquad H_a = \{\chi_a(z) f(x,y) : f \in H^*\}.$$

<u>Lemma 1.3</u>. The subspaces H_a, $a \in A$ are U_T invariant. Furthermore

$$H = \bigoplus_{j=1}^{k} \bigoplus_{\ell=0}^{s_j-1} H_{a_j^\ell}$$

is an orthogonal decomposition, where the action of U_T on H_0 is equivalent to the action of U_{T_1} on H^*.

<u>Proof</u>. The first statement follows from an obvious computation. To prove the second statement, we note that

$$A = \bigcup_{j=1}^{n} \mathcal{B}a_j$$

and that $\mathcal{B}a_j \cap \mathcal{B}a_{j'} = \emptyset$ if $j = j'$. Thus

$$\hat{A} = \{\chi_{a_j^\ell} : j = 1,\ldots,n;\ \ell = 0,\ldots,s_j-1\}.$$

The second statement then follows from elementary Fourier analysis. The last statement follows from the definitions of the spaces and operators involved. □

For each pair a_j^ℓ, $a_j^{\ell'} \in \mathcal{B}a_j$, there exists a (not necessarily unique) $b \in B$ such that $\sigma_b a_j^\ell = a_j^{\ell'}$, and by

Lemma 1.2, $a_j^{\ell}\sigma_b^* = a_j^{\ell'}$. We define the operator

$$S_{b,j,\ell} : H_{a_j^{\ell}} \to H_{a_j^{\ell'}}$$

by

$$(1.9) \qquad S_{b,i,\ell}(\chi_{a_j^{\ell}}(z)f(x,y)) = \chi_{a_j^{\ell}b^*}(z)f(x,by).$$

<u>Proposition 1.1</u>. Suppose $\sigma_b a_j^{\ell} = a_j^{\ell'}$ and that $\gamma(x)$ commutes with b for μ_0 almost every $x \in X_0$. Then

$$(1.10) \qquad S_{b,j,\ell} \circ U_T\Big|_{H_{a_j^{\ell}}} = U_T\Big|_{H_{a_j^{\ell'}}} \circ S_{b,j,\ell}.$$

We note that the condition on γ is automatically satisfied if B is abelian. Before proving Proposition 1.1, we state the following easy corollary which is the main result of this section.

<u>Corollary 1.1</u>. If B is abelian and if a and a' are in the same $\mathcal{B}$ orbit, then the spectrum of U_T in H_a is identical to the spectrum of U_T in $H_{a'}$.

<u>Proof of Proposition 1.1</u>. We have from (1.7), and (1.9) that

$$(1.11) \quad S_{b,j,\ell} \circ U_T\chi_{a_j^{\ell}}(z)f(x,y)$$
$$= S_{b,j,\ell}\chi_{a_j^{\ell}}(\phi(x,y)+z)f(T_0x,\gamma(x)y)$$
$$= \chi_{a_j^{\ell}\sigma_b^*}(z)\chi_{a_j^{\ell}}(\phi(x,by))f(T_0x,\gamma(x)by).$$

Similarly,

$$(1.12)\quad U_T \circ S_{b,j,\ell}\chi_{a_j^\ell}(z)f(x,y)$$

$$= U_T\chi_{a_j^\ell\sigma_b^*}(z)f(x,by)$$

$$= \chi_{a_j^\ell\sigma_b^*}(z)\chi_{a_j^\ell\sigma_b^*}(\phi(x,y))f(T_0x,b\gamma(x)y).$$

The equality of (1.11) and (1.12) follows from (1.5) and (1.6), and the fact that $\gamma(x)b = b\gamma(x)$ μ_0 almost everywhere. $\square$

§2. Examples

Two examples of $A - B$ extensions will be constructed here to demonstrate how results on spectral multiplicity can be achieved using this construction. The first of these examples will yield an infinite family of transformations T such that $\mathrm{card}(M_T) > 2$. Like the examples in Chapters I and II, these transformations will be double cyclic group extensions. The second example provides an alternate proof of part of the main result of Chapter I. It is included to show that results using this construction need not be confined to cases where the group A is cyclic.

