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APPROXIMATELY TRANSITIVE (2) FLOWS AND TRANSFORMATIONS HAVE SIMPLE SPECTRUM

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Introduction.

In [CW] A. Connes and E.J. Woods introduced a new property associated to group actions on Lebesgue spaces, called approximate transitivity. This property arises naturally in the context of hyperfinite von Neumann factors and in the study of nonsingular ergodic transformations. They showed, using von Neumann algebra techniques, that a transformation is orbit equivalent to an odometer of product type if and only if its Poincaré flow is approximately transitive. (A hyperfinite von Neumann factor is ITPFI if and only if the flow of weights is AT.)

One is led naturally to study this apparently new property of group actions in the context of ergodic theory. This was done to some extent in [CW], where they proved that all AT actions are ergodic, and measure-preserving AT transformations have zero entropy. Further properties of AT actions are discussed in [HW] and [H].

Approximate transitivity is an L^1 approximation property. The authors generalize the definition to the L^p case, as was done in [H], and concentrate on the L^2 case. The property called approximate transitivity (p) or AT(p) is introduced, and some properties of these actions are discussed.

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The main results of this paper are the following. First we prove that transitive free group actions are AT(p) for all $1 \le p < \infty$, and that for each p, the property of being AT(p) is an isomorphism invariant. We use these results to prove that to every odometer of product type, say $(X,\mathcal{B},\mu,G)$, we can associate a canonical $G \times \mathbb{R}$ action which is AT(p) for all p; from this we obtain a theorem stating that every AT(1) flow is isomorphic to a factor action of an AT(p) action for every $1 \le p < \infty$. In an earlier unpublished version of this paper the authors claimed that Poincaré flows for odometers of product type are AT(p) for all $p \in [1,\infty)$. The proof contained a gap, and this general question is still open. In some finite measure-preserving cases discussed in [H], it is true that AT(p) for $p = 1$ is equivalent to AT(p) for $p \in (1,\infty)$.

A study of properties of AT(2) action is done in §3. The main theorem of that section states that AT(2) flows and transformations have simple L^2 spectrum. As a corollary we obtain that finite-measure-preserving AT(2) flows and transformations have zero entropy. A result of independent interest proved in this section states that if T is an ergodic measure-preserving transformation, and F_t is its suspension flow with constant ceiling function, then T has simple spectrum if and only if F_t has simple spectrum.

We conclude the paper by studying examples of well known flows and transformations in ergodic theory to see which of these are and are not AT(2). Recent results of Choksi and Nadkarni [CN] prove that AT(2) transformations are generic in the space of nonsingular transformations. It remains to be determined, however, whether all approximately transitive (1) transformations are AT(2).

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§1. Definition and notation

We beign with the definition of Connes and Woods of an approximately transitive group action.

Definition 1.1. [CW] Let G be a Borel group, (X,μ) a Lebesgue measure space and $\alpha: G \longrightarrow \mathrm{Aut}(X,\mu)$ = {the group of nonsingular invertible automorphisms of (X,μ)}, a Borel homomorphism. We say that the action is approximately transitive (AT) if given $f_1,\cdots,f_n \in L^1_+(X,\mu)$ and $\epsilon > 0$, there exist $f \in L^1_+(X,\mu)$, $g_1,\cdots,g_m \in G$ and $\lambda_{jk} \geq 0$ such that $\|f_j - \sum_{k-1}^{m} \lambda_{jk} \cdot f \circ \alpha_{g_k} \frac{d\mu\alpha_{g_k}}{d\mu}\|_1 < \epsilon$ for each j. We also write AT(1) for approximate transitivity. In [H] it was shown that the approximating function f in the definition could be chosen to be a step (L^∞) function. We generalize this definition to the L^p case, and call it approximate transitivity (p) or AT(p), since in the L^2 case it provides a natural sufficient condition for simple spectrum. We fix any $p \in [1,\infty)$.

Definition 1.2. A Borel group G acting on a Lebesgue space (X,μ) is approximately transitive in the L^p norm, or AT(p), if given $f_1,\cdots,f_n \in L^p_+(X,\mu)$ and $\varepsilon > 0$, there exist $f \in L^p_+(X,\mu), g_1,\cdots,g_m \in G$ and $\lambda_{jk} \geq 0$ such that

$$\|f_j - \sum_{k-1}^{m} \lambda_{jk} \cdot f \circ \alpha_{g_k} \left[\frac{d\mu\alpha_{g_k}}{d\mu}\right]^{1/p} \|_p \leq \varepsilon$$

for each j. (This definition is equivalent to the one given in [H]).

