Monatsh. Math. 132, 237–253 (2001)

Monatshefte für
Mathematik

Modeling Ergodic, Measure Preserving Actions on $\mathbb{Z}^d$ Shifts of Finite Type

By

E. Arthur Robinson, Jr.[1] and **Ayşe A. Şahin**[2]

[1] George Washington University, Washington DC, USA
[2] DePaul University, Chicago IL, USA

(Received 21 July 1999; in revised form 4 November 2000)

Abstract. In this paper we show that $\mathbb{Z}^d$ shifts of finite type satisfying a strong topological mixing property are universal models for ergodic measure preserving $\mathbb{Z}^d$ dynamical systems.

2000 Mathematics Subject Classification: 28D15, 37B10
Key words: Ergodic theory, $\mathbb{Z}^d$ actions, entropy, symbolik dynamics

1. Introduction

The theory of multi-dimensional shifts of finite type is currently an area of great activity (see for example [5], [11], [13], [15], and [26]), due in part to some remarkable differences from the one-dimensional case. In this paper we consider a property that certain mixing multi-dimensional shifts of finite type have in common with one-dimensional shifts of finite type: a large and varied a set of invariant measures. Invariant measures for multi-dimensional shifts of finite type play an important role in statistical mechanics (see [2], [3]) and also in the theory of tiling dynamical systems (see [22]). However, the problems discussed in this paper are formulated most naturally in terms of *topological models* for abstract measure preserving dynamical systems.

Given a continuous $\mathbb{Z}^d$ action S on a compact metric space Y, let $\mathscr{M}(Y,S)$ denote the set of all S-invariant Borel probability measures on Y. Suppose that we want to model an ergodic measure preserving $\mathbb{Z}^d$ action (X,μ,T), up to metric isomorphism, by (Y,ν,S) for some $\nu \in \mathscr{M}(Y,S)$. The variational principle (see [14]) provides a necessary condition $h(X,\mu,T) \leqslant h(Y,S)$ relating the metric entropy of (X,μ,T) and the topological entropy of (Y,S). We say that (Y,S) is a *universal model* if for every aperiodic, ergodic, and measure preserving $\mathbb{Z}^d$ action (X,μ,T), with $h(X,\mu,T) < h(Y,S)$, there exists $\nu \in \mathscr{M}(Y,S)$ so that (X,μ,T) and

The first author wishes to thank the NSF for support from grant number DMS-9501103. He also thanks the Erwin Schrödinger Institute in Vienna, Austria and the North Dakota State University for their hospitality.

The second author wishes to thank the NSF for support from grant number DMS-9501103 and NDEPSCoR for support through NSF grant number OSR-9452892.

(Y, ν, S) are metrically isomorphic. In this paper we ask the question: when is a $\mathbb{Z}^d$ shift of finite type (Y, S) a universal model?

In the case $d = 1$, the Krieger Generator Theorem [9] states that topological mixing is a sufficient and necessary condition for a shift of finite type (Y, S) to be a universal model. In the theory of multi-dimensional shifts of finite type, one of the complications is that topological mixing does not guarantee positive topological entropy (see [12]) and thus a stronger mixing assumption is often necessary. In this paper we show that a mixing condition called the *uniform filling property*, together with an assumption about periodic points, is a sufficient condition for a multi-dimensional shift of finite type to be a universal model. Our main result is the following.

Theorem 1.1. *Let (Y, S) be a $\mathbb{Z}^d$ shift of finite type with the uniform filling property and with dense periodic points. Then (Y, S) is a universal model.*

When $d = 2$ it is known that the uniform filling property implies that periodic points are dense (see [27]), so we have the following.

Corollary 1.2. *Let (Y, S) be a $\mathbb{Z}^2$ shift of finite type with the uniform filling property. Then (Y, S) is a universal model.*

The uniform filling property (hereafter abbreviated UFP) is an essential ingredient throughout this paper. Roughly, it says there exists a finite set $F \subseteq \mathbb{Z}^d$ such that for any $y_1, y_2 \in Y$ and for any rectangle $R \subseteq \mathbb{Z}^d$ there exists $y \in Y$ such that $y = y_1$ on R and $y = y_2$ on the complement of $R + F$. There are many natural examples of shifts of finite type with the UFP, including the *iceberg model* [2] and the higher dimensional *golden mean* shift. The UFP implies topological mixing and positive topological entropy and yet it is weak enough to allow for interesting phenomena such as nonuniqueness of measures of maximal entropy (see [2]). However, there are also many well known examples of positive entropy mixing $\mathbb{Z}^d$ shifts of finite type that do not have the UFP. Thus the strong reliance of our arguments on the UFP raises the question of whether or not any of these are universal models. For example, we don't know if the 3 *color chessboard* shift (see [12]) or the shift corresponding to *domino tilings* of the plane (see [26]) are universal models.

The UFP is one of the many *specification properties* for $\mathbb{Z}^d$ shifts that appear in the literature. In principle, the UFP is a little weaker than Ruelle's *weak specification* [24], which is equivalent to Ward's *strong specification*[1] [27], and Burton and Stief's *strong irreducibility* [2]. Essentially, these properties differ from the UFP in that R need not be a rectangle (or even convex). We say these properties are stronger "in principal" because they are clearly philosophically the same as the UFP, even if not exactly so. In this paper we use the UFP because it is geometrically natural in the context of the Rohlin tower arguments that we use throughout.

Lightwood [11] (see also [10]) recently obtained a generalization of Krieger's topological embedding theorem for one-dimensional shifts of finite type to the $\mathbb{Z}^2$

[1]For Ruelle [24], *strong specification* means being able to specify *periodic orbits*, implying in particular that periodic points are dense, from which our Theorem 1.1 follows.

case – a result clearly related to ours, although aparently neither follows directly from the other. Lightwood requires the UFP, which he calls *square mixing* and he also requires an extension property that he calls *square filling*. Interestingly, such extension properties are unnecessary for our result. He shows (in the case $d = 2$) that square filling plus topological mixing implies the UFP (i.e., his square mixing), but that the UFP does not imply square filling [11]. Schmidt [25] studies a property called **n**-*specification* which includes the condition that the Gibbs relation is topologically transitive. The UFP actually implies that the Gibbs relation is minimal, but apparently it is not enough to imply **n**-specification without an additional condition like square filling. Also, **n**-specification does not imply the UFP [25].

To place our result in context, we give a brief overview of the theory of topological models. It is well known that every finite entropy ergodic measure preserving transformation T has a topological model (i.e., a measure preserving homeomorphism S of a compact metric space metrically isomorphic to T). A strong version of this fact is the Jewett-Krieger Theorem (see [4]), which says that S can be chosen to be strictly ergodic. Rosenthal [20] proved a generalization of the Jewett–Krieger Theorem to ergodic measure preserving $\mathbb{Z}^d$ actions. In such cases the topological model has a single invariant measure; the opposite of what we seek in this paper.

As we noted above, when $d = 1$, Krieger's Finite Generator Theorem shows that the full shift (on at least 2 symbols) is a universal model ([8], [9]), and the same is true for one-dimensional mixing shifts of finite type (see [4]). On the other hand, non-mixing shifts of finite type are not universal models. More generally, any S with a uniquely ergodic factor is not a universal model. In higher dimensions, $\mathbb{Z}^2$ full shifts are universal models by Rosenthal's Finite Generator Theorem [20]. For $d > 2$, Rosenthal's proof does not quite meet our stated entropy condition, but a proof of the general case follows from Kammeyer [6].