Both of the examples rely on the following notion:

Definition 2.1. An $A - B$ extension of a measure-preserving transformation T_0 of (X_0, μ_0) will be called special if

i) A is an m-group, i.e. $B = \mathbb{Z}/m$;

ii) the orbit of $0 \in A$ is the only $\mathcal{B}$ orbit with cardinality 1;

iii) the function $\eta(x) = \eta \in A$ is a constant function, so that $\phi(x,y) = \phi(y)$;

iv) if $R(\phi)$ denotes the range of ϕ, then for any a and a' in different $\mathcal{B}$ orbits,

$$\chi_a(R(\phi)) \cap \chi_{a'}(R(\phi)) = \emptyset. \tag{2.1}$$

A B-group A such that i) and ii) hold, and such that there exists an element $\eta \in A$ so that iv) also holds, will be called a special B group for η.

Proposition 2.1. Let A be a special B group for η. Then for a generic set of pairs (T_0,γ) in the sense of §5, Chapter I, the special $A-B$ extension T of T_0 corresponding to γ and η is ergodic, has continuous singular sepctrum, and M_T is equal to the reduced multiplicity of the action of B on A.

The proof of Proposition 2.1 is given in the next section. A similar result may be true in general for $A-B$ extensions when the group B is abelian. The genericity, in this case is defined in terms of the triples (T_0,γ,η).

We proceed to construct the two examples

Example 2.1. Let $A = \mathbb{Z}/n$ and $\mathcal{B} = \mathbb{Z}/n^x$, the group of units of A, acting on A by multiplication. It is well known that $\mathcal{B} = \{z \in \mathbb{Z}/n : (z,n) = 1\}$, and the function $\phi(n) = \mathrm{card}(\mathcal{B})$ is called the Euler function [13]. It is easy to determine the orbit structure of this action. Let

$$1 = d_1 < d_2 <\dots< d_k = n$$

be a complete set of divisors for n and let $d_j' = n/d_j$. For each j there is an orbit $\mathcal{B}d_j' = \{bd_j' : b \in \mathcal{B}\}$ of cardinality $\phi(d_j)$. $\mathcal{B}d_j'$ consists exactly of those elements of $\mathbb{Z}/n$ divisible by $d_{j'}$, but not divisible by any other divisors of n which do not divide $d_{j'}$. Furthermore, since

$$\sum_{j=1}^{k} \Phi(d_j) = n,$$

every orbit is of this type. Thus A is a $\mathcal{B}$ group.

Next, we consider the conditions i)-iv), $\mathcal{B}$ is cyclic [ ] if and only if $n = 2, 4, p^r$ or $2p^r$, where p is an odd prime and $r \geq 1$. The cases 2 and 4 are excluded as uninteresting, and the cases $2p^r$ are excluded since they do not satisfy ii). The cases $n = p$ are covered by Chapter I. Thus $n = p^r$ where p is an odd prime and $r > 1$, so that $B = \mathbb{Z}/m$ where $m = \Phi(p^r) = (p-1)p^{r-1}$. The reduced multiplicity for the action of is $(1, p-1, (p-1)p, \ldots, (p-1)p^{r-1})$.

Taking η to be any generator of $\mathcal{B}$, the condition iv) is satisfied since for any $a \in \mathcal{B}d_j'$, $a' \in R(\phi)$, $aa' \in \mathcal{B}d_j'$ and $\chi_a(a') = \exp 2\pi i aa'/n$.

Corollary 2.1. Let p be an odd prime and $r > 1$. There exists an ergodic measure-preserving transformation T with continuous spectrum such that

$$M_T = \{1, p-1, (p-1)p, \ldots, (p-1)p^{r-1}\}.$$

Example .2. Recall that for the finite field $GF(p^n)$ of order p^n, the multiplicative group of units $GF(p^n)^{\times}$ is isomorphic to the cyclic group $\mathbb{Z}/(p^n-1)$, [12]. (Here p denotes an arbitrary prime). Taking $A = GF(p^n)$ $B = \mathbb{Z}/(p^n-1)$ and $\mathcal{B} = GF(p^n)^{\times}$, which acts on A by multiplication, it is easy to see that A is a special B group, for any $\eta \neq 0$. The reduced multiplicity in this case is $(1,p^n-1)$.