Connes and Woods prove that a countable nonsingular ergodic amenable equivalence relation is orbit equivalent to an odometer of product type if and only if its associated ergodic flow is AT(1) [CW]. (We remark that these equivalence relations are generated by single ergodic transformations [CFW].) Their proof (and theorem) deals completely with von Neumann factors. An ergodic theoretic proof of one direction of the theorem is given in [H]; that proof is generalized in this paper to give the following proposition which is proved in the next section.

Proposition 1.3. The Poincaré flow of an odometer of product type is a factor action of an AT(p) group action for each $1 \le p < \infty$.

We define an odometer here, noting that odometers serve as prototypes for all orbit equivalence classes of countable nonsingular amenable ergodic equivalence relations [D], [S], [Kr].

Definition 1.4. Let $\{d_k\}_{k\ge 1}$ be a sequence of integers ≥ 1, and let $X_k = \{0,\cdots,d_k-1\}$. We define the Borel space $X = \prod_{k=1}^{\infty} X_k$, with $\mathcal{B}$ the σ-algebra of Borel sets on X. We let G_k denote the group of all cyclic permutations on X_k; then G_k also acts on X (by acting only on the k^{th} coordinate). Now by G we denote the group generated by all the G_k's; that is $G = \bigcup_{n=1}^{\infty}\left[\prod_{k=1}^{\infty} G_k\right]$. If we put any σ-finite Borel measure μ on $(X,\mathcal{B})$ we respect to which G acts ergodically, then we say $(X,\mathcal{B},G,\mu)$ is a measured odometer. If μ is a product measure of the form $\mu = \prod_{k=1}^{\infty}\mu_k$ with $\mu_k(X_k) = 1$ and $\mu_k(\{i\}) > 0$, then we say that $(X,\mathcal{B},G,\mu)$ is an *odometer of product type*, or a *product odometer.* One can check that the full group of G, denoted [G] is the same as the full group of the transformation T defined as follows:

Let $r(x) = \min\{k\ge 1: x_k < d_k - 1\}$, then

$$(Tx)_k = \begin{cases} 0 & \text{if } k < r(x) \\ x_k + 1 & \text{if } k = r(x) \\ x_k & \text{if } k = r(x) \end{cases}$$

hence the term odometer is appropriate for this action.

There is a canonical way to associate an ergodic flow to any measured odometer, and it has been proved by Krieger [Kr] that in the nonsingular and non-measure-preserving case this flow (up to metric isomorphism) provides a complete invariant for orbit equivalence classes of odometers. Thus flow is defined by first considering the G action on $X \times \mathbb{R}$ given by $(x,y) \longmapsto \left[gx, y+\log\frac{d\mu g}{d\mu}(x)\right]$ for each $g \in G$. In general this action is not ergodic, so we consider a measurable partition of $X \times \mathbb{R}$ which generates the σ-algebra $\mathcal{B}_0$ of all G-invariant sets up to sets of measure zero. The natural projection from $X \times \mathbb{R}$ to $(X \times \mathbb{R})/\mathcal{B}_0 \cong Y$ is a factor map; the desired flow is obtained from the $\mathbb{R}$-action $(x,y) \longmapsto (x,y+t)$ induced on the factor space Y. We remark that this $\mathbb{R}$-action is the same as the $G \times \mathbb{R}$ action given by:

$$\alpha_{(g,t)}(x,y) = \left[gx, y+t+\log\frac{d\mu g}{d\mu}(x)\right] \quad \text{for all } (g,t) \in G \times \mathbb{R},$$

$(x,y) \in X \times \mathbb{R}$ and then induced on the factor space Y, (since everything in the G direction collapses).

Definition 1.5. The factor action defined above is called the *Poincaré flow associated to the odometer*. (A complete account of this flow is given in [HO]).

We conclude this section by recalling the definition of simple spectrum for a flow

Definition 1.6. A nonsingular ergodic flow (F_t) on (Y,υ) has *simple spectrum* if the unitary representation U^t of $\mathbb{R}$ on $L^2(Y,\upsilon)$ defined by:

$$U^t f = f \circ F_t \cdot \left(\frac{d\upsilon F_t}{d\upsilon}\right)^{1/2}$$

has the property that there exists an element $f \in L^2(Y,\upsilon)$ such that $L^2(Y,\upsilon)$ = closure in L^2 of $\left\{\sum_{k=0}^{n} a_k U^{t_k} f;\ a_k \in \mathbb{C},\ t_k \in \mathbb{R}\right\}$, (cf. § 3 for a discussion of this and related definitions).

