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Mixing and spectral multiplicity

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Abstract. A mixing transformation is constructed which has non-simple spectrum of finite multiplicity. This example is based on a rank 1 mixing transformation and is constructed by cutting and stacking. It can be made to be mixing of all orders.

1. *Introduction*

Let T be an invertible ergodic measure preserving transformation of a Lebesgue probability space (X, μ), and let U_T be the induced unitary operator on the complex Hilbert space $L^2(X, \mu)$; $U_Tf(x)=f(Tx)$. The spectral multiplicity M_T of T is the maximal spectral multiplicity of U_T as defined in terms of the spectral theorem (cf. [**10**]). In this paper we consider the question of what values of M_T are possible for mixing transformations T.

It was proved some time ago by Oseledec [**13**] that non-simple spectrum of finite multiplicity (i.e. $1 < M_T < \infty$) occurs for certain weak mixing transformations T. Later, the methods of Oseledec were refined in [**15**] to obtain, for any given positive integer m, a weak mixing transformation T with $M_T = m$. Since then, other techniques have appeared for constructing different sorts of transformations with non-simple spectrum of finite multiplicity (cf. [**10**], [**16**], [**17**], [**6**], [**11**]). Without going into details we note that each of these fails in some fundamental way to be mixing.

Many mixing transformations have countable Lebesgue spectrum and therefore do not have finite multiplicity. No transformation with positive entropy can have finite multiplicity. On the other hand, there is a well known example of a mixing transformation with simple spectrum; the Ornstein rank 1 mixing transformation. In the present paper we apply the construction of Oseledec, mentioned above, to a rank 1 mixing transformation. With sufficient care, the resulting transformation is mixing (or even mixing of all orders), and admits an estimate of M_T. Our main result is the following:

THEOREM 1. *There exists a mixing transformation T with $2 \le M_T \le 6$. It is possible to insure that T is mixing of all orders.*

This theorem is proved in § 2.

Using a straightforward generalization of the ideas in this paper and a lemma from [**15**], it is possible to obtain a more general result. Given an integer $m \ge 2$, let $p(m)$ be the smallest prime number such that m divides $p(m)-1$ (the number $p(m)$ exists: [**15**]).

THEOREM 2. *There exists a mixing (or mixing of all orders) transformation T such that $m \leq M_T \leq m\, p(m)$.*

These results constitute a part of the author's 1983 University of Maryland Ph.D. dissertation [**16**], written under the supervision of Professor A. Katok. The author wishes to thank Professor Katok here for all of his help and encouragement. The present much simplified version of this paper was written at the MSRI in Berkeley, California.

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2. *Proof of theorem* 1

Let T_0 be an invertible transformation of $[0, 1/6]$, preserving normalized Lebesgue measure μ_0. Denote by $\mathbb{Z}/m$ the cyclic group of order m, with additive notation and identified in the usual way with the set $\{0, \ldots, m-1\}$. Given a Borel function $\gamma: [0, \frac{1}{6}] \to \mathbb{Z}/2$ we define T_1 on $[0, \frac{1}{6}] \times \mathbb{Z}/2$ by

$$T_1(x, y) = (T_0 x, \gamma(z) + y).$$

The transformation T_1 is called the $\mathbb{Z}/2$ *extension* of T_0 with *cocycle* γ. It preserves a natural probability measure which we denote by μ_1.

The construction of T_1 from T_0 is a special case of a construction known as *compact abelian group extension* (cf. [**10**]). We apply this construction a second time. Let $\varphi: \mathbb{Z}/2 \to \mathbb{Z}/3$ be defined by $\varphi(0) = 1$ and $\varphi(1) = 2$. Define the transformation T on $[0, \frac{1}{6}] \times \mathbb{Z}/2 \times \mathbb{Z}/3$ by

$$T(x, y, z) = (T_0 x, \gamma(x) + y, \varphi(y) + z).$$

Clearly, T is a $\mathbb{Z}/3$ extension of T_1. We denote the natural probability measure which T preserves by μ.

The transformation T has a special property first discovered by Oseledec [**13**].

LEMMA 1. *For any T_0 and γ, $M_T \geq 2$.*

Proof (cf. [**16**]). There exists a U_T invariant orthogonal decomposition

$$L^2([0, \tfrac{1}{6}] \times \mathbb{Z}/2 \times \mathbb{Z}/3, \mu) = H_0 \oplus H_1 \oplus H_2,$$

where

$$H_k = \{\chi_k(z) f(x, y): f \in L^2([0, \tfrac{1}{6}] \times \mathbb{Z}/2, \mu_1)\},$$

and

$$\chi_k(z) = \exp 2\pi i k z/3.$$

It follows from a straightforward computation that $V \circ U_T|_{H_1} = U_T|_{H_2} \circ V$, where

$$(V\chi_1 f)(x, y, z) = \chi_2(z) f(x, y+1)$$

This symmetry implies that $M_T \geq 2$. □

The next two lemmas will be helpful in establishing mixing for T.