§3. Skew Dynamical Systems and Approximation

In the following discussion, the measure spaces (X_0,μ_0) (X_1,μ_1) and (X,μ) will either be Lebesgue probability spaces or finite sets with normalized counting measure.

Definition 3.1. Let T be a measure-preserving transformation of (X,μ) and let $\theta : X \to C$ be a measurable function such that $|\theta(x)| = 1$ μ almost everywhere. Suppose that $H \subseteq L_2(X,\mu)$ is an invariant subspace for the unitary operator $V_{T,\theta}$ defined on $L_2(X,\mu)$ by

$$(3.1) \qquad V_{T,\theta}f(x) = \theta(x)f(Tx).$$

The operator $V_{T,\theta}\big|_H$ will be called a skew dynamical system.

The following is the most natural example of a skew dynamical system. Let T_1 be a measure-preserving transformation of (X_1,μ_1), let A be an abelian group, and let $\phi : X_1 \to A$ be a measurable function. Let T be the A extension of T_1 with cocycle ϕ:

$$T : (X,\mu) \to (X,\mu) \overset{\text{def}}{=} (X_1 \times A, \mu_1 \times \delta_A)$$

where

$$(3.2) \qquad T(w,z) = (T_1 w, \phi(w) + z).$$

We consider the subspace

$$H_a = \{\chi_a(z)f(w) : f \in L_2(X_1,\mu_1)\}.$$

The restriction

$$U_T\big|_{H_a} \tag{3.3}$$

is the skew dynamical system $V_{\overline{T}_1,\theta_a}$ where $\overline{T}_1$ is T_1 extended trivially to (X,μ) and the function $\theta_a(z,w) = \chi_a(\phi(w))$. We will call this sort of skew dynamical system an extended dynamical system.

We now consider a notion of approximation for extended dynamical systems.

<u>Definition 3.2</u>. Suppose that the transformation T_1 admits an m cyclic approximation by periodic transformations $T_{1,n}$ with speed $o(1/n)$ in the sense of §2 of Chapter I. Let ξ_n denote the partition whose elements are permuted by $T_{1,n}$. Let T denote the group extension (3.2) of T_1 for the cocycle ϕ. Suppose there exist ϕ_n, constant on the elements of ξ_n, such that $\|\phi_n-\phi\| = o(1/n)$. Let $T_{2,n}$ denote the group extension (3.2) of $T_{1,n}$ for the cocycle ϕ_n. We will say that the extended dynamical system $U_T\big|_{H_a}$ admits an m-cyclic approximation $U_{T_{2,n}}\big|_{H_a}$ with speed $o(1/n)$.

It is easy to see that the subspace H_a is invariant for both U_T and $U_{T_{2,n}}$. Results on approximations of transformations go through almost verbatim to the case of extended dynamical systems.

<u>Lemma 3.1</u>. Suppose the extended dynamical system $U_T|_{H_a}$ admits a cyclic approximation with speed $o(1/n)$. Then $U_T|_{H_a}$ has simple spectrum.

The proof of this lemma is identical to the proof of Theorem 3.1 in [18].

<u>Lemma 3.2</u>. Let $a_1, a_2 \in A$. Suppose each of the extended dynamical systems $U_T|_{H_{a_i}}$, $i = 1,2$, admist an m-cyclic approximation $H_{T_{2,n}}|_{H_{a_i}}$ with speed $o(1/n)$, with the following additional property: Le' $mq_n = \text{card}(\xi_n)$. For each n, the operators

$$U_{T_{2,n}}^{q_n}\Big|_{H_{a_i}}, \quad i = 1,2$$

have a completely discrete spectrum with eigenvalues $\Lambda_i = \{\lambda_{i,1},\ldots,\lambda_{i,k_i}\}$ independent of n. If $\Lambda_1 \cap \Lambda_2 = \emptyset$ then the maximal spectral types ρ_1 and ρ_2 of $U_T|_{H_{a_1}}$ and $U_T|_{H_{a_2}}$ are mutually singular.