§2 Approximate transitivity in the L^p norm.

We begin with a lemma which shows that transitive actions are AT(p).

Lemma 2.1. Let H be a metrizable locally compact abelian group which acts on itself by translation. The action is AT(p) for all $1 \le p < \infty$.

<u>Proof</u>. We show there exists an approximate identity for $L^p(H,d\omega)$, where $d\omega$ denotes Haar measure for the group H; that is, we prove the existence of a sequence of convolution operators on $L^p(H,d\omega)$ converging strongly to the identity. In particular, there exists a sequence of L^1 functions $\rho_k \ge 0$, $\|\rho_k\| = 1$ such that for all $f \in L^p(H,d\omega)$

$$f * \rho_k(h) = \int_G f(g)\rho_k(hg^{-1})d\omega(g) = \int_G f(hg^{-1})\rho_k(g)d\omega(g)$$

satisfies $f*\rho_k \longrightarrow f$ in $L^p(H,d\omega)$ as $k \longrightarrow \infty$. We define ρ_k as follows. Let B_k = ball of radius $1/k$ about $e \in H$, and let $\omega_k = \omega(B_k)$. We now define

$$\rho_k(h) = \begin{cases} \omega_k^{-1} & \text{if } h \in B_k \\ 0 & \text{if } h \notin B_k . \end{cases}$$

Then $\|\rho_k\|_1 = 1$, and we show that $T_k f = f*\rho_k$ is a bounded operator.

$$\|T_k f\|_p = \| \int_H f(hg^{-1})\rho_k(g)d\omega(g)\|_p$$

$$= \left[\int_H \left(\int_H f(hg^{-1})\rho_k(g)d\omega(g)\right)^p\right]^{1/p}$$

by Minkowski's integral inequality

$$\le \int_H \left(\int_H f(hg^{-1})\rho_k(g)^p d\omega(h)\right)^{1/p} d\omega(g)$$

$$\le \int_H |\rho_k(g)| \; \|f\|_p d\omega(g) = \|f\|_p .$$

We now suppose that $f \in L^p(H,d\omega)$ is continuous. Since $f(h)\cdot 1 = f(h)\cdot \int_H \rho_k(g)d\omega(g)$, we have

$$T_k f(h) - f(h) = \int_H \left[f(hg^{-1}) - f(h)\right]\rho_k(g)d\omega(g), \text{ so}$$

$$\|T_k f - f\|_p = \left[\int_H \left| \int_H \left[f(gh^{-1}) - f(h)\right]\rho_k(g)d\omega(g)\right|^p d\omega(h)\right]^{1/p}$$

$$\le \int_H \left[\int_H |f(hg^{-1}) - f(h)]\rho_k(g)|^p d\omega(h)\right]^{1/p} d\omega(g),$$

$$\le \int_H \left[\rho_k(g) \int_H |f_g(h) - f(h)|^p d\omega(h)\right]^{1/p} d\omega(g),$$

where $f_g(h) = f(hg^{-1})$, and the above is equal to

$$\int_H \rho_k(g)\|f_g(h) - f(h)\|_p d\omega(g)$$

$$\int_{B_k} \|f_g - f\|_p \omega_k^{-1} d\omega(g) + \int_{H\setminus B_k} \|f_g - f\|_p \cdot 0 .$$

By the continuity of $f, f_g(h) - f(h)$ is small for all $h \in B_k$ when k is large, so the above integral will be less than any fixed $\varepsilon > 0$ when k is large enough. Since the continuous functions are dense in $L^p(H, d\omega)$, and $\{T_k\}$ is a uniformly bounded sequence of operators, then it follows that $T_k \longrightarrow \text{Id}$ strongly on $L^p(H, d\omega)$.

To show that this implies approximate transitivity in the L^p norm is easy. Suppose we are given $\varepsilon > 0$ and $f_1, \cdots, f_n \in L^p_+(H, d\omega)$. We first choose k large enough so that $\|f_{j-\rho_k} * f_j\|_p < \varepsilon/4$, and then we choose $\rho_k = f \in L^p_+(H, d\omega)$. We then have

$$\|f_j - \int_H \lambda_j(g) f(hg^{-1}) d\omega(g)\| < \varepsilon/4 \quad \text{for each } j.$$

By approximating $\lambda_j = f_j$ by step functions, as in [CW], we can pass to a finite sum (cf. [H, Cor. 3.4] for details):

$$\|f_j - \sum_{k=1}^{s} \lambda_{jk} \cdot f(hg_k^{-1})\|_p < \varepsilon \quad \text{for each } j.$$

This proves the lemma. □

Our next lemma shows that the AT(p) property is invariant under nonsingular isomorphisms, so any transitive free action (i.e. even one which does not preserve Haar measure but leaves it quasi-invariant) is $AT(p)$.

