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TRANSFORMATIONS WITH HIGHLY NONHOMOGENEOUS SPECTRUM OF FINITE MULTIPLICITY†

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ABSTRACT

This paper studies a spectral invariant $\mathcal{M}_T$ for ergodic measure preserving transformations T called the *essential spectral multiplicities*. It is defined as the essential range of the multiplicity function for the induced unitary operator U_T. Examples are constructed where $\mathcal{M}_T$ is subject only to the following conditions: (i) $1 \in \mathcal{M}_T$, (ii) $\mathrm{lcm}(n, m) \in \mathcal{M}_T$ wherever $n, m \in \mathcal{M}_T$, and (iii) $\sup \mathcal{M}_T < +\infty$. This shows that D_T, defined $D_T = \mathrm{card}\, \mathcal{M}_T$, may be an arbitrary positive integer. The results are obtained by an algebraic construction together with approximation arguments.

§1. Introduction

In the last few years there has been a renewed interest in spectral multiplicity problems in ergodic theory. There are now several new constructions for ergodic measure preserving transformations T with nonsimple spectrum of finite multiplicity. In particular, recent results show that there exist transformations with arbitrary finite maximal spectral multiplicity [8], transformations with Lebesgue components of finite multiplicity [6], and mixing transformations with nonsimple spectrum of finite multiplicity [10]. Other examples with different properties appear in [4] and [2]. The history of spectral multiplicity problems is outlined in [8].

Usually the term spectral multiplicity in ergodic theory refers to the maximal spectral multiplicity, denoted for a finite measure preserving transformation T (of (X, μ)) by M_T. In this paper we will be concerned with a more general notion of spectral multiplicity: the set of all *essential spectral multiplicities* of T. We will denote this set by $\mathcal{M}_T$. In terms of the spectral theorem, (cf. [1]), applied to the

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induced unitary operator $U_T f(x) = f(T^{-1}x)$ on $L_2(X, \mu) \ominus \{\text{constants}\}$, $\mathcal{M}_T$ is defined as the essential range of the multiplicity function with respect to the maximal spectral type. The maximal spectral multiplicity is obtained from $\mathcal{M}_T$ by $M_T = \sup \mathcal{M}_T$. We also introduce a new spectral invariant D_T, defined by $D_T = \operatorname{card} \mathcal{M}_T$, called the *degree of nonhomogeneity* of the spectrum. This follows the usual terminology where, when $\mathcal{M}_T = \{k\}$, T is said to have homogeneous spectrum.

At least implicitly, $\mathcal{M}_T$ has been studied for a long time. It is well known that many common examples in ergodic theory (e.g., irrational rotations, Bernoulli shifts, affine transformations) have $\mathcal{M}_T = \{1\}$, $\{+\infty\}$ or $\{1, +\infty\}$. Although in general there are no known restrictions on $\mathcal{M}_T$ (and it seems unlikely that there are any), only isolated examples of the possibilities for $\mathcal{M}_T$ have even been found. In addition to those listed above, there are the examples with nonsimple spectrum of finite multiplicity in [8], [6], [2], which all satisfy $\mathcal{M}_T = \{1, k\}$ for some k (and any k is possible, [8]). There are some examples due to A. Katok [4], where $\mathcal{M}_T$ satisfies certain interesting estimates, including $1 \notin \mathcal{M}_T$ and $M_T < \infty$, but where $\mathcal{M}_T$ is not completely determined (cf. also [10]). Also one special class of T where $\mathcal{M}_T$ is both interesting and can be determined exactly is the class of Gaussian transformations (cf. [1]). If T is ergodic Gaussian and $\mathcal{M}_T \neq \{+\infty\}$ then it is known that $\mathcal{M}_T$ is a multiplicative sub-semi group, with identity, of the natural numbers. The case $\mathcal{M}_T = \{1\}$ does occur. Otherwise, interesting $\mathcal{M}_T$ always has $D_T = +\infty$ and $M_T = +\infty$.

In this paper we construct a different special class of transformations T with many possibilities for $\mathcal{M}_T$, but this time with $D_T < +\infty$ and $M_T < +\infty$. Within our class, $\mathcal{M}_T$ is subject only to the following mild restrictions: (i) $1 \in \mathcal{M}_T$, (ii) if $m_1, m_2 \in \mathcal{M}_T$ then $\operatorname{lcm}(m_1, m_2) \in \mathcal{M}_T$ and (iii) $M_T < +\infty$. Thus we obtain many new examples. In particular, there exist transformations with arbitrary finite D_T. For D_T large, we say the spectrum is highly non-homogeneous. The cases

$$\mathcal{M}_T = \{1, p-1, p(p-1), \ldots, p^r(p-1)\},$$

where p is an odd prime, first appeared in the author's dissertation [9].

The construction in this paper is a generalization of that in [8], but more elaborate in several respects to facilitate computing $\mathcal{M}_T$ rather than just M_T. In particular, the upper bounds on $\mathcal{M}_T$ are obtained in a new way: by showing that the spectrum is simple on certain U_T invariant subspaces and then showing how these subspaces fit together. The basic technique is the theory of approximation by periodic transformations (cf. [5]).

A few words on the notation. We denote the cyclic group of order m by $\mathbf{Z}/m$

and the circle by **T**. Transformations T will always be assumed to be invertible measure preserving transformations of Lebesgue probability spaces. Sets and functions will always be measurable. The characteristic function of B is denoted 1_B. The notation U_T will be used both for the induced unitary operator on $L_2(X,\mu)$ and its restriction to $L_2(X,\mu)\ominus\{\text{constants}\}$.