Theorem 1.1 can be interpreted as a multi-dimensional generalization of Krieger's theorem for one-dimensional mixing shifts of finite type. However, our proof requires different techniques than either the one-dimensional case or the proofs in [20] and [6]. With the former, the differences are due to the more complicated combinatorics involved in higher dimensional shifts of finite type. With the latter the differences are due to the fact that we are working with a subshift, and hence our copying arguments are more complicated. The proof in this paper uses the theory of *joinings* in the style of Burton–Rothstein [1], as presented by Rudolph [23]. We show that in a certain space of joinings the measures supported on the graphs of isomorphisms are a dense G_δ set. The joinings approach is well suited to the multi-dimensional setting because convergence is handled by the Baire Category Theorem, avoiding the need to nest different approximation steps. The main ingredient in the argument is a combinatorial result called the *Copying Lemma*. The core of the paper involves obtaining a Copying Lemma valid for $\mathbb{Z}^d$ shifts of finite type satisfying the UFP.

Our interest in this problem was inspired by a problem in the theory of tiling dynamical systems. One can consider a tiling dynamical system to be an $\mathbb{R}^d$ symbolic dynamical system, and there is a natural notion of a shift of finite type in

this context (see [16]). Rudolph's one-dimensional Two Step Coding Theorem says that a finite entropy ergodic measure preserving flow can be represented as a flow under a function with just two return times [21]. This can be interpreted to mean that any flow can be modeled by an invariant measure on a one-dimensional full tiling shift on two tiles. Rudolph's proof has two parts. The first part is a tiling lemma, which shows that the orbits of the flow can be tiled by orbits of the tiling system. The second part is a generator theorem, which shows that the partition into tiles (the "time-zero" partition for the tiling dynamical system) can be chosen to be a generating partition. In the case $d > 1$, Rudolph constructed a tiling dynamical system that is an $\mathbb{R}^2$ shift of finite type, and he obtained a tiling lemma for that system [22]. However, he did not prove a generator theorem for $d > 1$. Such a generator theorem would be very interesting, but a proof is not immediate. In particular, the Rudolph tiling shift does not satisfy the uniform filling property.

The organization of the paper is as follows. In the next section we provide the notation and definitions which we will be using in the paper. In Section 3 we discuss the uniform filling property and some of its consequences. Section 4, where we prove the Copying Lemma, is where we do most of the work. Finally in Section 5 we show how the Copying Lemma gives the proof of the main theorem. This proof is standard for the Burton–Rothstein method. We include it for the sake of completeness.

The authors wish to acknowledge helpful conversations with Bob Burton, Nelson Markley, Daniel Rudolph and Klaus Schmidt. In particular, D. Rudolph suggested the Burton–Rothstein approach.

2. Notation and Definitions

For a vector $\vec{v} \in \mathbb{R}^d$ let $\|\vec{v}\| = \max|v_i|$. We say $\vec{v} < \vec{w}$ if $v_i < w_i$ for $i = 1, \ldots, d$. If $v_i > n$ for all i we write $\vec{v} > n$. For $\vec{v} < \vec{w}$ we define the *box*

$$[\vec{v}, \vec{w}] = \prod_{j=1}^{d} [v_i, w_i],$$

with similar notation for (half) open boxes. We write $B_n = [-\vec{n}, \vec{n}]$ for $\vec{n} = (n, n, \ldots n), n \geqslant 0$. For a real number r, let $[\vec{v} - r, \vec{w} + r] = \prod_{j=1}^{d} [v_i - r, w_i + r]$. When $r > 0$ this is the box $[\vec{v}, \vec{w}]$ together with a *collar* of thickness r. When $r < 0$, $[\vec{v} - r, \vec{w} + r] \subset [\vec{v}, \vec{w}]$.

Let (X, μ, T) be a measure preserving $\mathbb{Z}^d$ action. A *Rohlin tower* with *shape* $R \subset \mathbb{Z}^d$ and *base* $F \subset X$, is a disjoint union $T^R F = \bigcup_{\vec{v} \in R} T^{\vec{v}} F$. If $\mu(T^R F) > 1 - \delta$, the tower has *error* $< \delta$. For $E \subset F$ we call $T^R E$ a *slice* of the tower $T^R F$. We call $T^R\{x\}, x \in X$ the *slice based at* x . If $S \subset R$ we call $T^S F$ a *subtower* of $T^R F$. The *Rohlin Lemma* says that if (X, μ, T) is a periodic and ergodic then for any $\delta > 0$ and for any $n \in \mathbb{N}$, there is a Rohlin tower with shape B_n and error $< \delta$ [7].

Let A be a finite set (the *alphabet*) and let $Y_A = A^{\mathbb{Z}^d}$. For $\vec{w} \in \mathbb{Z}^d$ we denote the $\vec{w}$th entry of $y \in Y_A$ by $y[\vec{w}]$. Let S be the $\mathbb{Z}^d$ *shift* on Y_A defined by $(S^{\vec{v}} y)[\vec{w}] = y[\vec{w} + \vec{v}]$ for $\vec{v}, \vec{w} \in \mathbb{Z}^d$. In the product topology Y_A is compact, metrizable, and the shift S is a continuous $\mathbb{Z}^d$ action. A $\mathbb{Z}^d$ *subshift* (Y, S) is the restriction of S to a closed S-invariant subspace $Y \subset Y_A$.

A *shift of finite type* (abbreviated as SFT) is a subshift (Y, S) consisting of those elements of (Y_A, S) that omit a given finite collection of finite blocks. To make this precise we need a little terminology. If $R \subseteq \mathbb{Z}^d$, we call $\mathbf{b} \in A^R$ a *block* with *shape* R. A block is *finite* if R is finite. The block obtained by restricting $y \in Y_A$ to R is denoted $y[R]$. We denote the complement of a shape R by R^c.

If $\mathscr{F} = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ is a finite collection of finite blocks with shapes $R_1, \ldots, R_n$, we define the *shift of finite type* (SFT) $Y_{\mathscr{F}} = \{y \in Y_A : (S^{\vec{v}} y)[R_j] \neq \mathbf{f}_j$ for any $\mathbf{f}_j \in \mathscr{F}$ and $\vec{v} \in \mathbb{Z}^d\}$. We refer to $\mathscr{F}$ as the set of *forbidden blocks*. Given $\mathscr{F}$, let $m = \max_j\{\mathrm{diam}(R_j)\}$ with respect to $\|\cdot\|$. Let $s = (m+1)/2$ if m is odd and $s = m/2$ if m is even. We call s the *step-size* of $Y = Y_{\mathscr{F}}$. Without loss of generality, we can assume that every $\mathbf{f} \in \mathscr{F}$ satisfies $\mathbf{f} \in A^{B_s}$. We say a block $\mathbf{b}$ of shape R is *extendable* if $\mathbf{b} = y[R]$ for some $y \in Y$.