<u>Lemma</u> 2.2. Suppose that the Borel group G has an $AT(p)$ action $\alpha: G \longrightarrow \mathrm{Aut}(X,\mu)$, and there exists a measure $\upsilon \sim \mu$ and another action $\beta: G \longrightarrow \mathrm{Aut}(X,\upsilon)$ such that the actions are isomorphic. Then the action $\{\beta_g\}_{g\in G}$ is also $AT(p)$, for any $1 \le p < \infty$.

<u>Proof</u>. By our hypotheses, there exists $\varphi:(X,\mu) \longrightarrow (X,\upsilon)$ an invertible (a.e.) map such that $\varphi(\alpha_g x) = B_g(\varphi x)$ for every $g \in G$, μ-a.e. $x \in X$, and $\upsilon\varphi \sim \mu$. We obtain operators on the appropriate L^p spaces from α, β, and φ as follows.

We define for each $g \in G$ the operator

$$A_g: L^p(X,\mu) \longrightarrow L^p(X,\mu) \quad \text{by} \quad A_g f(x) = f(\alpha_g x)\left[\frac{d\mu\alpha_g}{d\mu}(x)\right]^{1/p}$$

$$B_g: L^p(X,\upsilon) \longrightarrow L^p(X,\upsilon)$$

given by $B_g f(x) = f(\beta_g x)\left[\dfrac{d\upsilon\beta_g}{d\upsilon}(x)\right]^{1/p}$ for each $g \in G$, μ or υ - a.e. $x \in X$. We get an intertwining operator from φ, the map $U_\varphi: L^p(X,\upsilon) \longrightarrow L^p(X,\mu)$ defined by $U_\varphi f(x) = f(\varphi x)\left[\dfrac{d\upsilon\varphi}{d\mu}\right]^{1/p}$ for each $g \in G$, μ or υ - a.e. $x \in X$. It is easy to check that $A_g U_\varphi f = U_\varphi B_g f$ for all $f \in L^p(X,\upsilon)$; that is, the diagram commutes:

$$\begin{array}{ccc} L^p(X,\mu) & \xrightarrow{\;A_g\;} & L^p(X,\mu) \\ U_\varphi \uparrow & & \\ L^p(X,\upsilon) & \xrightarrow{\;B_g\;} & L^p(X,\upsilon). \end{array}$$

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since each element of U_ℓ can be identified with an element of G which does not affect any coordinates of $x \in X$ after x_ℓ.

Then using $f \in L^p_+(X\times\mathbb{R},\upsilon), \lambda_{jk}$, and $(g_k, t_k) \in G \times \mathbb{R}$ obtained above, we have

Suppose that we are given $f_1,\cdots,f_n \in L_+^P(x,\upsilon)$ and $\varepsilon > 0$. Then we consider $U_\varphi f_1,\cdots,U_\varphi f_n \in L_+^P(x,\mu)$ and we can find $g_1,\cdots,g_m$, $\lambda_{jk} \geq 0$ and $h \in L_+^P(x,\mu)$ with $h = U_\varphi f_1$ for some $f \in L_+^P\ x,\mu)$ with $h = U_\varphi f$ for some $f \in L_+^P(x,\upsilon)$ such that $\|U_\varphi f_j - \sum_{k=1}^{m} \lambda_{jk} \cdot A_{g_k} h\|$.

Equivalently,

$$\|U_\varphi f_j - \sum_{k=1}^{m} \lambda_{jk} \cdot A_{g_k}\ U_\varphi f\| = \|U_\varphi f_j - \sum_{k=1}^{m} \lambda_{jk}\ U_\varphi B_{g_k} f\|_p < \varepsilon.$$

Using the linearity of U_φ and the fact that it is norm preserving, it is clear that the action given yb β on (X,υ) is AT(p).□

We now turn to the product odometers introduced in §1. We can write the space $X = \bar{X}_n \times \underline{X}^n$ by defining

$$\bar{X}_n = \prod_{k=1}^{n} X_k \quad \text{and} \quad \underline{X}^n = \prod_{k=n+1}^{\infty} X_k.$$