LEMMA 2. *If T_1 is weak mixing and T is ergodic, then T is weak mixing.*

Proof. We note that $U_T|_{H_0}$ is equivalent to U_{T_1}. If T is not weak mixing then there is an eigenfunction g for U_T with eigenvalue $\lambda \neq 1$. Since T_1 is weak mixing, the projection of g to H_0 is trivial. It follows that g has a non-trivial projection g' in either H_1 or H_2 which is also an eigenvector for λ. Suppose $g' \in H_1$. Then $Vg' \in H_2$ is another eigenfunction with eigenvalue λ. This implies that g'/Vg' is a non-constant invariant function, contradicting the ergodicity of T. □

Given γ, let $\bar{\gamma} = \gamma - 1$ and let $\bar{T}_1$ denote the transformation constructed using $\bar{\gamma}$ instead of γ in the definition of T_1. The next lemma is a special case of a theorem of Jones and Parry [7].

LEMMA 3. *If T_0 is weak mixing and both T_1 and $\bar{T}_1$ are egodic, then both T_1 and $\bar{T}_1$ are weak mixing.*

Proof. Consider the U_{T_1} invariant orthogonal decomposition

$$L^2([0, \tfrac{1}{6}] \times \mathbb{Z}/2, \mu_1) = H^0 \oplus H^1,$$

where $H^k = \{\chi_k(y) f(x): f \in L^2([0, \tfrac{1}{6}], \mu_0)\}$ and $\chi_k(y) = \exp \pi iky$. Here, $U_{T_1}|_{H^0}$ is equivalent to U_{T_0}. Thus, given any eigenfunction g with eigenvalue $\lambda \neq 1$, the projection of g to H^0 is trivial, and consequently $g(x, y) = \chi_1(y) f(x)$. This implies $\chi_1(\gamma(x)) f(T_0 x) = \lambda f(x)$ and $f^2(T_0 x) = \lambda^2 f^2(x)$.

Since T_0 is weak mixing, $\lambda^2 = 1$ and $f^2(x) = c = \text{constant}$ μ_0-a.e. The ergodicity of T_1 rules out $\lambda = 1$. On the other hand, if $\lambda = -1$, let ψ be a $\mathbb{Z}/2$-valued measurable function such that $C^{-1} f(x) = \exp 2\pi i \psi(x)$. Then the eigenvalue equation for g implies that

$$\gamma(x) + \psi(T_0 x) = 1 + \psi(x),$$

or, in other words,

$$\bar{\gamma}(x) = \psi(x) - \psi(T_0 x).$$

One can easily verify that this contradicts the ergodicity of $\bar{T}_1$. □

Now we will show how to construct T_0, T_1 and T. Note that by definition, T_1 and T will be determined by T_0 and γ. In fact, we will construct all three of these transformations simultaneously by cutting and stacking. We refer to [5] for the details of this method of construction.

Recall that a transformation T is said to be *rank* r (cf. [8]) if there exists a cutting and stacking construction for T with r towers at each step, and no construction with fewer towers. Such a construction will be called a rank r construction. A special case of a rank r construction is what we will call a *homogeneous rank* r construction. It may be described as follows:

After construction step n we assume that there are r n-towers $\mathcal{Y}_1^n, \ldots, \mathcal{Y}_r^n$, of equal width and equal height. In step $n+1$, each tower $\mathcal{Y}_k^n$ is cut into p_n columns $\mathcal{Y}_{k,1}^n \cdots \mathcal{Y}_{k,p_n}^n$, of equal width. Then, for $k = 1, \ldots, r$ and $j \leq p_n$, we stack $t_{n,j} \geq 0$ spacers on top of the column $\mathcal{Y}_{k,j}^n$ to obtain an augmented column $\bar{\mathcal{Y}}_{k,j}^n$.

Next we describe the stacking. For each $j < p_n$ let σ_j^n be a given permutation of the set $\{1, \ldots, r\}$. Then, for each $k = 1, \ldots, r$ and $j < p_n$, we stack the column $\bar{\mathcal{Y}}_{\sigma_j^n(k), j+1}^n$ on top of the column $\bar{\mathcal{Y}}_{k,j}^n$. In this manner we obtain r new towers, which we number in such a way that the base of $\mathcal{Y}_k^{n+1}$ is a subset of the base of $\mathcal{Y}_k^n$. This completes step $(n+1)$.