3. The Uniform Filling Property and Periodic Points

Definition 3.1. A SFT (Y, S) satisfies the *uniform filling property* (UFP) with filling length $l > 0$ if

(i) card $(Y) > 1$, and

(ii) for any $y_1, y_2 \in Y$, and any box $[\vec{v}, \vec{w}]$, there exists $y \in Y$ such that

$$y[[\vec{v}, \vec{w}]] = y_1[[\vec{v}, \vec{w}]] \quad \text{and} \quad y[[\vec{v} - l, \vec{w} + l]^c] = y_2[[\vec{v} - l, \vec{w} + l]^c].$$

In particular, this says it is possible to interpolate between two extendable blocks with shapes $[\vec{v}, \vec{w}]$ and $[\vec{v} - l, \vec{w} + l]^c$ using a thickness l *filling collar* $\partial_l[\vec{v}, \vec{w}] = [\vec{v} - l, \vec{w} + l] \backslash [\vec{v}, \vec{w}]$. Notice that l does not depend on the size of $[\vec{v}, \vec{w}]$. In fact, the UFP implies a stronger statement: that one can interpolate between infinitely many extendable blocks, using infinitely many filling collars, provided the collars are sufficiently well separated. We will use this stronger version of the UFP throughout this paper. We refer the reader to [19], Lemma 3.2, for a formal statement of the stronger property and the proof that it is equivalent to the UFP.

In [2] it is shown that strong irreducibility implies positive entropy, and the same proof works for the UFP.

Theorem 3.2. ([2], Prop. 1.12) *If a SFT (Y, S) has the UFP then (Y, S) is topologically mixing and $h(Y, S) > 0$.*

Let (Y, S) be a SFT, and let $\{\vec{e}_i\}$ denote the standard basis for $\mathbb{Z}^d$. For $\vec{k} = (k_1, \ldots, k_d) > 0$ we say $y \in Y$ is a *periodic point of period* $\vec{k}$ if for $\vec{v}_i = k_i \vec{e}_i$ we have $T^{\vec{v}_i} y = y$. We say that $\vec{k}$ is the *least period* if y does not have period $\vec{k}'$ for any $\vec{k}' < \vec{k}$. We call $[0, \vec{k}]$ the *fundamental domain*, and $y[[0, \vec{k}]]$ the *fundamental block* of y. If $\vec{k} > n$, we say y has period greater than n. We say (Y, S) has *dense periodic points* if the set of periodic points is dense in Y. It is known that a $\mathbb{Z}^d$ SFT with UFP, $d = 2$, has dense periodic points (see [27]). It is unknown whether this is true for $d > 2$ (probably not), but every known example does have this property.

For the proof of Theorem 1.1 we need periodic points with a certain geometry. To describe this geometry we define the *eccentricity* of a box by

$$\mathrm{ecc}([\vec{v}, \vec{w}]) = \frac{\min_i\{v_i - w_i\}}{\max_j\{v_j - w_i\}}.$$

We say a sequence of boxes $\{[\vec{v}_t, \vec{w}_t]\}$ has *bounded eccentricity* if there exists $0 < \alpha \leqslant 1$ such that $\mathrm{ecc}([\vec{v}_t, \vec{w}_t]) \geqslant \alpha$ for all t. We will need the following lemma, which we shall not prove.

Lemma 3.3. *Suppose (Y, S) is a SFT with the UFP and dense periodic points. Then for all n there is a periodic point y_n with period greater than n, such that the sequence $[0, \vec{k}_n]$ of fundamental domains has bounded eccentricity. If $d = 2$ it suffices to assume only that (Y, S) has the UFP.*

4. The Copying Lemma

The following Copying Lemma is a version of the "Ornstein Copying Lemma" as presented in [23]. First, some notation.

A *process* is a $\mathbb{Z}^d$ action (X, μ, T) together with a finite partition Q on X where the elements of Q are labeled by the symbols from a finite alphabet A. For $a \in A$ we write $Q(x) = a$ if $x \in Q_a \in Q$. For $R \subset \mathbb{Z}^d$ we let $Q^R = \bigvee_{\vec{v} \in R} T^{\vec{v}} Q$ and we write $Q^R_{\mathbf{a}}, \mathbf{a} \in A^R$, for the elements of Q^R. When $R = B_n$ we write $Q^R = Q^n$.

The (T, Q)-*name* of $x \in X$, denoted $\phi_Q(x)$, is the element $y \in Y_A$ such that $y[\vec{v}] = Q(T^{\vec{v}} x)$ for $\vec{v} \in \mathbb{Z}^d$. Given a shape $R \subset \mathbb{Z}^d$, the (T, Q, R)-name of $x \in X$ is $\phi_Q(x)[R]$. The (T, Q, n)-name is the name corresponding to the shape B_n.

Given a SFT $(Y, S) \subseteq (Y_A, S)$, we say a partition Q is a *Markov partition* of *type* (Y, S) if for μ a.e. $x \in X$, the (T, Q)-name of x satisfies $\phi_Q(x) \in Y$. We define a map $\phi_Q : \mathscr{M}(X, T) \to \mathscr{M}(Y, S), \phi_Q(\mu)(E) = \mu(\{x \in X : Q(x) \in E\})$.

A partition Q on (X, μ) is μ, ϵ contained in a σ-algebra $\mathscr{A}, Q \overset{\epsilon}{\underset{\mu}{\subset}} \mathscr{A}$, if there exists a partition $Q' \subset \mathscr{A}$ with $\mu(Q \Delta Q') = \{x : Q(x) \neq Q'(x)\} < \epsilon$. The *weak* topology* on $\mathscr{M}(X, T)$ is given by the metric

$$d(\nu, \nu') = \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} \sum_{C \in Q_i} |\nu(C) - \nu'(C)|,$$

where $\{Q_i\}$ is a *generating separating tree* of partitions on X (see [22]). On $\mathscr{M}(Y, S)$ we use $Q_i = P^i$, where P denotes the time-zero partition on Y. In this case we recover the classical weak* topology on $\mathscr{M}(Y, S)$.

We denote the set of *joinings* of two ergodic $\mathbb{Z}^d$ actions (X, μ, T) and (Z, γ, U) by $J((X, \mu, T), (Z, \gamma, U))$ (see [22]). The ergodic joinings are denoted J_e.

Theorem 4.1. (The Copying Lemma) *Consider a $\mathbb{Z}^d$ SFT (Y, S) with the uniform filling property and dense periodic points. Let P denote the time-zero partition on (Y, S). Let (X, μ, T, Q) be a nonatomic ergodic process and $\nu \in \mathscr{M}(Y, S)$ such that*

$$h(Y, \nu, S) > h(X, \mu, T, Q). \tag{4.1}$$

Suppose $\rho \in J_e((Y, \nu, S), (X, \mu, T))$ and let $\epsilon > 0$ be given. Then there exists a type (Y, S) Markov partition $\tilde{P}$ on (X, μ, T) such that

$$d((Y \times X, \rho, S \times T, P \times Q), (X, \mu, T, \tilde{P} \vee Q)) < \epsilon \tag{4.2}$$

and

$$Q \overset{\epsilon}{\underset{\mu}{\subset}} \bigvee_{\vec{v} \in \mathbb{Z}^d} T^{\vec{v}} \tilde{P}. \tag{4.3}$$

In the next few sections we prove some preliminary results needed for the proof of Theorem 4.1. The proof itself is in Section 4.4.

4.1. The $\bar{d}$-metric. Despite the fact that the conclusion of the Copying Lemma is a weak* topology result, we use the strictly stronger $\bar{d}$-metric in the proof.

Given partitions Q and R on X and Z respectively, define the partitions $\bar{Q} = Q \times Z$ and $\bar{R} = X \times R$ on $X \times Z$. We set $\bar{Q}\Delta\bar{R} = \{(x,z) : \bar{Q}(x,z) \neq \bar{R}(x,z)\} = \{(x,z) : Q(x) \neq R(z)\}$. For $n \geqslant 0$ the $\bar{d}^n$*-distance* between processes is defined by

$$\bar{d}^n((X,\mu,T,Q),(Z,\gamma,U,R)) = \min\ \{\lambda(\overline{Q^n}\Delta\overline{R^n}) : \lambda \in J_e((X,\mu,T),(Z,\gamma,U))\}.$$

For $\nu, \nu' \in \mathscr{M}(Y,S)$ we write $\bar{d}^n(\nu,\nu') = \bar{d}^n((Y,\nu,S,P),(Y,\nu',S,P))$. Notice that $\bar{d}^0 = \bar{d}$, the usual $\bar{d}$ metric. All of these metrics are equivalent.