<u>Proof</u>. We first consider the operator $U_T|_{H_{a_1}}$ which, for simplicity, we denote by U_1 the speed of approximation implies (cf. [18], Theorem 3.?) that for any $g^1 \in H_{a_1}$, there exist functions $g^1_{n,j} \in H_{a_1}$. $j - 1,\ldots,k_1$ such that

$$g^1 = g^1_{n,1} + \ldots + g^1_{n,k_1}$$

and

$$\| U_1^{q_n} g - \sum_{j=1}^{k_1} \lambda_{1,j} g^1_{n,j} \| = \varepsilon_{1,n} \to 0.$$

Let $g^1 \in H_{a_1}$ be a function of maximal spectral type for U_1 in H_{a_1} such that $\|g\| = 1$. Let ρ_1 be a measure representing the maximal spectral type of U_1 in H_{a_1} such that $\rho_1(\mathbb{T}) = 1$.

$$(3.4) \qquad \left| \int_{-\pi}^{\pi} e^{iq_n t} d\rho_1(t) - \sum_{j=1}^{k_1} \|g^1_{n,j}\|^2 \lambda_{1,j} \right|$$

$$= \left| (U_j^{q_n} g, g) - \left(\sum_{j=1}^{k_1} \lambda_{1,j} g^1_{n,j}, g \right) \right|$$

$$\leq \| U_1^{q_n} q - \sum_{j=1}^{k_1} \lambda_{1,j} g^1_{n,j} \| = \varepsilon_{1,n} \to 0.$$

In a similar way,

$$(3.5) \qquad \left| \int_{-\pi}^{\pi} e^{iq_n t} \, d\rho_2(t) - \sum_{j=1}^{k_2} \|g^2_{n,j}\|^2 \lambda_{2,j} \right| = \varepsilon_{2,n} \to 0,$$

where ρ_2 is a probability measure representing the maximal spectral type of U_2 on H_{a_2}, and $g^2_{n,1} + \ldots + g^2_{n,k_2}$ is a function of maximal spectral type.

We first consider the case

$$\left| \int_{-\pi}^{\pi} e^{it} \, d\rho(t) - 1 \right| < \varepsilon,$$

so that

$$1 - \varepsilon < \int_{-\pi}^{\pi} \cos t d\rho(t) \leq (1 - \rho(-\delta,\delta))\cos \delta + \rho(-\delta,\delta)$$

$$\leq \delta^2/2 \; \rho(-\delta,\delta) + 1 - \delta^2/2 + \delta^4/24,$$

and

$$\rho(-\delta,\delta) \geq 1 - \delta^2/12 - 2\varepsilon/\delta^2.$$

Taking $\delta = \varepsilon^{1/4}$, we obtain

$$\rho(-\varepsilon^{1/4},\varepsilon^{1/4}) \geq 1 - \frac{25}{12}\varepsilon^{1/2}. \tag{3.6}$$

Let $\varepsilon_n = \max(\varepsilon_{1,n},\varepsilon_{2,n})$ and $d = \mathrm{dist}(\Lambda_1,\Lambda_2)$. For sufficiently large n, $\varepsilon_n^{1/4} < d$. Define

$$\Omega_{n,j} = \bigcup_{e^{itq_n}\in\Lambda_i} (t - \varepsilon_n^{1/4} q_n^{-1}, t + \varepsilon_n^{1/4} q_n^{-1}).$$

For all n, $\Omega_{1,n} \cap \Omega_{2,n} = \emptyset$. Furthermore, by applying (3. 5) and either (3.4) or (3.5),

$$\rho_i(\Omega_{n,i}) \geq \sum_{j=1}^{k_1} \|g_{n,j}^i\|^2 (1 - \frac{25}{12}\varepsilon_{i,n}^{1/2})$$

$$= 1 - \frac{25}{12}\varepsilon_{i,n}^{1/2}, \quad i = 1,2.$$

Thus $\rho_1(\Omega_{n,1}) \to 1$ as $n \to \infty$ and $\rho_2(\Omega_{n,2}) \to 1$ as $n \to \infty$. It follows that $\rho_1 \perp \rho_2$.