The results in this paper constitute a generalization of a part of the author's 1983 University of Maryland dissertation [9], written under the direction of Prof. A. Katok. The author wishes to thank Prof. Katok for all of his useful advice.

§2. Algebraic framework

Given a finite abelian group A, we apply the structure theorem to obtain a fixed decomposition of the form

$$A=\bigoplus_{j=1}^{l}\mathbf{Z}/n_j. \tag{2.1}$$

Then for $a,b\in A$, we define

$$\chi_a(b)=\exp 2\pi i\sum_{j=1}^{l}a_jb_j/n_j \tag{2.2}$$

where $a_j\in\mathbf{Z}/n_j$ in (2.1) and $\mathbf{Z}/n_j$ is identified with $\{0,\dots,n_j-1\}$. Let $\hat{A}$ denote the dual group of A identified with $\{\chi_a\colon a\in A\}$. The mapping $a\to\chi_a$ is an isomorphism between A and $\hat{A}$. By an automorphism α of A we mean an abelian group automorphism. Given an automorphism α, there is a unique automorphism $\bar{\alpha}$ of A (called the adjoint of α), satisfying $\chi_{\bar{\alpha}a}(b)=\chi_a(\alpha b)$ for all $a,b\in A$. In the cases of primary interest in this paper A will actually have a ring structure and α will be implemented by multiplication by a unit. However, our construction is a little more general than this.

The α-orbit $\mathcal{O}$ of an element $a\in A$ is defined as $\mathcal{O}=\{\alpha^l a\colon l\in\mathbf{Z}\}$. We say the a has α-order l if $l=\operatorname{card}\mathcal{O}$. Let us define $\mathcal{M}_\alpha=\{l\colon l$ is the α-order of some $a\in A\}$. Note that $\mathcal{M}_\alpha=\mathcal{M}_{\bar{\alpha}}$. We say α is *separating* if for any $a,a'\in A$ which belong to different $\bar{\alpha}$ orbits, there exists an α-orbit $\mathcal{O}$ with

$$\chi_a(\mathcal{O})\cap\chi_{a'}(\mathcal{O})=\varnothing \tag{2.3}$$

where $\chi_a(\mathcal{O})$ denotes the image of $\mathcal{O}$ under χ_a. We call α *proper* if it fixes only $0\in A$. Any cyclic group (except $\mathbf{Z}/2$) has separating proper automorphisms, and the automorphism $\left(\begin{smallmatrix}1&2\\1&0\end{smallmatrix}\right)$ acting on $\mathbf{Z}/3\oplus\mathbf{Z}/3$ by matrix multiplication is separating and proper. This shows that there are non-cyclic examples. A complete classification will not concern us here. The next lemma shows that there are enough cyclic examples for our purposes.

LEMMA 2.1. *Suppose $\mathcal{M}$ is a finite set of natural numbers such that* (i) $1 \in \mathcal{M}$ *and* (ii) *whenever* $m_1, m_2 \in \mathcal{M}$, $\mathrm{lcm}(m_1, m_2) \in \mathcal{M}$. *Then there exists a cyclic group* $\mathbf{Z}/n$ *and an element* $b \in \mathbf{Z}/n$ *such that the automorphism* $\alpha(z) = bz$ *is separating, proper, and satisfies* $\mathcal{M}_\alpha = \mathcal{M}$. *The order of* α *is* $\mathrm{lcm}\, \mathcal{M}$.

PROOF. For p prime, the multiplicative group of units $(\mathbf{Z}/p)^\times$ of $\mathbf{Z}/p$ is isomorphic to $\mathbf{Z}/p-1$. Let $m \mid p-1$ and let $b \in \mathbf{Z}/p$ be a generator of the subgroup H of $(\mathbf{Z}/p)^\times$ isomorphic to $\mathbf{Z}/m$. For $\alpha(z) = bz$, the α-orbits correspond to $0 \in \mathbf{Z}/m$ and the cosets of H.

Let $m_1, \ldots, m_l \in \mathcal{M}$, $m_j \neq 1$, be a minimal set of generators for $\mathcal{M}$ with respect to the operation lcm. For each m_j, $j = 1, \ldots, l$, let p_j be the smallest prime so that $m_j \mid p_j - 1$ and p_j is not equal to p_k for any $k < j$. This is possible by the Dirichlet Theorem on primes in an arithmetic progression. Let $A = \bigoplus_{j=1}^{l} \mathbf{Z}/p_j$ and note that

$$A = \mathbf{Z}/n, \qquad n = \prod_{j=1}^{l} p_j.$$

We define α on $\mathbf{Z}/n$ as $\alpha = \bigoplus_{j=1}^{l} \alpha_j$, where α_j is chosen for p_j and m_j as above. It follows that α is proper and $\mathcal{M}_\alpha = \mathcal{M}$. Furthermore, α may be realized by $\alpha(z) = bz$ where b is a unit in $\mathbf{Z}/n$. Also, α is separating, since any $a, a' \in \mathbf{Z}/n$ with $a' \neq b^k a$ satisfies (2.3), where $\mathcal{O}$ is the orbit of $1 \in \mathbf{Z}/n$. The final statement is trivial. □

Our main theorem is that for each algebraic example there is a corresponding ergodic theoretic example.

THEOREM 2.2. *For any separating automorphism α of a finite abelian group A there exists an ergodic transformation T with $\mathcal{M}_T = \mathcal{M}_\alpha$. If in addition α is proper then T can be made weak mixing.*

COROLLARY 2.3. *For each finite set $\mathcal{M}$ of positive integers satisfying* (i) $1 \in \mathcal{M}$ *and* (ii) *whenever* $m_1, m_2 \in \mathcal{M}$, $\mathrm{lcm}(m_1, m_2) \in \mathcal{M}$, *there exists a weak mixing transformation T with $\mathcal{M}_T = \mathcal{M}$.*

COROLLARY 2.4. *For each positive integer d there exists a weak mixing transformation T with $D_T = d$.*

For the remainder of this section we set up the basic construction and prove some preliminary lemmas. Most of the proof is postponed until the next section.