Similarly, the product measure μ can be written as $\mu = \bar{\mu}_n \times \underline{\mu}^n$, with $\bar{\mu}_n = \prod_{k=1}^{n} \mu_k$ $\underline{\mu}^n = \prod_{k=n+1}^{\infty} \mu_k$. Also by $\bar{G}_n$ we denote the group

generated by $G_1,\cdots,G_n$ and by $\underline{G}^n$ the group generated by G_{n+1},

$G_{n+2},\cdots$, so $G = \bar{G}_n \oplus \underline{G}^n$. We remark that $\bar{G}_n$ acts freely and transitively on $\bar{X}_n$, leaving $\bar{\mu}_n$ quasi-invariant. Furthermore if we consider the action of $G_n \times \mathbb{R}$ on $\bar{X}_n \times \mathbb{R}$ given by $\alpha_{(g,t)}(x,y) = (gx, y+t+\log\frac{d\bar{\mu}_n}{d\bar{\mu}_n}(x))$ for each $(g,t) \in \bar{G}_n \times \mathbb{R}$, $(x,y) \in \bar{X}_n \times \mathbb{R}$, we see that this action is transitive and free. If we put the finite measure $\nu_n = \bar{\mu}_n \times e^{-y^2} dy$ on $X_n \times \mathbb{R}$, then applying Lemma 2.2 tells us that this action is AT(p) with respect to ν_n. This is used in the proof of Proposition 1.3.

We remark that it was proved in [CW] and in [H] that the factor action of an AT(1) action is AT(1); this proof does not work for AT(p) actions if $p > 1$ unless the action is finite measure-preserving (because of the presence of a Radon-Nikodym derivative which does not cancel). However we can use the idea of the proof in [H] in the following proposition.

Proposition 1.3. Let $(X,\mathcal{B},\mu,G)$ denote an odometer of product type. Then the $G \times \mathbb{R}$ defined by $\alpha_{(g,t)}(x,y) = (gx, y+t+\log\frac{d\mu g}{d\mu}(x))$ for each $(g,t) \in G \times \mathbb{R}$, $(x,y) \in X \times \mathbb{R}$ is AT(p) for every $1 \le p < \infty$. Consequently the Poincaré flow of an odometer of product type is the factor action of an AT(p) group action for each $p \in [1,\infty)$.

Before proving the proposition, we state and prove a corollary which gives an interesting characterization of approximately transitive flows using the theorem of Connes and Woods.

Corollary 2.3. An ergodic nonsingular flow is AT(1), or approximately transitive, if and only if it is a factor flow of an action is AT(p) for each $p \in [1,\infty)$.

Proof: (⇒) We assume that a nonsingular ergodic flow F_t is AT(1). Then by the theorem of [CW], F_t is the Poincaré flow of an odometer of product type. That is, F_t is a factor action of the $G \times \mathbb{R}$ action on $X \times \mathbb{R}$ defined above. By Proposition 1.3, F_t is the factor action of an AT(p) action for every $1 \le p < \infty$.

(⇔) We now assume that the flow F_t is the factor action of an action which is AT(p) for every $1 \le p < \infty$. By [CW], it follows that F_t itself is AT(1). □

We now turn to the proof of Proposition 1.3, using all notation as defined above.

Proof. Assume we are given $f_1,\ldots,f_n \in L_+^p(X\times\mathbb{R},\nu)$ and $\varepsilon > 0$. We can approximate each f_j in the L^p norm by a step function of $X \times \mathbb{R}$ whose support in X is a finite number of cylinders. More precisely, we find a positive integer ℓ dependent on ε, and functions $f_j^{(\ell)}(x,y) = f_j^{(\ell)}(x_1,\ldots,x_\ell,\cdot,y)$ (its value depends only on the first ℓ coordinates of $x \in X$), and such that $\|f_j - f_j^{(\ell)}\|_p < \varepsilon/2$ for each $j = 1,\ldots,n$.