We will need the following lemma.

Lemma 4.2. (Perturbation Lemma) *Let (Y,S) be a $\mathbb{Z}^d$ SFT with the UFP. Let $\nu \in \mathscr{M}(Y,S)$ be ergodic and suppose $\delta > 0$ and $n \geqslant 1$. Then there exists $\nu' \in \mathscr{M}(Y,S)$ ergodic so that*

$$\bar{d}^n(\nu,\nu') < 2\delta,$$

and

$$h(Y,\nu',S) \geqslant h(Y,\nu,S) + \delta(h(Y,S) - h(Y,\nu,S)).$$

This is an improvement of Theorem 2.2 in [19]. It is obtained by applying Theorem 1.1 of [18] to find a weakly mixing measure of nearly maximal entropy, and then using [19].

4.2. Genericity. We first define the notation for the frequency with which a block **a** occurs in a larger block **b**.

Definition 4.3. For $\epsilon > 0$ let n_0 be such that $\sum_{k=n_0}^{\infty} \frac{1}{2^k} < \epsilon$. For $n > n_0$, a block $\mathbf{b} \in A^{B_n}$ is ϵ *generic* if for all $0 \leqslant m \leqslant n_0$ and all $\mathbf{a} \in A^{B_m}$

$$|\delta(\mathbf{a},\mathbf{b}) - \mu(\mathbf{a})| < \epsilon, \tag{4.4}$$

where

$$\delta(\mathbf{a},\mathbf{b}) = \frac{|\{\vec{v} \in B_{n-m} : \mathbf{b}[\vec{v} + B_m] = \mathbf{a}\}|}{|B_n|} \tag{4.5}$$

and $\mu(\mathbf{a}) = \mu(Q_{\mathbf{a}}^m)$.

One-dimensional versions of the next two results can be found in [23] and their higher dimensional generalizations are immediate.

Theorem 4.4. (Theorems 7.13 and 7.15 in [23]) *Let (X,μ,T,Q) be an ergodic process. Given $\epsilon > 0$, there exists an $n_1 \in \mathbb{N}$ such that for any $n \geqslant n_1$, and for any Rohlin tower with shape B_n, base F, and error $< \epsilon$, for ϵ a.e. $x \in F, \mathbf{a} = \phi_{Q^n}(x)[B_n]$ satisfies*

(i) **a** *is ϵ-generic, and*

(ii) $\left|\frac{-1}{|B_n|}\log\left(\frac{\mu(F_{\mathbf{a}})}{\mu(F)}\right) - h(T,\mu,Q)\right| < \epsilon,$

where $F_{\mathbf{a}} = Q_{\mathbf{a}}^n \cap F$.

Theorem 4.5. ([23], Theorem 7.4) *Let (X,μ,T,Q) and (Y,ν,S,P) be ergodic processes. Let $\mathbf{a}_1,\dots,\mathbf{a}_r \in A^{B_n}$ be ϵ_1-generic n-blocks for the first process. Suppose that there are r disjoint Rohlin towers in (Y,ν,S) with bases $F_1,\dots,F_r$ such that*

$$\nu\left(\bigcup_{j=1}^{r}\bigcup_{\vec{v}\in B_n} S^{\vec{v}}F_j\right) > 1-\epsilon_2,$$

for some $\epsilon_2>0$, and such that for all $j\in\{1,\dots,r\}$ and $y\in F_j$, we have $\phi_P(y)[B_n]=\mathbf{a}_j$. Then

$$d((X,\mu,T,Q),(Y,\nu,S,P)) < \epsilon_1+\epsilon_2.$$

4.3. Markers. Fix (Y,S), a SFT with the UFP. *Let s* denote its step size, and l its filling length. To construct markers, we first establish that there is some block whose occurrences in words from Y can be erased. To do this we use the density of periodic points and the UFP to show that there are periodic points in Y with the same period but disjoint orbits.

For $\vec{k}\in\mathbb{Z}^d$ with $\vec{k}>0$ and $s,l\in\mathbb{N}$ we set

$$\partial^s[0,\vec{k}) = [0,\vec{k})\backslash[s,\vec{k}-s), \text{ and}$$
$$\partial^l[s,\vec{k}-s) = [s,\vec{k}-s)\backslash[s+l,\vec{k}-(s+l))$$

Lemma 4.6. *There exists $\vec{k}\in\mathbb{Z}^d$ so that if $y\in Y$ is a periodic point with period $\vec{k}$ then there exists a periodic point $y'\in Y$, also with period $\vec{k}$ such that*

(i) the orbit of y' is disjoint from the orbit of y and

(ii) $y[\partial^s[0,\vec{k})] = y'[\partial^s[0,\vec{k})]$.

Proof. By the UFP and Theorem 3.2 there is an $n_0\in\mathbb{N}$ such that if $\vec{k}>n_0$ then we can find $\mathbf{a}'\in A^{[s+l,\vec{k}-(s+l))}$ such that for all $\vec{v}\in\mathbb{Z}^d$

$$S^{\vec{v}}y \notin P_{\mathbf{a}'}^{[s+l,\vec{k}-(s+l))}. \tag{4.6}$$

Define $\mathbf{p}'$ by setting

$$\begin{aligned}\mathbf{p}'[\partial^s[0,\vec{k})] &= y[\partial^s[0,\vec{k})],\\ \mathbf{p}'[[s+l,\vec{k}-(s+l))] &= \mathbf{a}',\end{aligned} \tag{4.7}$$

and using the uniform filling property to determine $\mathbf{p}'[\partial^l[s,\vec{k}-s)]$.

It follows from (4.7) that $\mathbf{p}'$ is the fundamental block of a periodic point $y'\in Y$, and from (4.6) that the orbits of y and y' are disjoint. □

We now show that if y is a periodic point with a large enough fundamental block, we can erase copies of that fundamental block from the name from a given partition of type (Y,S). Furthermore, we can do this in such a way that the change will be on a set of small measure and so that the new partition will still be of type (Y,S).

Lemma 4.7. *Let (X,μ,T,Q) be an ergodic nonatomic process, with Q a type (Y,S) Markov partition, and let $\vec{k}\in\mathbb{Z}^d$ be given by Lemma 4.6. Given $\epsilon>0$, there is an $m_0\in\mathbb{N}$ such that for any $y\in Y$ with period $\vec{k}$, there is a type (Y,S) Markov partition Q' of X such that $\mu(Q\Delta Q')<\epsilon$, and $\phi_{Q'}(x)\in Y$ does not contain the block $\mathbf{f}=y[[0,(2m_0+1)\vec{k})]$ for μ-a.e. $x\in X$.*

Proof. Apply Lemma 4.6 to obtain a periodic point $y' \in \mathbb{Z}^d$ with period $\vec{k}$ such that (i) and (ii) hold. Since (X, μ, T, Q) is nonatomic, there is an $m_0 \in \mathbb{N}$ such that for all $\mathbf{c} \in B^{2m_0+1}$, $\mu(Q_{\mathbf{c}}^{2m_0+1}) < \frac{\epsilon}{|[0,\vec{k})|}$.