Of course, the numbers p_n and $t_{n,j}$ must be chosen in such a way that the total measure is finite (cf. [**12**]). This being the case, the resulting transformation T defined by the construction has a certain rank 1 transformation as a factor (i.e. the transformation constructed with a single tower at each step, using p_n and $t_{n,j}$). It is well known that rank 1 transformations are ergodic (cf. [**5**]). We will now give a sufficient condition for the ergodicity of the transformation obtained from a homogeneous rank r construction, $r>1$.

Let $k, l=1,\ldots,r$ and define

$$g^n_{k,l}=(p_n-1)^{-1}\#\{j<p_n:\sigma^n_j(k)=l\}.$$

Let G^n be the $r\times r$ matrix with entries $g^n_{k,l}$. For $J\subseteq\{1,\ldots,r\}$, let J^c denote the complement of J. Assuming $J, J^c\neq\emptyset$, let us define

$$E_J(G^n)=\sum_{\substack{k\in J\\ l\in J^c}} g^n_{k,l},$$

$$E(G^n)=\min_{J,J^c\neq\emptyset} E_J(G^n).$$

We will say that G^n is *δ-ergodic* if $E(G^n)>\delta>0$.

The δ-ergodicity of G^n can be interpreted in terms of stacking during step $n+1$. Namely, if the n-towers are divided into any two collections, then a fraction at least δ of the columns from one collection will be stacked on top of columns for the other during stacking in step $n+1$.

It follows from a standard argument (cf. [**8**] or [**16**]) that if G^n is δ-ergodic for infinitely many n, then the transformation T obtained from the construction is ergodic.

In the case $r=2$ these considerations are particularly simple. Suppose T_0 is rank 1 and let $\mathcal{Y}^n$ be the n-tower for T_0. Let us define $\tilde{\mathcal{Y}}^n$ to be the tower obtained from $\mathcal{Y}^n$ by removing the top level. A $\mathbb{Z}/2$-valued Borel function γ will be called *rank* 1 if for each n it is constant on every level of $\tilde{\mathcal{Y}}^n$. One can easily see that in this case the transformation T_1 constructed from T_0 and γ will admit a homogeneous rank 2 construction. Furthermore, every homogeneous rank 2 construction is of this type.

Let us define $\gamma_n=\gamma|_{\tilde{\mathcal{Y}}_{n+1}\setminus\tilde{\mathcal{Y}}_n}$. The functions γ_n have disjoint support (in the obvious sense) and $\gamma=\sum_{n=2}^{\infty}\gamma_n$. Furthermore, γ is rank 1 if and only if γ_n is rank 1 for each n. For a rank 1 γ, γ_n determines how the columns from the two n-towers are exchanged in building the $(n+1)$-towers during the homogeneous rank 2 construction for T_1. In particular, the 2×2 matrix G^n is determined by γ_n, and it can always be made 1-ergodic by an appropriate choice of γ_n. It follows that for any rank 1 T_0 there is a rank 1 γ so that T_1 is ergodic.

Let N be a set of positive integers and let $\gamma^N=\sum_{n\in N}\gamma_n$. Observing that $\bar{\gamma}$ is rank 1 if and only if γ is, we are led to the next lemma.

LEMMA 4. *Suppose T_0 is rank 1 and weak mixing. Let N be an arbitrary infinite set of positive integers. Then there exists a choice of a rank 1 γ^N such that for any choice of rank 1 γ^{N^c}, if $\gamma=\gamma^N+\gamma^{N^c}$ then the $\mathbb{Z}/2$ extension T_1 of T_0 with cocycle γ is weak mixing.*

Proof. Since N is infinite, it contains two disjoint infinite subsets N_1 and N_2. The steps $n \in N_1$ are used to make T_1 ergodic and the steps $n \in N_2$ are used to make $\bar{T}_1$ ergodic. An application of lemma 3 implies that T_1 is weak mixing. □

In a similar way, if γ is rank 1 then T will admit a homogeneous rank 6 construction. (We will consider the details of this construction in the proof of lemma 7 below.) It follows that in this case, the rank of T is at most 6. Chacon [4] has proved that for any rank r transformation T, $M_T \leq r$. This provides the desired estimate for M_T:

LEMMA 5. *If T_0 and γ are rank* 1, *then* $2 \leq M_T \leq 6$.