Since the action of $\overline{G}_\ell \times \mathbb{R}$ is AT(p) with respect to ν_ℓ, we identify each $f_j^{(\ell)}$ with the function it represents in $L_+^p(\overline{X}_\ell \times \mathbb{R},\nu_\ell)$, and then we can find elements $g_1,\ldots,g_m \in \overline{G}_\ell$, $t_1,\ldots,t_m \in \mathbb{R}$, $\lambda_{jk} \geq 0$ and $f \in L_+^p(\overline{X}_\ell \times \mathbb{R},\nu_\ell)$ satisfying:

$$\|f_j^{(\ell)} - \sum_{k=1}^{m} \lambda_{jk} \cdot f \circ \alpha_{(g_k,t_k)}\left[\frac{d\nu_\ell\alpha(g_k,t_k)}{d\nu_\ell}\right]^{1/p}\|_p < \varepsilon$$

for each j. Then we simply regard f as a function on $X \times \mathbb{R}$ by $f(x,y) = f(x_1,\ldots,x_\ell,y)$ and we use the fact that μ, being a product measure, gives us this nice identify: for all $(g,t) \in \overline{G}_\ell$

$$\left[\frac{d\nu_\ell\alpha_{(g,t)}}{d\nu_\ell}(x_1,\ldots,x_\ell,y)\right]^{1/p} = \left[\frac{(d\overline{\mu}_\ell \times e^{-y^2}dy)\alpha_{(g,t)}}{d\overline{\mu}_\ell \times e^{-y^2}dy}(x_1,\ldots,x_\ell,y)\right]^{1/p}$$

$$\left[\frac{(d\overline{\mu}_\ell \times \mu^\ell) \times e^{-y^2}dy\alpha_{(g,t)}}{d\overline{\mu}_\ell \times \mu^\ell) \times e^{-y}dy}(x_1,\ldots,x_\ell,y)\right]^{1/p}$$

$$= \left[\frac{d\nu\alpha_{(g,t)}}{d\nu}(x,y)\right]^{1/p} \quad \text{for all } (x,y) \in X \times \mathbb{R},$$

since each element of U_ℓ can be identified with an element of G which does not affect any coordinates of $x \in X$ after x_ℓ.

Then using $f \in L_+^p(X\times\mathbb{R},\upsilon),\lambda_{jk}$, and $(g_k,t_k) \in G \times \mathbb{R}$ obtained above, we have equivalence by the unitary equivalence class of U^t. The measure class of σ is called the *maximal spectral type*. The *multiplicity function* $m(\lambda) = \dim H_\lambda$ is determined uniquely σ-a.e.

A cyclic subspace j of $\mathcal{H}$ corresponds to a measurable choice of a 1-dimensional subspace J_λ of each H_λ, $\lambda \in \mathbb{R}$, in some spectral representation of U^t. U^t has simple spectrum if and only if $m(\lambda) = 1$ for σ-a.e. $\lambda \in \mathbb{R}$.

Given a unitary operator U on $\mathcal{H}$ we may regard its powers U^n, $n \in \mathbb{Z}$ as a unitary representation of $\mathbb{Z}$. The definitions of cyclic subspace and simple spectrum generalize to this case in an obvious way, and there is also a spectral representation, similar to that for U^t, except that $\mathbb{R}$ is replaced with the circle $\mathbb{T}$.

Given a nonsingular transformation $\mathbb{T}$ (or a nonsingular measurable flow F_t) of a Lebesgue space (X,μ) we construct the *induced unitary operator* $U_T f(x) = f(Tx)\left(\frac{d\mu T}{d\mu}\right)^{1/2}$ (*strongly continuous unitoary representation* $U_F^t f(x) = f(F_t x)\left(\frac{d\mu F_t}{d\mu}(x)^{1/2}\right)$), on $L^2(X,\mu)$, and say T (resp. F_t) has *simple spectrum* if U_t (resp. U_F^t) has simple spectrum. We not that this definition is equivalent to Definition 1.6.

Lemma 3.1 was first obtained by Katok and Stepin [KS] for Z. We give the easy proof for $\mathbb{R}$, noting that it can be generalized to type I groups using a result of Riley [R1].

<u>Lemma</u> <u>3.1</u> Suppose U^t does not have simple spectrum. Then there exist orthonormal vectors $\varphi_1, \varphi_2 \in \mathcal{H}$ such that for any cyclic subspace J of $\mathcal{H}$,

$$d^2(\varphi_1,J) + d^2(\varphi_2,J) \geq 1, \quad (*)$$

where d denotes the distance from a vector to a subspace.

<u>Proof</u>. Consider the spectral representation for U^t and let $M = \{\lambda \in \mathbb{R}: m(\lambda(> 1\}$. Since the spectrum of U^t is not simple $\sigma(M) > 0$, and by changing to an equivalent measure we may assume

$\sigma(M) = 1$. For each $\lambda \in M$, measurably choose an orthonormal pair $\varphi_1(\lambda), \varphi_2(\lambda) \in H_\lambda$ and define $\varphi_1(\lambda) = \varphi_2(\lambda) = 0$ for $\lambda \not\in M$. It is easy to see that φ_1 and φ_2 are orthonormal in $\mathcal{H}$.