Let $\mathbf{f}$ be as in the statement, and note that $\mathbf{f}$ consists of $(2m_0+1)^d$ occurrences of the block $\mathbf{a} = y[[0,\vec{k})]$. We eliminate all the occurrences of the block $\mathbf{f}$ from (X, μ, T, Q) as follows. We construct a sliding block code Ψ with a window of shape $[0, (2m_0 + 2\vec{k})]$. Ψ looks to see if it is in the central copy of $\mathbf{a}$ in an occurrence of the block $\mathbf{f}$ in $\phi_Q(x)$. If it is, Ψ replaces the current symbol from $\mathbf{a}$ with the corresponding symbol from $\mathbf{b} = y'[[0,\vec{k})]$. Otherwise, it copies the current symbol. This defines a new partition Q' satisfying $\phi_{Q'}(x) = \Psi(\phi_Q(x))$.

We claim that the effect of this sliding block code is to write entire copies of $\mathbf{b}$, with no overlaps, and otherwise just to copy the symbols already present. To prove this, it suffices to note that since $\mathbf{a}$ is the fundamental block of y, any two occurrences of the block $\mathbf{a}$ cannot overlap. By (ii) $\Psi(\phi_Q(x)) \in Y$, so Q' is type (Y, S) Markov.

In addition we have

$$\begin{aligned}
\mu(Q\Delta Q') &= \mu\{x : \phi_Q(x) \in S^{-\vec{v}} P_{\mathbf{f}}^{[0,(2m_0+1)\vec{k})} \text{ such that } \vec{v} \in [0,\vec{k}) + m_0\vec{k}\} \\
&= \sum_{\vec{v}\in[0,\vec{k})+m_0\vec{k}} \mu\{x : \phi_Q(x) \in S^{-\vec{v}} P_{\mathbf{f}}^{[0,(2m_0+1)\vec{k})}\} \\
&= |[0,\vec{k})| \cdot \mu(\phi_Q^{-1}(P_{\mathbf{f}}^{[0,(2m_0+1)\vec{k})})) \\
&= |[0,\vec{k})| \cdot \mu(Q_{\mathbf{f}}^{[0,(2m_0+1)\vec{k})}) \\
&< \epsilon.
\end{aligned} \tag{4.8}$$

We now claim that there are no occurrences of $\mathbf{f}$ in any Q' name. For if there is such, then this copy of $\mathbf{f}$ must overlap one of the newly painted copies of $\mathbf{b}$. Since the orbit of y' is disjoint from the orbit of y, no complete copy of $\mathbf{a}$ from $\mathbf{f}$ can be contained in a cluster of $\mathbf{b}$'s. Hence most of $\mathbf{f}$ must intersect some block adjacent to $\mathbf{b}$. In fact, because of the size of $\mathbf{f}$, no matter how $\mathbf{f}$ intersects a newly painted $\mathbf{b}$, some block of shape $[0,\vec{k})$ adjacent to $\mathbf{b}$ will be contained in $\mathbf{f}$. In particular, this adjacent block will be partitioned by four copies of the block $\mathbf{a}$ from $\mathbf{f}$. But any newly painted copy of $\mathbf{b}$ is surrounded only by $\mathbf{a}$'s or $\mathbf{b}$'s. If the adjacent block is $\mathbf{a}$ itself, then this contradicts the least periodicity of the point y. If the adjacent block is $\mathbf{b}$, this contradicts the fact that y and y' have disjoint orbits. □

The fact that we can erase a block $\mathbf{f}$ is not enough to give us a marker: it is possible to accidentally recreate $\mathbf{f}$ while painting in the filling collars to interpolate between blocks which themselves don't see $\mathbf{f}$. We defer describing the actual marker to the proof of Theorem 4.1, where it will be easier to motivate the particular geometry that it will have.

4.4. Proof of Theorem 4.1. By Lemmas 3.3 and 4.6 we can find a vector $\vec{k} \in \mathbb{Z}^d$ and periodic points $y, y' \in Y$, both with period $\vec{k}$, such that the orbit of y' is disjoint from the orbit of y and

$$y[\partial^s[0,\vec{k})] = y'[\partial^s[0,\vec{k})]. \tag{4.9}$$

Using Lemma 4.7 we can assume, without loss of generality, that there is a positive integer m_0 such that the block $\mathbf{f} = y[[0, (2m_0+1)\vec{k})]$ does not appear in $\phi_P(y)$ for ν-a.e. $y \in Y$.

Now let $\epsilon > 0$ be given. Fix α such that

$$0 < \alpha < h(Y, \nu, S, P) - h(X, \mu, T, Q) \tag{4.10}$$

and

$$0 < \eta_1 < \min\left\{\frac{\epsilon}{16}\alpha, \frac{\epsilon}{8}\right\}, \ 0 < \eta_2, \eta_3 < \epsilon/4. \tag{4.11}$$

Let $m = 3(2m_0+1) + 2\|\vec{k}\| + 2(l+s)$. Let $n_1 \in \mathbb{N}$ be chosen so that

$$n_1 > \sqrt[d]{\frac{2}{\alpha}\log\left(\frac{8}{\epsilon}\right)}, \tag{4.12}$$

and

$$\frac{2d(2m+6l+6s)}{n_1} < \eta_3, \tag{4.13}$$

and also so that

$$\text{Theorem 4.4 holds for } (Y \times X, \rho, S \times T, P \times Q) \text{ with } \epsilon = \eta_1. \tag{4.14}$$

Let $n = n_1 + 2(m+3l+3s)$ and use the Rohlin Lemma to construct a Rohlin tower $T^{B_n}F$ with error $< \eta_2$. Consider the subtower $T^{B_{n_1}+\vec{c}}F$, where $\vec{c} = (m+3l+3s)(1,\ldots,1)$. Let $G = T^{\vec{c}}F$ be the base of this subtower. By the choice of n_1, the error of the subtower is $< \eta_2 + \eta_3$. We call $T^{B_n \setminus (B_{n_1}+\vec{c})}F$ the *collar tower* of $T^{B_n}F$. By painting η_1-generic names from the process $(Y \times X, \rho, S \times T, P \times Q)$ onto a large slice of $T^{B_{n_1}}G$, we are going to construct a type (Y, S) Markov partition $\tilde{P}$ on X that will closely approximate Q.

Consider the tower $(S \times T)^{B_n}(\bar{F})$ in $Y \times X$, where $\bar{F} = Y \times F$. Note that the base of the *inner tower* $(S \times T)^{B_{n_1}+\vec{c}}(\bar{F})$ is $\bar{G} = Y \times G$. By (4.14) there is $\bar{G}_0 \subset \bar{G}$ such that $\rho(\bar{G}_0) > (1-\eta_1)\rho(\bar{G})$, and for all $(y,x) \in \bar{G}_0$ we have that

$$(\mathbf{a}, \mathbf{b}) = \phi_{P\times Q}((y,x))[B_{n_1}] \tag{4.15}$$

is η_1-generic for $(Y \times X, \rho, S \times T, P \times Q)$. This implies $\mathbf{a}$ and $\mathbf{b}$ are η_1-generic for (Y, ν, S) and (X, μ, T) respectively. Furthermore, by (4.14) for $\bar{G}_{\mathbf{b}} = Y \times (Q_{\mathbf{b}}^{n_1} \cap G)$ and $\bar{G}_{\mathbf{a}} = P_{\mathbf{a}}^{n_1} \times G$, we have that

$$\left|\frac{-1}{|B_{n_1}|}\log\left(\frac{\rho(\bar{G}_{\mathbf{a}})}{\rho(\bar{G})}\right) - h(X, \mu, T, Q)\right| < \eta_1 \tag{4.16}$$

and

$$\left|\frac{-1}{|B_{n_1}|}\log\left(\frac{\rho(\bar{G}_{\mathbf{b}})}{\rho(\bar{G})}\right) - h(Y, \nu, S, P)\right| < \eta_1. \tag{4.17}$$

We define the sets of n_1-blocks:

$$\mathbf{J} = \phi_Q(\pi_2(\bar{G}_0))[B_{n_1}] \tag{4.18}$$

and

$$\mathbf{I} = \phi_P(\pi_1(\bar{G}_0))[B_{n_1}], \tag{4.19}$$

where π_1 and π_2 denote the projections of $Y \times X$ to Y and X respectively.