Let α be an automorphism of A, $\bar{\alpha}$ the adjoint automorphism, and m = the order of α = the order of $\bar{\alpha}$. We write $A_1 = \mathbf{Z}/m$ and $A_2 = A$, with δ_1 and δ_2

denoting normalized Haar measure on A_1 and A_2. Let T_0 be a transformation of (X_0, μ_0). For $i = 1, 2$ let $\gamma_i: X_0 \to A_i$ and define

$$(X_i, \mu_i) = (X_{i-1} \times A_i, \mu_{i-1} \times \delta_i).$$

Let us define transformations T_1 and T_2 on (X_1, μ_1) and (X_2, μ_2):

$$T_1(x, y) = (T_0 x, \gamma_1(x) + y) \tag{2.4}$$

and

$$T_2(x, y, z) = (T_0 x, \gamma_1(x) + y, \alpha^y \gamma_2(x) + z). \tag{2.5}$$

T_1 and T_2 satisfy the following general lemma (true in general for finite abelian group extensions, cf. [8]).

LEMMA 2.5. *For $i = 1, 2$ there exists a U_{T_i}-invariant orthogonal decomposition*

$$L_2(X_i, \mu_i) = \bigoplus_{k \in A_i} H_k^i$$

where

$$H_k^i = \{f \in L_2(X_i, \mu_i): f(x, w)\chi_k(w)\tilde{f}(x) \text{ some } \tilde{f} \in L_2(X_{i-1}, \mu_{i-1})\}.$$

Furthermore, $U_{T_i} \big| H_0^i$ is unitarily equivalent to $U_{T_{i-1}}$.

In addition to the above, the transformation T_2 has the following special property which generalizes a method of Oseledec [7] for obtaining transformations with nonsimple spectrum (i.e. $M_T > 1$):

LEMMA 2.6. *If $a, a' \in A$ lie in the same $\bar{\alpha}$-orbit then $U_{T_2}\big|_{H_a^2}$ and $U_{T_2}\big|_{H_{a'}^2}$ are unitarily equivalent.*

PROOF. This is essentially the same as Lemma 2.1 in [8]. We define $S: H_a^2 \to H_{\alpha a}^2$ by $(S\chi_a f)(x, y, z) = \chi_{\bar{\alpha} a}(z) f(x, y+1)$. Then the Lemma follows from the equation $U_{T_2}\big|_{H_{\bar{\alpha}a}^2} \circ S = S \circ U_{T_2}\big|_{H_a^2}$, which follows from (2.5) and

$$\chi_a(\alpha^{y+1}\gamma_2) = \chi_a(\alpha\alpha^y\gamma_2) = \chi_{\bar{\alpha}a}(\alpha^y\gamma_2). \qquad \square$$

In the next section we will show that under certain conditions: (i) for each $a \in A$, $U_{T_2}\big|_{H_a^2}$ has a continuous spectrum with spectral multiplicity 1, and (ii) if $a, a' \in A$ lie in different $\bar{\alpha}$-orbits then $U_{T_2}\big|_{H_a^2}$ and $U_{T_2}\big|_{H_{a'}^2}$ have mutually singular maximal spectral types.

Theorem 2.2 will follow.

We conclude this section with a characterization of T_2. Let G denote the semi-direct product group $\mathbf{Z}/m \times_\alpha A$, i.e., the group of pairs $(y,z) \in \mathbf{Z}/m \times A$ with multiplication $(y',z')(y,z) = (y'+y, \alpha^y z' + z)$. Then T_2 is just the finite nonabelian G extension of T_0 with cocycle (γ_1, γ_2): $X_0 \to G$ (cf. [10]). In the case $A = \mathbf{Z}/n$ (our main concern), G is the semi-direct product of cyclic groups. Such groups are called metacyclic groups.

§3. Approximation theory

The proof of Theorem 2.2 is based on Katok and Stepin's theory of approximation by periodic transformations [5] (cf. also [4]). We begin with some preliminaries.

For $i = 1,2$ let $\mathcal{A}_i$ denote the set of all γ_i: $X_0 \to A_i$, with the topology given by the "L_1-norm": $\|\gamma_i\|_1 = \mu_i\{x: \gamma_i(x) \neq 0\}$. The product space $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is given the product topology. A *partition* ξ of (X,μ) is a finite disjoint collection of measurable sets with $\mu(\bigcup_{c \in \xi} c) = 1$. A set E is called ξ-measurable if, up to sets of measure 0, it is a union of elements of ξ. Similarly, a function $\gamma_i \in \mathcal{A}_i$, $i = 1,2$, is called ξ-measurable if all of its level sets are ξ-measurable. A sequence of partitions ξ_n of (X,μ) is called *generating* if for any set E there exist ξ_n-measurable sets E_n so that $\mu(E \triangle E_n) \to 0$ as $n \to \infty$ (where $\triangle$ denotes the symmetric difference). We denote this by $\xi_n \to \varepsilon$.

Definition 3.1. A transformation T admits a *good periodic approximation* (T_n, ξ_n) if

(1) ξ_n is a partition on (X,μ) with q_n elements of equal measure, such that $\xi_n \to \varepsilon$,

(2) T_n is a sequence of transformations with $T_n B \in \xi_n$ for every $B \in \xi_n$ (we say T_n permutes ξ_n), and

(3) $\Sigma_{B \in \xi_n} \mu(TB \triangle T_n B) = o(1/q_n)$ as $n \to \infty$.