For a cyclic subspace J of $\mathcal{H}$ and $h_1, h_2 \in J$ we have

$$\|\varphi_1 - h_1\|^2 + \|\varphi_2 - \varphi_2\|^2$$

$$\geq \int_M (\|\varphi_1(\lambda) - h_1(\lambda)\|^2 + (\|\varphi_2(\lambda) - h_2(\lambda)\|^2) d\sigma(\lambda)$$

$$\geq \int_M (d^2_{H_\lambda}(\varphi_1(\lambda)\ J_\lambda)(\lambda)\|^2 + (\|\varphi_2(\lambda)\|^2) d\sigma(\lambda).$$

As easy computation shows that for a Hilbert space of dimension at least two, the inequality (*) holds for any orthonormal pair and any 1-dimensional subspace. An application of this fact to the integrand for each λ yields the result. □

Let T be an ergodic measure preserving transformation on a Lebesgue probability space (X,μ) and let $(Y,\gamma) = (X\times[0,1],\mu\times ds)$, where ds is Lebesgue measure. the suspension F_t of T is the measure preserving flow on (Y,γ) defined by

$$F_t(x,s) = (T^k x, r)$$

where k and 4 are determined by the conditions $t + s = k + r, k \in \mathbb{Z}$ and $r \in [0,1)$. Let σ_F, m_F, σ_T and m_T denote the maximal spectral types and multiplicities for U_F^t and U_T respectively. Denote by exp the mapping $\mathbb{R} \longrightarrow \mathbb{T}$, $\exp(\lambda) = e^{2\pi i\lambda}$. We prove the following result of independent interest about the spectrum of t and F_t.

<u>Lemma 3.2.</u> The flow F_t defined above has simple spectrum if an only if T has simple spectrum.

<u>Proof.</u> Let H_n be the subspace of $L^2(Y,\gamma)$ of functions with spectral representation supported on the interval $I_n = [-n,-n+1] \subset \mathbb{R}$. The subspaces H_n, $n \in \mathbb{Z}$, form a U_F^t invariant orthogonal decomposition of $L^2(Y,\gamma)$, and $U_F^t|_{H_n}$ has a spectral representation with spectral type $\sigma_n = \chi_{I_n}\sigma_F$ and multiplicity $m_n = \chi_{I_n} m_F$. The

unitary operator $U_F^1 \mid_{H_n}$ has spectral type σ_n and multiplicity m_n, after identifying I_n with $\mathbb{T}$ by exp. Thus, it suffices to show that for each n, $U_F^1 \mid_{H_n}$ is equivalent to U_T.

For $f \in L^2(Y,\gamma)$, let $Mf(x,s) = e^{-2\pi is}f(x,s)$. M is unitary, and because $e^{-2\pi is}$ is an eigenfunction for the eigenvalue 1 of U_F^t,

$$U_F^t Mf(x,s) = e^{-2\pi it} MU_F^t f(x,s). \qquad (\#)$$

In particular, M commutes with U_F^t.

We now show that $MH_n = H_{n+1}$ or equivalently $P_{H_n} = M^{-1}P_{H_{n+1}}M$, where P_{H_n} denotes the prejection onto H_n. Let $P(\lambda)$ denote projection to the functions with spectral representation supported on $(-\infty,\lambda]$. $P(\lambda)$ is determined for σ_F - a.e. $\lambda \in \mathbb{R}$ by the condition that

$$(U_F^t f,g) = \int_{\mathbb{R}} e^{2\pi it\lambda} d(P(\lambda)f,g)$$

hold for all $f,g \in L^2(Y,\gamma)$. We have

$$(U_F^t Mf,Mg) = \int_{\mathbb{R}} e^{2\pi it\lambda} d(P(\lambda)Mf,Mg)$$

and by (#)

$$(U_F^t Mf,Mg) = e^{-2\pi it}(MU_F^t f,Mg) = e^{-2\pi it} MU_F^t f,g)$$

$$= e^{-2\pi it} \int_{\mathbb{R}} e^{2\pi it\lambda} d(P(\lambda)f,g)$$

$$= \int_{\mathbb{R}} e^{2\pi it\lambda} d(P\lambda+1)f,g) .$$

Thus

$$M^{-1}P(\lambda)M = P(\lambda+1),$$

and since $P_{H_n} = (Id - P(-n)\ P(-n+1),$

$$M^{-1}P_{H_{n+1}}M = P_{H_n}.$$

Next we consider the subspaces J_n of $L^2(Y,\gamma)$ defined as follows. Let W^t be a fixed strongly continuous unitary representation of $\mathbb{R}$ on $L^2(X,\mu)$ such that $W^1 = U_T$; (such a W^t exists by the spectral theorem). the subspace J_n will consist of functions of the form