For every $\mathbf{b} \in \mathbf{J}$ we can associate a subset $\mathbf{I}(\mathbf{b})$ of $\mathbf{I}$ such that for every $\mathbf{a} \in \mathbf{I}(\mathbf{b})$ there is a point $(y, x) \in \bar{G}_0$ such that

$$\phi_Q(x)[B_{n_1}] = \mathbf{b} \tag{4.20}$$

and

$$\phi_P(y)[B_{n_1}] = \mathbf{a}. \tag{4.21}$$

The goal is to paint the pure $\mathbf{b}$-slice of the tower $T^{B_{n_1}}(\pi_1(\bar{G}_0))$ by a name from $\mathbf{I}(\mathbf{b})$ in a one-to-one way.

Suppose $\psi : \mathbf{J}_0 \to \mathbf{I}$ is a one to one map, and suppose $\mathbf{J}_0 \subset \mathbf{J}$ is the maximal domain of all such maps.

Lemma 4.8. *There exists $\eta_4 < \frac{\epsilon}{4}$ such that*

$$\mu\{x \in G : \phi_Q(x)[B_{n_1}] \in \mathbf{J}_0\} > (1 - \eta_4)\mu(G). \tag{4.22}$$

The proof of this result is identical to the corresponding part of the argument in Theorem 7.17 in [23].

Now we are ready to construct the partition $\tilde{P}$ on (X, μ, T). The main result in [17], Theorem 1.1, is that if (Y, S) has the UFP (or even a weaker condition) then every aperiodic, ergodic and measure preserving $\mathbb{Z}^d$ action (X, μ, T) has a type (Y, S) Markov partition. Let us denote this partition by R. By Lemma 4.7, we can assume without loss of generality that for μ-a.e. $x \in X$ the block $\mathbf{f} = y[[0, (2m_0 + 1)\vec{k})]$ does not occur in $\phi_R(x)$.

For points $x \in G$ with $\phi_Q(x)[B_{n_1}] \notin \mathbf{J}_0$, we keep the R labelling on the entire slice $T^{B_n}(T^{-\vec{c}}x)$. We also keep the R labels for points in $(T^{B_n}F)^c$. For $\mathbf{b} \in \mathbf{J}_0$, we paint the corresponding pure $\mathbf{b}$ slice of the subtower, namely the slice with base $G_{\mathbf{b}} = Q^{n_1}_{\mathbf{b}} \cap G$, with the n_1-block $\psi(\mathbf{b})$.

The collar tower will be painted as follows. Recall that the collar tower is of width $m + 3l + 3s$. This collar consists of nested collars around $\psi(\mathbf{b})$, as in Figure 1. The collar of width $m + 2l + 2s$ will be called the *marker collar*. On either side of the marker collar is a collar of width s. On each of these s-collars we keep the R labels. The innermost l-collar will enable us to use the uniform filling property to interpolate between $\psi(\mathbf{b})$ and the innermost s-collar. Let M denote the shape consisting of 2^d translates of $B_{\frac{m}{2}+l+s}$, situated at the 2^d corners of the marker collar (see Figure 1). All the points in the marker collar, except for those in M, will also keep their R labels. The indices in M will be painted with blocks that will serve as markers.

To define these marker blocks, we first put $\mathbf{f}_1 = y[[0, 3(2m_0 + 1)\vec{k})]$. The marker block, denoted $\mathbf{g}$, will be a copy of $\mathbf{f}_1$ surrounded by a collar consisting of copies of the block $\mathbf{c} = y''[[0, \vec{k})]$ (see Figure 2). By (4.9) this block is extendable.

We paint this marker block in the central copy of $B_{\frac{m}{2}}$ in each of 2^d copies of $B_{\frac{m}{2}+l+s}$ making up the shape M. Note that there is room for an l-collar around $\mathbf{g}$ inside each $B_{\frac{m}{2}+l+s}$. Outside $\mathbf{g}$ and this l-collar, we keep the R-labels, and we use the

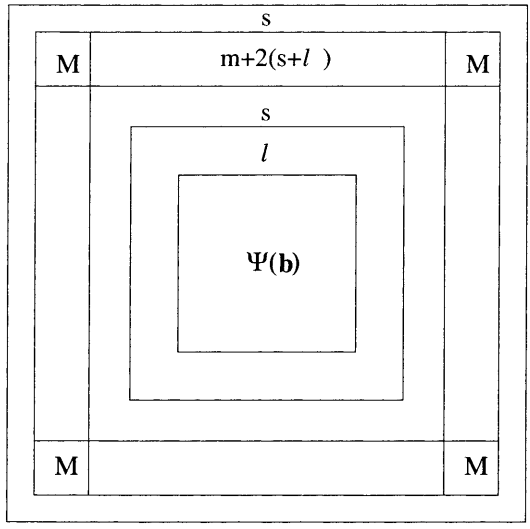

Fig. 1. The collar tower

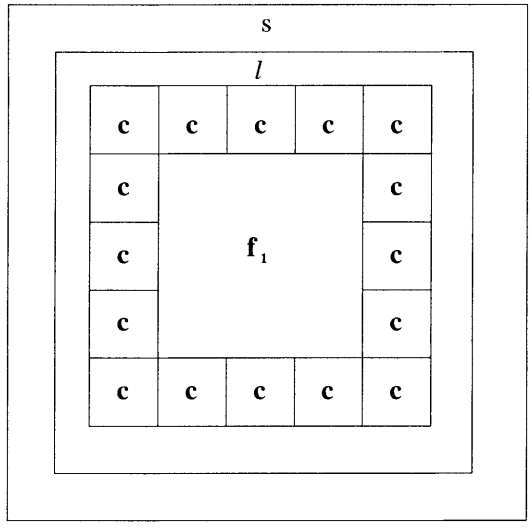

Fig. 2. The marker block **g**. Here $\mathbf{c} = y''[[0, \vec{k})]$

l-collar, with the uniform filling property, to interpolate (see Figure 2). By the uniform filling property, all these n_1-names on the towers are extendable.

Let $x \in X$, and note that we have described a name $\phi(x)$ for all times except those in the outermost l-collar of the collar tower. Outside this collar we have the names corresponding to the partition R, and inside we have extendable n_1 names. We use the UFP to interpolate and obtain $\phi(x) \in Y$. We define the partition $\tilde{P}$ by $\phi_{\tilde{P}} = \phi$.