Note. To say for a sequence ω_n that $\omega_n = o(1/q_n)$ means $\lim_{n\to\infty} q_n\omega_n = 0$.

Let us now regard T_n as a permutation of ξ_n and consider its cyclic structure. If T_n has a single cycle and satisfies Definition 3.1 then we say that (T_n, ξ_n) is a *good cyclic approximation* for T. If, instead, (T_n, ξ_n) has m cycles of equal length we say it is a *good m-cyclic approximation.*

The set $\mathcal{U}$ of all transformations of (X_0, μ_0) may be given the weak topology (cf. [3]). A property of T_0 is called *generic* in $\mathcal{U}$ if there exists a dense G_δ subset $\mathcal{U}'$ of $\mathcal{U}$ such that every $T \in \mathcal{U}'$ has the given property. Halmos shows in [3] that weak mixing (and hence ergodicity) is a generic property. Katok and Stepin [5] show that the property that T_0 admits a good cyclic approximation is generic. We will always assume T_0 satisfies both of these properties, another generic condition, and let $(T_{0,n}, \xi_{0,n})$ be a fixed good cyclic approximation for T_0.

Given a $\xi_{0,n}$-measurable pair $(\gamma_1^n, \gamma_2^n) \in \mathcal{A}$, if we replace T_0 with $T_{0,n}$ and (γ_1, γ_2) with (γ_1^n, γ_2^n) in (2.4) and (2.5), we obtain the transformations which we call $T_{1,n}$ and $T_{2,n}$. We can also "lift" $\xi_{0,n}$ in the obvious way to partitions $\xi_{1,n}$ and $\xi_{2,n}$ on (X_1, μ_1) and (X_2, μ_2). It is clear that $T_{1,n}$ and $T_{2,n}$ permute $\xi_{1,n}$ and $\xi_{2,n}$.

A pair $(\gamma_1, \gamma_2) \in \mathcal{A}$ is called *admissible* (corresponding to (γ_1^n, γ_2^n)) if there exists a $\xi_{0,n}$-measurable sequence $(\gamma_1^n, \gamma_2^n) \in \mathcal{A}$ such that for $i = 1,2$

$$\| \gamma_i^n - \gamma_i \|_1 = o(1/q_n^2). \tag{3.1}$$

This is more than enough to insure that $T_{1,n}$ and $T_{2,n}$ satisfy the conditions of Definition 3.1.

For an "approximation step" $(T_{0,n}, \xi_{0,n})$ let us now consider the various different cyclic structures for $T_{1,n}$ which correspond to different choices of γ_1^n. We say that a given property of $T_{1,n}$ is *attainable* if for any $\xi_{0,n}$-measurable γ_1^n there exists a $\xi_{0,n}$-measurable γ_1', which differs from γ_1^n on at most 2 elements of $\xi_{0,n}$, and such that transformation $T_{1,n}$ constructed from γ_1' instead of γ_1^n has the given property.

For example we have the following:

LEMMA 3.2. *The property that $T_{1,n}$ is m-cyclic is attainable.*

PROOF. Choose any $B \in \xi_{0,n}$ and let $B' \in \xi_{1,n}$ be defined $B' = B \times \{0\}$. Define

$$\Gamma_1^n(l, x) = \begin{cases} 0 & \text{if } l = 0, \\ \sum_{j=0}^{l-1} \gamma_1^n(T_{0,n}^j x) & \text{if } l > 0. \end{cases} \tag{3.2}$$

Then one has $T_{1,n}^{q_n} B' = B \times \{k\}$ where $k = \Gamma_1^n(q_n, x)$ (k is independent of x since $T_{0,n}$ is cyclic). $T_{1,n}$ is m-cyclic if and only if $k = 0$. If $k \neq 0$, we define $\gamma_1' = \gamma_1^n - k\chi_B$, which has $\Gamma_1^n(q_n, x) = 0$ and differs from γ_1^n only on B. □

For $l \in \mathbf{Z}/m$ the transformation $T_{1,n}$ is called *l-satisfactory* if the transformation $R_l \circ T_{1,n}$ is cyclic, where $R_l(x, y) = (x, y + l)$.

LEMMA 3.3. *For any $l \in \mathbf{Z}/m$ the property that $T_{1,n}$ is l-satisfactory is attainable.*

PROOF. This is similar to Lemma 3.3. Let $S = R_l \circ T_{1,n}$. $S^{q_n} B \times \{0\} = B \times \{k\}$ for some $k \in \mathbf{Z}/m$. S is cyclic if and only if k generates $\mathbf{Z}/m$. If it does not, we modify γ_1^n on B so that it does. □

Properties of $T_{2,n}$ are treated in much the same way. A property is called *attainable* for $T_{2,n}$ if it can be attained by modifying any $\xi_{0,n}$-measurable $(\gamma_1^n, \gamma_2^n) \in \mathcal{A}$ on at most 2 elements of $\xi_{0,n}$. The property which we will need requires a preliminary discussion.