We now study the combinatorics of extended dynamical systems which arise from special $A - B$ extensions of transformations T_0 admitting cyclic approximations with speed $o(1/n)$.

Let T_0 be a cyclic permutation of a q element set X_0 with normalized counting measure μ_0, and let $(X_1,\mu_1) = (X_0 \times \mathbb{Z}/m, \mu_0 \times \delta_{\mathbb{Z}/m})$. By Chapter I, Lemma 4.1 and 4.2 there are cocycles $\gamma : X_0 \to \mathbb{Z}/m$ so that the $\mathbb{Z}/m$ extension T_1 of T_0 corresponding to γ is either type 1 or type 2, in the sense of Chapter I Definitions 5.1 and 5.2.

Let A be a special B group for $B = \mathbb{Z}/m$ and let T be the special $A-B$ extension corresponding to the pair (T_0,γ) above.

<u>Lemma 3.3</u>. Let $f_1 \in L_2(X_1,\mu_1)$ be the characteristic function of $x_1 \in X_1$. Let $a \in A$ and $f(x,y,z) = \chi_a(z)f_1(x,y)$. If γ is type 1 then the vectors $U_T^j f$, $j = 0,\dots,mq-1$ are orthogonal.

<u>Proof</u>. This follows easily from the following facts: $U_T\big|_{H_a}$ is a skew dynamical system; T_1 is a cyclic permutation; f_1 is the characteristic function of a single element of X_1. □

<u>Lemma 3.4</u>. Suppose γ is type 2, and let $a \in A$. Then the set of eigenvalues for the linear transformation $U_T^q\big|_{H_a}$ is $\chi_a(R(\phi))$. Thus if a and a' lie on different orbits of B, the linear transformations $U_T^q\big|_{H_a}$ and $U_T^q\big|_{H_{a'}}$ have disjoint sets of eigenvalues.

Proof. The first statement follows from the proof of Proposition 4.1 in Chapter I and the definition of a type 2 cocycle, Chapter I Definition 5.2. The second statement follows from iv) of Definition 2.1. □

Proof of Proposition 2.1. It follows from Chapter I, §6, that for a generic set of pairs (T_0,γ), T_1 has a good approximation such that the approximating cocycles γ_u alternate between type 1 and type 2. This, combined with Lemma 3.3 and Lemma 3.4 implies that, alternately in n, the hypotheses of Lemma 3.1 and Lemma 3.2 are satisfied.

Lemma 3.1 implies that the spectrum in each subspace H_a, $a \in A$, is simple. Lemma 3.2 and Lemma 3.4 imply that if a and a' lie in different orbits of the action of B, the spectrum of U_T in H_a is singular with respect to the spectrum of U_T in H_a.

Since all the spectra are disjoint from the spectrum in H_0, there are no invariant functions outside H_0. By Chapter I, Lemma 5.2, T_2 has continuous spectrum and so the only invariant functions in H_0 are constants. This implies that T is ergodic. By iii) of Definition 3.1, and Proposition 1.1, every H_a, $a \neq 0$ has a spectrum equivalent to the spectrum in at least one other $H_{a'}$, there can be no eigenfunctions. If there were, then the multiplicity of the associated eigenvalue would be at least 2, contradicting the ergodicity of T. Thus each subspace H_a has simple continuous spectrum, disjoint from the spectra on subspaces $H_{a'}$ where a and a' lie on different $\mathcal{B}$ orbits. It follows from Corollary 1.1 that M_T is the reduced multiplicity of the $\mathcal{B}$ action. □

Corollary 3.1. For any odd prime p and r > 1 there exists an interval exchange transformation T such that

$$M_T = \{1, p-1, (p-1)p, \ldots, (p-1)p^r\}.$$

The corollary can be proven easily by making the appropriate modifications of the arguments in §7 of Chapter I.

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