Now consider the partition $\tilde{P} \vee Q$ on (X, μ, T). For all x outside an η_4 proportion of G the $(T, \tilde{P} \vee Q, n_1)$-name of $x \in G$ is a pair $(\mathbf{b}, \psi(\mathbf{b}))$ that is an η_1-generic name for $(Y \times X, \rho, S \times T, P \times Q)$. The error of the tower $T^{B_{n_1}} G$ is $< \eta_2 + \eta_3$. It follows from Theorem 4.5, (4.11), and Lemma 4.8 that

$$d((Y \times X, \rho, S \times T, P \times Q), (X, \mu, T, \tilde{P} \vee Q)) < \eta_1 + \eta_2 + \eta_3 + \eta_4 < \epsilon,$$

and (4.2) follows.

For (4.3) we show that for a set of x of large measure, given the $(T, \tilde{P})$-name of x, we can reconstruct its (T, Q, n)-name. In particular, these points will be exactly the points x that lie in $T^{B_{n_1}} x'$ for some $x' \in G$ such that $T^{B_{n_1}} x'$ has been painted by a new name.

To read the (T, Q, n_1)-name of such a point, given its $(T, \tilde{P})$-name, we examine $\phi_{\tilde{P}}(x)[B_{4n}]$ for occurrences of a copy of M painted with 2^d complete copies of $\mathbf{g}$. We claim that for $x \in T^{B_{n_1}} G$, this will happen exactly once. To prove this it suffices to show that $\mathbf{g}$ cannot be created accidentally during the painting process, namely to show that

(i) none of the generic names contain copies of $\mathbf{g}$,

(ii) the interpolation collars do not contain copies of $\mathbf{g}$,

(iii) the generic names, the interpolation collars, and the ambient names do not combine to create new copies of $\mathbf{g}$, and

(iv) the copies of $\mathbf{g}$ painted in the marker collar, together with their interpolation collar, do not combine with the ambient name to create another copy of $\mathbf{g}$.

It follows from our construction that there are no copies of $\mathbf{f}$ in any of the ambient names, or in the generic names that are painted in. Also, the interpolation collar is too small to contain a copy of $\mathbf{f}$. This proves (i)–(iii), and it remains to prove (iv).

Suppose, to the contrary, that a copy of $\mathbf{g}$ and its interpolation collar combine with the ambient name to create another copy of $\mathbf{g}$, say $\mathbf{g}'$. Note that since y and y' have disjoint orbits, no copy of $y'[[0, \vec{k})]$ can lie inside $\mathbf{f}_1$. Note also that the copies of $y[[0, \vec{k})]$ from $\mathbf{g}$ and $\mathbf{g}'$ must be aligned. Hence the only possibility is that $\mathbf{g}$ and $\mathbf{g}'$ overlap along a common border of copies of $y'[[0, \vec{k})]$. But this is impossible since $\mathbf{f}$ does not occur in the ambient name.

It follows that for $x \in T^{B_{n_1}} G$ we can locate the occurrences of $\mathbf{g}$ in $\phi_{\tilde{P}}(x)$ with no ambiguity. Then using the location of M, relative to the origin in the name of x, we can determine the exact location of the base point of the tower slice containing x. Call the base point of this slice x'. By construction the $(T, \tilde{P}, B_{n_1})$-name of $T^{\vec{c}} x', \mathbf{h} = \phi_{\tilde{P}}[B_{n_1}](T^{\vec{c}} x')$, will be in $\psi(\mathbf{J}_0)$. Hence, the (T, Q, B_{n_1})-name of $T^{\vec{c}} x'$ is unambiguously determined by $\psi^{-1}(\mathbf{h})$.

The set of points x for which the (T, Q, B_{n_1})-name of $T^{\vec{c}} x$ cannot be determined are:

(i) those points in $(T^{B_n} F)^c$, which by the construction of the tower $T^{B_n} F$ is a set of measure $< \eta_2$;

(ii) those points in the collar tower, by (4.13) a set of measure $< \eta_3$; and,

(iii) those points in a tower slice based at a point x', such that $\phi_Q(T^{\vec{c}} x')[B_{n_1}] \notin \mathbf{J}_0$, by (4.22) a set of measure $< \eta_4$.

By (4.11) this is a set of measure $< \eta_2 + \eta_3 + \eta_4 < \epsilon$.

5. The Proof of Theorem 1.1

Let (X,μ,T) be a free ergodic measure preserving action of $\mathbb{Z}^d$ and let (Y,S) be a $\mathbb{Z}^d$ SFT with UFP, dense periodic points, and such that $h(X,\mu,T)<h(Y,S)$. We need to show that there exists $\nu\in\mathscr{M}(Y,S)$ so that (Y,ν,S) and (X,μ,T) are metrically isomorphic.

Let $\{Q_i\}$ be a generating separating tree for (X,μ). Let

$$K=\bigcup_{\nu\in\mathscr{M}(Y,S)} J((Y,\nu,S),(X,\mu,T)),$$

and note K is convex and weak* compact. We denote by K_e the ergodic elements of K, and set $K'=\{\rho\in K_e : h(Y,\pi_1(\rho),S)>h(X,\mu,T)\}$.

By [18] there exists $\nu^*\in\mathscr{M}(Y,S)$ so that (Y,ν^*,S) is weakly mixing and $h(Y,\nu^*,S)>h(X,\mu,T)$. Thus $\nu^*\times\mu\in K_e$ and $K'\neq\emptyset$. Let $\bar{K}'$ denote the closure of K' in the weak* topology.

Let

$$\mathscr{O}(i)=\{\rho\in K' : P\times X \underset{\rho}{\overset{1/i}{\subset}} Y\times\mathscr{F} \quad\text{and}\quad \mathscr{B}\times X \underset{\rho}{\overset{1/i}{\supset}} Y\times Q_i\}.$$

We claim that the sets $\mathscr{O}(i)$ are open and dense in $\bar{K}'$.

Assuming the claim, since $\bar{K}'$ is a Baire space, we have that $\mathscr{O}=\cap_i\mathscr{O}(i)\subset\bar{K}'$ is a dense G_δ. Further, it is clear that for any $\hat{\rho}\in\mathscr{O}$

$$P\times X \underset{\hat{\rho}}{\subset} Y\times\mathscr{F} \tag{5.1}$$

and also

$$\mathscr{B}\times X \underset{\hat{\rho}}{\supset} Y\times Q_i \tag{5.2}$$

for all i. Since P is a generating partition for $(Y,\pi_1(\hat{\rho}),S)$, and since $\mathscr{F}=\bigvee_{i=1}^{\infty}Q_i$, (5.1) and (5.2) imply

$$\mathscr{B}\times X \underset{\hat{\rho}}{=} Y\times\mathscr{F}.$$

By [23], Theorem 6.8 it follows that $\hat{\rho}$ is supported on the graph of a metric isomorphism $\phi:(X,\mu,T)\to(Y,\pi_1(\hat{\rho}),S)$. Taking $\nu=\pi_1(\hat{\rho})$ we have (Y,ν,S) and (X,μ,T) are metrically isomorphic, and the result follows.

Comment: Let $\mathscr{D}=\pi_1(\mathscr{O})$. Then $\mathscr{D}\subset\mathscr{M}(Y,S)$ is weak* dense, and for any $\nu\in\mathscr{D}$, (Y,ν,S) and (X,μ,T) are metrically isomorphic.

We now prove the claim. It is clear that the sets $\mathscr{O}(i)$ are open. To argue that they are dense we proceed as follows. Let $\rho\in K'$ and let $\epsilon>0$ be given.