For $(x, y, z) \in X_2$ it follows from (2.5) that the third coordinate Π_3 of $T_{2,n}^{q_n}(x, y, z)$ is given by

$$\Pi_3(T_{2,n}^{q_n}(x, y, z)) = z + \alpha^y \sum_{k=0}^{q_n-1} \alpha^{\Gamma_1^n(k,x)} \gamma_2^n(T_{0,n}^k x) \tag{3.3}$$
$$\stackrel{\text{def}}{=} z + \alpha^y(\Gamma_2^n(x)).$$

By (3.2),

$$\Gamma_1^n(k+1, x) = \Gamma_1^n(k, T_{0,N}x) + \Gamma_1^n(1, x) \tag{3.4}$$
$$= \Gamma_1^n(k, T_{0,n}x) + \gamma_1^n(x)$$

and so by (3.3),

$$\Gamma_2^n(T_{0,n}x) = \alpha^{-\gamma_1^n(x)} \sum_{k=0}^{q_n-1} \alpha^{\Gamma_1^n(k+1,x)} \gamma_2^n(T_{0,n}^{k+1} x). \tag{3.5}$$

Assuming $T_{1,n}$ is m-cyclic, so that $\Gamma_1^n(q_n, x) = 0$, and since $T_{0,n}^{q_n} = I$, (3.5) becomes $\Gamma_2^n(T_{0,n}x) = \alpha^{-\gamma_1^n(x)}\Gamma_2^n(x)$. This shows that $\Gamma_2^n(T_{0,n}x)$ and $\Gamma_2^n(x)$ belong to the same α-orbit $\mathcal{O}$. Since $T_{0,n}$ is cyclic, $\Gamma_2^n(x) \in \mathcal{O}$ for all x, and furthermore, by (3.3),

$$\Pi_3(T_{2,n}^{q_n}(x, y, z)) - z \in \mathcal{O} \tag{3.6}$$

for all $(x, y, z) \in X_2$.

For a given α-orbit $\mathcal{O}$, we say that $T_{2,n}$ is *$\mathcal{O}$-satisfactory* if $T_{1,n}$ is m-cyclic and (3.6) holds for $\mathcal{O}$.

LEMMA 3.4. *For any approximation step and any α-orbit $\mathcal{O}$, the property that $T_{2,n}$ is $\mathcal{O}$-satisfactory is attainable.*

PROOF. First we apply Lemma 3.2 to make $T_{1,n}$ m-cyclic. then we define

$$\bar{\Gamma}_2^n(x) = \sum_{k=1}^{q_n-1} \alpha^{\Gamma_1^n(k,x)} \gamma_2^n(T_{0,n}^k x),$$

so that $\Gamma_2^n(x) = \gamma_2^n(x) + \bar{\Gamma}_2^n(x)$. We fix an $x \in X_0$ and modify γ_2^n on the element $B \in \xi_{0,n}$ containing x so that $\Gamma_2^n(x) \in \mathcal{O}$. □

The next lemma shows how various approximation properties imply corresponding ergodic properties. It is the key to proving Theorem 2.2.

LEMMA 3.5. *Let T_0 be weak mixing with a good cyclic approximation $(T_{0,n}, \xi_{0,n})$ and let $(\gamma_1, \gamma_2) \in \mathcal{A}'$ be admissible, corresponding to the sequence (γ_1^n, γ_2^n).*

(i) *If $T_{1,n}$ is 0-satisfactory for infinitely many n, then $U_{T_2}|_{H_a^2}$ has spectral multiplicity 1 for each $a \in A$.*

(ii) *If in addition to* (i), *α is separating, and for each α-orbit $\mathcal{O}$ there exist infinitely many n with $T_{2,n}$ $\mathcal{O}$-satisfactory, then $U_{T_2}|_{H_a^2}$ and $U_{T_2}|_{H_{a'}^2}$ have mutually singular spectral types for any $a, a' \in A$ belonging to different $\tilde{\alpha}$ orbits.*

(iii) *If in addition to* (i) *and* (ii), *α is proper, and for each $l \in \mathbf{Z}/m$, $T_{1,n}$ is l-satisfactory, then T_2 is weak mixing.*

PROOF. (i) (cf. [5], Theorem 3.1) By passing to the subsequence where $T_{1,n}$ is 0-satisfactory we have a good cyclic approximation $(T_{1,n}, \xi_{1,n})$ for T_1. Let $B_n \in \xi_{1,n}$ and

$$C_n = \bigcap_{k=0}^{mq_n - 1} T_1^{-k}(T_{1,n}^k B_n \cap T_1^k B_n)$$

so that for $0 \leqq k < mq_n$, $T_1^k C_n \subseteq T_{1,n}^k B_n$. If

$$S(q_n) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{k=0}^{mq_n - 1} \mu(T_1 T_{1,n}^k B_n \,\triangle\, T_{1,n}^{k+1} B_n),$$

then (cf. [5])

$$\mu(B_n \setminus C_n) \leqq S(q_n) \leqq o(1/q_n). \tag{3.7}$$

Let $\bar{B}_n^k = T_{1,n}^k B_n \cap (\gamma_1^n - \gamma_1)^{-1}(0) \cap (\gamma_2^n - \gamma_2)^{-1}(0)$, so that for $j = 1, 2$, $0 \leqq k < mq_n$,

$$\mu(T_{1,n}^k B_n \setminus \bar{B}_n^k) \leqq \sum_{i=1}^{2} \| \gamma_i^n - \gamma_i \|_1 . \tag{3.8}$$

Letting $D_n = \bigcap_{k=0}^{mq_n - 1} T_1^{-k}(T_1^k C_n \cap \bar{B}_n^k)$, we have $D_n \subseteq C_n$ and by (3.1), (3.7) and (3.8), $\mu(B_n \setminus D_n) \leqq \sum_{i=1}^{2} mq_n \| \gamma_i^n - \gamma_i \|_1 + S(q_n) = o(1/q_n)$. Thus

$$\frac{\mu(B_n \setminus D_n)}{\mu(B_n)} = q_n \mu(B_n \setminus D_n) \leqq q_n o(1/q_n) \xrightarrow{n \to \infty} 0. \tag{3.9}$$

Letting $\xi'_{1,n} = \{T_1^k D_n : 0 \leqq k < mq_n - 1\}$, we have by (3.9) and $\xi_{1,n} \to \varepsilon$ that $\xi'_{1,n} \to \varepsilon$.