Our next task is to show that this construction can be carried out so that T is mixing. We observe that since T_0 is a factor of T, T cannot be mixing (or mixing of all orders) unless T_0 is. D. Ornstein [**12**] has shown that rank 1 mixing transformations exist, and D. Rudolph [**19**] has generalized the argument to show that there exist rank 1 transformations which are mixing of all orders. From now on we will assume that T_0 is of this type.

Remark. It turns out that this assumption probably entails no loss of generality. S. Kalikow [**9**] has shown that any rank 1 mixing transformation (2-fold mixing in Kalikow's terminology) is actually 3-fold mixing. It is likely that the proof can be strengthened to obtain mixing of all orders.

The next lemma was proved by J. P. Thouvenot (unpublished) for $k=2$ and D. Rudolph [**20**] for $k>2$. It is used to 'lift' the mixing property.

LEMMA 6. *A weak mixing compact abelian group extension of a k-fold mixing transformation is k-fold mixing.*

Note. We are using the terminology wherein 2-fold mixing is identical to mixing.

Remark. Similar lemmas are true for some stronger mixing properties: the K-property [**14**] and the Bernoulli property [**18**].

The final step in the proof of theorem 1 is the following lemma:

LEMMA 7. *There exists an infinite set M of positive integers with the following property. For any infinite subset M_1 of M there exists a choice of a rank* 1 γ^{M_1} *such that for any choice of rank* 1 $\gamma^{M_1^c}$ *the transformation T constructed from T_0 and $\gamma = \gamma^{M_1} + \gamma^{M_1^c}$ is ergodic.*

Proof. Note that T_0 and T share the parameters p_n and $t_{n,j}$. Let us define

$$k_n = \#\{j < p_n : t_{n,j} > 0\}.$$

Since T_0 is assumed to be mixing, $\limsup_{n\to\infty} k_n/(p_n-1) = 2\theta > 0$, and $p_n \to \infty$ as $n \to \infty$ (i.e. there are both necessary conditions for a rank 1 transformation to be mixing). We define

$$M = \{n : k_n > \theta(p_n - 1) > 12\}$$

and note that M is infinite.

We will show that if $n \in M$, there exists a rank 1 γ_n such that the 6×6 matrix G'' for T is δ-ergodic. We begin with the following observation. Let α and β be the permutations of $\{1, \ldots, 6\}$ defined by $\alpha(k) = k+3 \pmod 6$, (where 0 and 6 are identified), and

$$\beta(k) = \begin{cases} k+1 & \text{if } k \text{ is odd}, \\ k-1 & \text{if } k \text{ is even}. \end{cases}$$

Let $a_n = \#\{j < p_n : \sigma_j^n = \alpha \text{ or } \alpha^2\}$ and $b_n = \#\{j < p_n : \sigma_j^n = \beta\alpha \text{ or } \beta\alpha^2\}$. Then G^n is δ-ergodic for any $\delta \le (p_n - 1)^{-1} \min(a_n, b_n)$. This can easily be seen by identifying G^n with a graph and observing how α, α^2, $\beta\alpha$ and $\beta\alpha^2$ contribute to the edge weights.

Now for $w = (x, y, z) \in [0, \frac{1}{6}] \times \mathbb{Z}/2 \times \mathbb{Z}/3$, let $\pi_1(w) = x$, $\pi_2(w) = (y, z)$ and $\tilde{\pi}(w) = y + 2z + 1$. Let L_1 and L_2 be any pair of levels of $\bar{\mathscr{Y}}_{k,j}^n$ such that $TL_1 = L_2$. As long as L_2 is not the top level of $\bar{\mathscr{Y}}_{k,p_n}^n$, $\pi_1 L_1$ and $\pi_1 L_2$ are levels of $\tilde{\mathscr{Y}}^n$ in the construction of T_0. Because it is rank 1, γ is constant on each of them. It follows from the definition of T that for $w = (x, y, z) \in L_1$, if $c = \gamma(\pi_1 L_1)$, then $\pi_2(Tw) = (c + y, \varphi(y) + z)$. Moreover,

$$\tilde{\pi}(Tw) = \begin{cases} \alpha(\tilde{\pi}w) & \text{if } c = 0, \\ \beta\alpha(\tilde{\pi}w) & \text{if } c = 1. \end{cases}$$

In other words, if $\gamma = 0$ on the top level of $\bar{\mathscr{Y}}_{k,j}^n$, we stack $\bar{\mathscr{Y}}_{k+1,\alpha(j)}^n$ on top of it. If $\gamma = 1$ on the top level of $\bar{\mathscr{Y}}_{k,j}^n$, we stack $\bar{\mathscr{Y}}_{k+1,\beta\alpha(j)}^n$ on top of it.