Note that by hypothesis there is $\alpha>0$ such that

$$h(Y,S)-h(X,\mu,T)>\alpha>0. \tag{5.3}$$

Fix α and choose $i>0$ large enough that Q_i satisfies

$$h(X,\mu,T)-h(X,\mu,T,Q_i)<\frac{\epsilon\alpha}{12}. \tag{5.4}$$

We will show there is a joining $\rho'\in\mathscr{O}(i)$ such that $d(\rho,\rho')<\epsilon$.

For $0<x<1$ let $H(x) = -x\log x - (1-x)\log(1-x)$. Choose $0<\epsilon_1<\frac{\epsilon}{2}$ small enough that

$$H\left(\frac{\epsilon_1}{4}\right) + \frac{\epsilon_1}{4}\log(|A|) < \frac{\epsilon\alpha}{12}. \tag{5.5}$$

Apply Theorem 4.1, the Copying Lemma, with $\epsilon = \epsilon_1$ and $Q = Q_i$ to copy ρ into (X,μ,T,Q_i). Let $\tilde{P}$ be the resulting type (Y,S) Markov partition of (X,μ) with

$$d((Y\times X,\rho,S\times T,P\times Q_i),(X,\mu,T,\tilde{P}\vee Q_i)) < \epsilon_1 \tag{5.6}$$

and

$$Q \overset{\frac{\epsilon_1}{2}}{\underset{\mu}{\subset}} \bigvee_{\vec{v}\in\mathbb{Z}^d} T^{\vec{v}}\tilde{P}. \tag{5.7}$$

By (5.7) there exists N_1 so that if $n \geqslant N_1$ then

$$Q_i \overset{\frac{\epsilon_1}{4}}{\underset{\mu}{\subset}} \tilde{P}^n. \tag{5.8}$$

Since $h(X,\mu,T,\tilde{P}) = h(X,\mu,T,\tilde{P}^n)$, it follows from (5.8) and Lemma 5.10 in [23] that for $n \geqslant N_1$,

$$h(X,\mu,T,\tilde{P}^n) \geqslant h(X,\mu,T,Q_i) - H\left(\frac{\epsilon_1}{4}\right) - \frac{\epsilon_1}{4}\log(|A|). \tag{5.9}$$

Let $\nu_1 = \phi_{\tilde{P}}\mu \in \mathscr{M}(Y,S)$ and put $\epsilon_2 = \epsilon/6$. It follows from Lemma 4.2 that there exists $\nu_2 \in \mathscr{M}(Y,S)$ with

$$h(Y,\nu_2,S) \geqslant h(Y,\nu_1,S) + \epsilon_2(h(Y,S) - h(Y,\nu_1,S)) \tag{5.10}$$

and $\bar{d}^n((X,\mu,T,\tilde{P}),(Y,\nu_2,S,P)) = \bar{d}^n(\nu_2,\nu_1) < 2\epsilon_2$. Let ρ' be the ergodic joining that achieves this $\bar{d}^n$-distance. Thus in particular $\rho'(\tilde{P}^n \times Y\Delta X\times P^n) = \bar{d}^n(\nu_2,\nu_1)$.

Now define $\gamma \in J_e((X,\mu,T),(Y\times X,\rho_1,S\times T))$ by

$$\gamma(A\times B\times C) = \rho'(B\times(A\cap C)).$$

A calculation shows that

$$\gamma((X\times(P\times Q_i)^n)\Delta((\tilde{P}\vee Q_i)^n\times(Y\times X))) = \bar{d}^n(\nu_2,\nu_1), \tag{5.11}$$

which implies

$$\bar{d}^n((X,\mu,T,\tilde{P}\vee Q_i),(Y\times X,\rho',S\times T,P\times Q_i)) = \bar{d}^n(\nu_2,\nu_1) < 2\epsilon_2. \tag{5.12}$$

It follows from the relation between d and $\bar{d}^n$ (see [23]) that there is N_2 such that if $n \geqslant N_2$, (5.12) implies

$$d((X,\mu,T,\tilde{P}\vee Q_i),(Y\times X,\rho',S\times T,P\times Q_i)) < 3\epsilon_2. \tag{5.13}$$

Now let $n \geqslant \max\{N_1,N_2\}$. Then by (5.6), (5.13) and the choice of ϵ_1 and ϵ_2,

$$d((Y\times X,\rho,S\times T,P\times Q_i),(Y\times X,\rho',S\times T,P\times Q_i)) < \epsilon,$$

i.e., ρ' is weak* close to ρ.

To see that $\rho' \in K'$, note that $h(Y, \pi_1(\rho'), S) = h(Y, \pi_1(\rho'), S, P) = h(Y, \nu_2, S, P) = h(Y, \nu_2, S)$, so by (5.10)

$$h(Y, \pi_1(\rho_1), S) \geqslant h(Y, \nu_1, S) + \epsilon_2(h(Y, S) - h(Y, \nu_1, S)). \tag{5.14}$$

Since $\nu_1 = \phi_{\tilde{P}}\mu$, it follows from (5.9) that (5.14) is

$$\geqslant h(X, \mu, T, Q_i) - H\left(\frac{\epsilon_1}{4}\right) - \frac{\epsilon_1}{4}\log(|A|) + \epsilon_2(h(Y, S) - h(Y, \nu_1, S)),$$

and by (5.3), (5.4), (5.5), our choice of ϵ_1 and α we have

$$\begin{aligned}
&> h(X, \mu, T, Q_i) - H\left(\frac{\epsilon_1}{4}\right) - \frac{\epsilon_1}{4}\log(|A|) + \epsilon_2\alpha \\
&> h(X, \mu, T, Q_i) + \left(\epsilon_2 - \frac{\epsilon}{12}\right)\alpha \\
&> h(X, \mu, T, Q_i) + \frac{\epsilon}{6}\alpha \\
&> h(X, \mu, T).
\end{aligned}$$

Thus we conclude that $\rho' \in K'$ as desired.

We now argue that $\rho' \in \mathcal{O}(i)$, namely that

$$P \times X \underset{\rho'}{\overset{\frac{1}{i}}{\subset}} Y \times \mathscr{F} \tag{5.15}$$

and

$$\mathscr{B} \times X \underset{\rho'}{\overset{\frac{1}{i}}{\supset}} Y \times Q_i. \tag{5.16}$$

For (5.15), we note that $\tilde{P}$ is $\mathscr{F}$ measurable, so (5.12) implies

$$P \times X \underset{\rho'}{\overset{2\epsilon_2}{\subset}} Y \times \mathscr{F},$$

and by the choice of ϵ and ϵ_2 we have $2\epsilon_2 < \frac{\epsilon}{3} < \frac{1}{i}$.

To see (5.16), we recall that by (5.8) and the choice of n there is a partition $Q' \subset \tilde{P}^n$ such that $\mu(Q'\Delta Q_i) < \frac{\epsilon_2}{4} < \frac{\epsilon}{8}$. Let us define $P' \subset P^n$ to be the partition on (Y, ν) whose atoms have the same P^n names that the atoms of Q' have in $\tilde{P}^n$. Then

$$\rho'(P' \times X\Delta Y \times Q_i) = \mu(Q'\Delta Q_i) < \frac{\epsilon}{8} < \frac{1}{i}.$$

Since P generates $\mathscr{B}$, we are done. □

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Authors' addresses: E.A. Robinson, Jr., Department of Mathematics, George Washington University, Washington, DC 20052, e-mail: robinson@gwu.edu; A.A. Şahin, Department of Mathematics, North Dakota State University, Fargo, ND 58105. Current address: Department of Mathematical Sciences, DePaul University; 2320 N. Kanmore; Chicago, IL 60614, e-mail: asahin@condor.depaul.edu