Let $H_{a,n}^2 = \{f(x, y)\chi_a(z) : f$ is $\xi'_{1,n}$-measurable$\}$. Then $H_{a,n}^2 \subseteq H_a^2$ and since $\xi'_{n,1} \to \varepsilon$ it follows that for any $\delta > 0$, $H_{a,n}^2$ is δ-dense in the unit ball of H_a^2 for n sufficiently large. We define $h_n = 1_{D_n} \chi_a$, and let $H(h_n)$ denote the cyclic subspace generated by h_n. Since $H_{a,n}^2 \subseteq H(h_n) \subseteq H_a^2$, $H(h_n)$ is also δ-dense in the unit ball of H_a^2 for n sufficiently large. It follows from a standard argument (cf. [5] Lemma 3.1) that $U_{T_2}|_{H_a^2}$ has simple spectrum.

(ii) Since α is separating, for any a, a' in different $\bar{\alpha}$-orbits there exists an α orbit $\mathcal{O}$ so that (2.3) holds.

Let n be such that $T_{2,n}$ is $\mathcal{O}$-satisfactory. Then $T_{1,n}$ is m-cyclic, the length of each cycle being q_n. Thus for $B \in \xi_{1,n}$, $T_{1,n}^{q_n}B = B$. Let $h = 1_B\chi_a$. Then by (3.6),

$$\begin{aligned} U_{T_{2,n}}^{q_n}h(x,y,z) &= \chi_a(\alpha^y\Gamma_2^n(x)+z)1_B(x,y) \\ &= \chi_a(\alpha^y\Gamma_2^n(x))\chi_a(z)1_B(x,y) \\ &= \lambda h(x,y,z), \end{aligned}$$

where $\lambda \in \chi_a(\mathcal{O})$.

Now since any $g_n \in H_{a,n}^2$ is a finite linear combination of functions of the type h corresponding to different $B \in \xi_{1,n}$, it follows that $\chi_a(\mathcal{O})$ is the set of eigenvalues for $U_{T_{2,n}}^{q_n}|_{H_{a,n}^2}$.

Let g be a vector of maximal spectral type for $U_{T_2}|_{H_a^2}$. Assume $\|g\|_2 = 1$, and let ρ_g denote the corresponding spectral (probability) measure. Since $\xi_{n,1} \to \varepsilon$, we can find $g_n \in H_{a,n}^2$, $\|g_n\|_2 = 1$, with $\|g - g_n\|_2 \to 0$ as $n \to \infty$. The function g_n has a unique eigenfunction expansion, i.e.:

$$g_n = \sum_{\lambda \in \chi_a(\mathcal{O})} g_{n,\lambda}$$

with $U_{T_{2,n}}^{q_n}g_{n,\lambda} = \lambda g_{n,\lambda}$. furthermore, there exist $g'_{n,\lambda} \in H_a^2$ with $g = \sum_{\lambda \in \chi_a(\mathcal{O})} g'_{n,\lambda}$, $\|g'_{n,\lambda}\|_2 = \|g_{n,\lambda}\|_2$, and

$$\lim_{n\to\infty} \|g'_{n,\lambda} - g_{n,\lambda}\|_2 = 0. \tag{3.10}$$

Let us denote by $\rho_{n,\lambda}$ the spectral measure associated with $g'_{n,\lambda} \in H_a^2$. Then $\rho_g = \sum_{\lambda \in \chi_a(\mathcal{O})} \rho_{n,\lambda}$ and $\rho_{n,\lambda}(\mathbf{T}) = \|g_{n,\lambda}\|_2^2$. For each $\lambda \in \chi_a(\mathcal{O})$ we have

$$\begin{aligned} \varepsilon_n &\overset{\text{def}}{=} \left| \int_{-\pi}^{\pi} e^{iq_n t} d\rho_{n,\lambda}(t) - \|g_{n,\lambda}\|_2^2 \lambda \right| \\ &= |(U_{T_2}^{q_n}g'_{n,\lambda}, g'_{n,\lambda}) - (U_{T_{2,n}}^{q_n}g_{n,\lambda}, g_{n,\lambda})| \\ &\leq \|U_{T_2}^{q_n}g'_{n,\lambda} - U_{T_{2,n}}^{q_n}g_{n,\lambda}\|_2 + \|g'_{n,\lambda} - g_{n,\lambda}\|_2 \qquad (3.11) \\ &\leq \|U_{T_2}^{q_n}g_{n,\lambda} - U_{T_{2,n}}^{q_n}g_{n,\lambda}\|_2 + 2\|g'_{n,\lambda} - g_{n,\lambda}\|_2. \end{aligned}$$

It follows from the proof of (i) above that $\|U_{T_2}^{q_n}g_{n,\lambda} - U_{T_{2,n}}^{q_n}g_{n,\lambda}\|_2 \to 0$ as $n \to \infty$ so that $\varepsilon_n \to 0$ as $n \to \infty$.