By induction, moving up the columns, one finds that each permutation σ_n^j, $j < p_n$, belongs to the group generated by α and β. Since α and β satisfy the relations $\alpha^2 = \beta^3 = \alpha\beta\alpha\beta = 1$, this group is isomorphic to the symmetric group S_3.

Now for $j < p_n$, let L_1, L_2, and L_3 be the top three levels of $\bar{\mathscr{Y}}_{k,j}^n$, $T^2 L_1 = TL_2 = L_3$, and assume $t_{n,j} > 0$. Then $\pi_1 L_2$ and $\pi_1 L_3$ are in the support of γ_n, and $\sigma_j^n = \omega_1\omega_2\omega_3$ where

$$\omega_1 = \begin{cases} \alpha & \text{if } \gamma_n = 0 \text{ on } \pi_1 L_3 \\ \beta\alpha & \text{if } \gamma_n = 1 \text{ on } \pi_1 L_3, \end{cases}$$

$$\omega_2 = \begin{cases} \alpha & \text{if } \gamma_n = 0 \text{ on } \pi_1 L_2 \\ \beta\alpha & \text{if } \gamma_n = 1 \text{ on } \pi_1 L_2, \end{cases}$$

and $\omega_3 \in S_3$. It follows that by varying the value of γ_n on $\pi_1 L_3$ and $\pi_1 L_2$, σ_j^n can be made equal to any of : $\omega_1\omega_2\omega_3$, $\beta\omega_1\omega_2\omega_3$, $\omega_1\beta\omega_2\omega_3$ or $\beta\omega_1\beta\omega_2\omega_3$. It is easy to check that one of these is always α or α^2 and one is always $\beta\alpha$ or $\beta\alpha^2$.

We have assumed for this that $t_{n,j} > 0$. If $n \in M$, this happens k_n times, so that for an appropriate choice of γ_n, $a_n + b_n = k_n$ and $|a_n - b_n| \le 1$. Thus,

$$a_n,\, b_n > k_n/3 > \frac{\theta}{3}(p_n - 1) > 4$$

and

$$(p_n - 1)^{-1} \min(a_n, b_n) > \theta/3.$$

This implies that G^n is $\theta/3$ ergodic. □

3. *Discussion*

The weak mixing results in [**15**] together with the results in this paper make the following conjecture plausible.

CONJECTURE. *For any given m there exists a mixing transformation T with $M_T = m$.*

The conjecture would follow, for example, from the existence of a transformation T with simple Lebesgue spectrum. In that case one would have, $M_{T^m} = m$, where T^m denotes the m'th iterate of T. Although a complete solution to the Lebesgue spectrum problem does not seem close at hand, Mathew and Nadkarni **[11]** have recently made some encouraging progress by constructing a mixed spectrum example with a Lebesgue component of multiplicity 2 in the spectrum.

To see why the mixing construction in this paper is not enough to prove the conjecture, it is informative to see how it can be modified to obtain a weak mixing example. The modification is interesting in its own right because it provides an explicit cutting and stacking construction for a weak mixing transformation with $M_T = 2$. With suitable changes, one can also obtain arbitrary finite M_T. (The examples obtained are essentially the same as those in **[15]**.)

Let $p_n \to \infty$ and define:

$$t_{n,j} = \begin{cases} 0 & \text{if } n \text{ is even and } j < p_n, \\ 1 & \text{otherwise.} \end{cases}$$

The rank 1 transformation constructed with these parameters is weak mixing (cf. **[3]**). Furthermore, if we choose γ rank 1, and having the additional property that for infinitely many construction steps, most of the top of one of the two towers for T_1 is mapped into the base of the other, then it follows from **[1]** that T_1 is rank 1. In a similar way, it is possible to guarantee simultaneously that T is rank 2. Applying the same arguments as in the mixing case, we can obtain a weak mixing T with $M_T = 2$.

Now we show that in the mixing case, T cannot be rank 2. K. Berg **[2]** has proved that any rank 1 mixing transformation is prime. This implies that T_1 cannot be rank 1 since it has T_0 as a factor. Recent work by D. Ullman (U.C. Berkeley dissertation) shows that if a rank 2 mixing transformation has a factor, it is unique and is rank 1 mixing. Therefore, T cannot be rank 2 and mixing. Of course this does not imply that $M_T > 2$, but it suggests that the techniques needed to estimate M_T more exactly in the mixing case are more difficult than simple rank considerations.

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