Starting with the intervals $(t-\delta, t+\delta) \subseteq \mathbf{T}$, let us define

$$E^{\lambda}_{n,\delta} = \bigcup_{e^{iq_n t}=\lambda} (t-\delta, t+\delta). \tag{3.12}$$

Letting θ be the smallest positive number with $\lambda = e^{iq_n\theta}$ and letting $\rho'_{n,\lambda} = \| g_{n,\lambda} \|_2^{-2} \rho_{n,\lambda}$, we have

$$\begin{aligned} \varepsilon_n &= \left| \int_{-\pi}^{\pi} e^{iq_n t} d\rho_{n,\lambda}(t) - \| g_{n,\lambda} \|_2^2 \lambda \right| \\ &= \| g_{n,\lambda} \|_2^2 \left| \int_{-\pi}^{\pi} e^{iq_n(t-\theta)} d\rho'_{n,\lambda}(t) - 1 \right| . \end{aligned} \tag{3.13}$$

Then for small δ, with $\delta_n = q_n\delta$, (3.13) implies

$$\begin{aligned} 1 - \varepsilon_n \| g_{n,\lambda} \|_2^{-2} &\leqq \int_{-\pi}^{\pi} \cos q_n(t-\theta) d\rho'_{n,\lambda}(t) \\ &\leqq \rho'_{n,\lambda}(E^{\lambda}_{n,\delta}) + (1 - \rho'_{n,\lambda}(E^{\lambda}_{n,\delta}))\cos \delta_n \\ &\leqq \delta_n^2 \rho'_{n,\lambda}(E^{\lambda}_{n,\delta})/2 + 1 - \delta_n^2/2 + \delta_n^4/24, \end{aligned}$$

so that

$$\rho'_{n,\lambda}(E^{\lambda}_{n,s}) \geqq 1 - \delta_n^2/12 - 2\varepsilon_n/(\delta_n^2 \| g_{n,\lambda} \|_2^2).$$

Taking

$$\delta = \varepsilon_n^{1/4} q_n^{-1} \| g_{n,\lambda} \|_2^{-1/2}, \tag{3.14}$$

we have

$$\begin{aligned} \rho_{n,\lambda}(E^{\lambda}_{n,\delta}) &= \| g_{n,\lambda} \|_2^2 \rho'_{n,\lambda}(E^{\lambda}_{n,\delta}) \\ &\geqq \| g_{n,\lambda} \|_2^2 - (25/12) \| g_{n,\lambda} \|_2 \varepsilon_n^{1/2}. \end{aligned} \tag{3.15}$$

For δ as above, let us define

$$F^a_n = \bigcup_{\lambda \in \chi_a(\mathcal{O})} E^{\lambda}_{n,\delta}. \tag{3.16}$$

Then by (3.15) and the definition of $g_{n,\lambda}$,

$$\begin{aligned} \rho_g(F^a_n) &\geqq \sum_{\lambda \in \chi_a(\mathcal{O})} \rho_{n,\lambda}(E^{\lambda}_{n,\delta}) \\ &\geqq \sum_{\lambda \in \chi_a(\mathcal{O})} \| g_{n,\lambda} \|_2^2 - K\varepsilon_n^{1/2} \\ &= 1 - K\varepsilon_n^{1/2} \end{aligned} \tag{3.17}$$

(and K depends only on the cardinality of $\chi_a(\mathcal{O})$).

Now let us repeat the construction for $U_{T_2}|_{H^2_{a'}}$, letting g' be a unit vector of maximal spectral type with corresponding spectral measure $\alpha_{g'}$. We have by (3.16),

$$\rho_{g'}(F_n^{a'}) \geqq 1 - K'\varepsilon_n'^{1/2}, \tag{3.18}$$

where $\varepsilon_n' \to 0$ as $n \to \infty$.

It follows from (3.11), (3.12), (3.14) and (3.16) that

$$F_n^a \cap F_n^{a'} = \varnothing \tag{3.19}$$

for sufficiently large n. Then (3.17), (3.18) and (3.19) imply that ρ_g and $\rho_{g'}$ are mutually singular.

(iii) For each $l \in \mathbf{Z}/m$ there are infinitely many n such that $R_l \circ T_{1,n}$ is cyclic. Thus $R_l \circ T_{1,n}$ constitutes a good cyclic approximation for $R_l \circ T_1$. By [5] Corollary 2.1, $R_l \circ T_1$ is ergodic for each l. By a straightforward generalization of [10] Lemma 3, since T_0 is weak mixing and each $R_l \circ T_1$ is ergodic, T_1 is weak mixing.

Now let us suppose $U_{T_2} f = \lambda f$ for some $f \in H^2_a$. By Lemma 2.5 there exists $f' \in H^2_{a'}$, $a' = \bar{\alpha} a$, with $U_{T_2} f' = \lambda f'$. The function $g = f'/f \in H^2_{a''}$, for some a'', is invariant, and by the assumption that α is proper, g is not constant. By (ii) above, $a'' = 0$, but by the second part of Lemma 2.4 this contradicts the fact that T_1 is weak mixing. □

The next lemma shows that $(\gamma_1, \gamma_2) \in \mathscr{A}$ satisfying the hypotheses of Lemma 3.5 exist. In fact, they are actually generic.

LEMMA 3.6. *Let T_0 admit a good cyclic approximation $(T_{0,n}, \xi_{0,n})$. Then there is a subsequence (T_{0,n_k}, ξ_{0,n_k}) and a dense G_δ subset $\mathscr{A}'$ of $\mathscr{A}$ such that each pair $(\gamma_1', \gamma_2') \in \mathscr{A}'$ is admissible, with the following additional property: For any $l \in \mathbf{Z}/m$, infinitely many $T_{1,n}$ are l-satisfactory, and for any α-orbit $\mathcal{O}$, infinitely many $T_{2,n}$ are $\mathcal{O}$-satisfactory.*

PROOF. We follow the method of [4] closely. Let $\mathscr{T}$ be the disjoint union of all $l \in \mathbf{Z}/m$ and all α-oribits $\mathcal{O}$. It follows from Lemmas 3.3 and 3.4 that for any $t \in \mathscr{T}$ and any $\xi_{0,n}$-measurable $(\gamma_1, \gamma_2) \in \mathscr{A}$ there exists a $\xi_{0,n}$-measurable $(\bar{\gamma}_1, \bar{\gamma}_2) \in \mathscr{A}$ with $\|\gamma_i - \bar{\gamma}_i\| < 3/q_n$ such that $T_{i,n}$, $i = 1$ or $i = 2$, is t-satisfactory. Since $\xi_{0,n} \to \varepsilon$, $q_n \to \infty$, so we have for any $j > 0$ there exists $n = n(j)$ sufficiently large for the following: For any $(\gamma_1, \gamma_2) \in \mathscr{A}$, $t \in \mathscr{T}$ there exists a $\xi_{0,n}$-measurable $(\bar{\gamma}_1, \bar{\gamma}_2) \in \mathscr{A}$ with $\|\gamma_i - \bar{\gamma}_i\|_1 < 1/j$, $i = 1, 2$, such that $T_{1,n}$ or $T_{2,n}$ is t-satisfactory. Let us fix such a $\bar{\gamma}_i$ in each case. This choice defines a function F_i with $\bar{\gamma}_i = F_i(\gamma_1, \gamma_2, t, j)$.

Let $S'(q) = 1/q^3$, so that $S'(q) = o(1/q^2)$ as $q \to \infty$. We define

$$G(\gamma_1, \gamma_2, t, j) = \{(\gamma'_1, \gamma'_2) \in \mathcal{A}: \text{for } n = n(j), \|\gamma'_i - \bar{\gamma}_i\|_1 < S'(q_n), i = 1, 2\}, \tag{3.20}$$

which is clearly a nonempty open subset of $\mathcal{A}$. Let

$$\mathcal{G}_{t,J} = \bigcup_{j=J}^{\infty} \bigcup_{(\gamma_1, \gamma_2) \in \mathcal{A}} G(\gamma_1, \gamma_2, t, j)$$

which is open and dense, so that

$$\mathcal{A}' = \bigcap_{t \in \mathcal{T}} \bigcap_{J=0}^{\infty} \mathcal{G}_{t,J}$$

is dense G_δ.

Now (γ'_1, γ'_2) belongs to $\mathcal{A}'$ if, for each $t \in \mathcal{T}$, it belongs to an infinite sequence of neighborhoods $G(\gamma^k_1, \gamma^k_2, t, j_k)$ with $j_k \to \infty$. We define the sequence (T_{0,n_k}, ξ_{0,n_k}) to be the subsequence $(T_{0,n(j_k)}, \xi_{0,n(j_k)})$. Clearly, the sequence $\bar{\gamma}^k_i \overset{\text{def}}{=} F_i(\gamma^k_1, \gamma^k_2, t, j_k)$ is ξ_{0,n_k}-measurable and has subsequences which are satisfactory in every way. Furthermore, by (3.20) and the definition of S', $\|\bar{\gamma}^k_i - \gamma'_i\|_1 = o(1/q^2_{n(j_k)})$ as $k \to \infty$, so that (γ'_1, γ'_2) is admissible, corresponding to $(\bar{\gamma}^k_1, \bar{\gamma}^k_2)$. □

Proof of Theorem 2.2. First assume only that α is separating. Choose $(\gamma_1, \gamma_2) \in \mathcal{A}'$. By Lemmas 3.6 and 3.5(i), each $U_{T_2}|_{H^2_a}$ has spectral multiplicity 1. For each $\bar{\alpha}$ orbit $\mathcal{O}$, let $H^2_{\mathcal{O}} = \bigoplus_{a \in \mathcal{O}} H^2_a$. Then by Lemma 2.5, $U_{T_2}|_{H^2_{\mathcal{O}}}$ has spectral multiplicity uniformly equal to card $\mathcal{O}$. By Lemma 3.5(ii) $U_{T_2}|_{H^2_{\mathcal{O}}}$ and $U_{T_2}|_{H^2_{\mathcal{O}'}}$ have mutually singular spectral types for all pairs of orbits $\mathcal{O} \neq \mathcal{O}'$. It follows that $\mathcal{M}_T = \mathcal{M}_\alpha$. The ergodicity of T_2 follows from (ii) and from the ergodicity of T_1, which follows from the fact that T_1 admits a good cyclic approximation ([5] Corollary 2.1).

Now assume that α is proper. Then by Lemma 3.4(iii) $U_{T_2}|_{H^2_a}$, $a \neq 0$, has continuous spectrum, and $U_{T_2}|_{H^2_0}$ has only the eigenvalue 1 corresponding to the constants. This implies T_2 is weak mixing. □

Theorem 2.1 can immediately be strengthened as follows:

Corollary 3.7 (Genericity). *For all T_0 in a dense G_δ subset $\mathcal{U}'$ of $\mathcal{U}$ there is a dense G_δ subset $\mathcal{A}'_{T_0}$ of $\mathcal{A}$ such that the corresponding transformations T_2 satisfy Theorem 2.2.*

With a similar but more elaborate argument along the lines of [8, Proposition 6.1], one can obtain the following alternative genericity statement.

COROLLARY 3.8. *There is a dense G_δ subset $\mathcal{W}$ of $\mathcal{U} \times \mathcal{A}$ (in the product topology) such that the corresponding transformations T_2 satisfy Theorem* 2.1.

Using the methods of [8, §7] it is possible to prove that transformations satisfying Theorem 2.2 can be realized within the class of interval exchange transformations. Using the arguments of [10, §3], transformations satisfying Theorem 2.2 can also be constructed by cutting and stacking.

Finally, we note that a close look at the proof of Theorem 2.2 reveals that the transformations T_0, T_1 and T_2 are all non-mixing, rigid, and have singular spectrum (cf. [4], [5]).

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