$f(x,s) = e^{-2\pi i n s} W^s g(x)$. An easy computation shows that $\bigoplus_{n\in\mathbb{Z}} J_n$ forms a U_F^t invariant orthogonal decomposition of $L^2(Y,\gamma)$. Thus for each m and n the projections P_{J_n} and P_{H_n} commute. Furthermore, $U_F^t|_{J_m}$ is equivalent to U_T and $MJ_m = J_{m+1}$.

Let $K_{n,m} = H_n \cap J_m$ and note that since P_{H_n} and P_{J_m} commute, $L^2(Y,\nu) = \bigoplus_{n,m\in\mathbb{Z}} K_{n,m}$. We define $R_n : H_n \longrightarrow J_n$ by $R_n|K_{n,m} = M^{n-m}$, so that $R_n K_{n,m} = K_{2n-m,n}$, and extend to H_n by linearity. It is clear that R_n intertwines U_F^1 on H_n and J_n, establishing the desired equivalences. □

<u>Proposition</u> <u>3.3</u>. Let F_t be an AT(2) flow. Then F_t has simple spectrum.

<u>Proof.</u> Suppose the spectrum is not simple and choose φ_1 and φ_2 according to Lemma 3.1. Write

$$\varphi_j = \varphi_j^1 - \varphi_j^2 + i\varphi_j^3 - i\varphi_j^4$$

$j = 1,2$, where $\varphi_j^k \ge 0$. Given $\varepsilon > 0$ there exist $f \in L_+^2(X,\mu)$, $\lambda_{j\ell}^k \ge 0$ and $t_\ell \in \mathbb{R}$, $j = 1,2, k = 1,\cdots,4$, $\ell = 1,\cdots,p$ such that

$$\left\|\varphi_j^k - \sum_{\ell-1}^{p} \lambda_{\ell-1}^k f \circ F_{t_\ell}\left(\frac{d\mu F_{t_\ell}}{d\mu}\right)^{1/2}\right\|_2 < \varepsilon/8$$

for all j,k,ℓ. Thus there exist $h_1, h_2 \in H(f)$ such that

$$\|\varphi_1 - h_1\|_2 + \|\varphi_2 - h_2\|_2 < \varepsilon,$$

contradicting Lemma 3.1. □

<u>Remarks</u>. 1. Proposition 3.3 is true for nonsingular flows, transformations, and type I group actions which are AT(2); that is, it holds for the same actions as those for which Lemma 3.1 is true.

2. In particular, the $G \times \mathbb{R}$ action defined in Proposition 1.3 has simple spectrum, so the Poincaré flow of an odometer of product type is always a factor action of an action with simple spectrum.

3. Using a proof similar to [H, Thm. 3.2] it can be shown that a factor action of a finite measure-preserving AT(2) action is AT(2) (and therefore has simple spectrum), but the general question is still open. Therefore it is not yet known whether an AT(1) flow has simple spectrum.

§4 Examples

We conclude with a composition of some of the implications of the fact that AT(2) transformations and flows have simple spectrum. For any finite measure-preserving action, we see easily that AT(p) implies AT(q) for $q < p$; similarly if an action is not AT(2), then it is not AT(p) for any $p \geq 2$.

Connes and Woods [CW] show that an AT measure-preserving transformation has zero entropy. We obtain that result for AT(2), and in addition we obtain:

Corollary 4.1. A measure-preserving AT(2) flow or transformation has zero entropy.

Proof: Flows with positive entropy have an invariant subspace in L^2 with countable Lebesgue spectrum [CFS]. □

We also obtain some zero entroy examples of non-AT(2) transformations and flows.

Corollary 4.2. The following are not AT(2):

(i) horocycle flows on surfaces of constant negative curvature;

(ii) time t maps of horocycle flows on surfaces of constant negative curvature;

(iii) ergodic nilflows without totally discrete spectrum (cf. [AGH]);

(iv) ergodic affine transformations on nilmanifolds without totally discrete spectrum;

(v) measure-preserving transformations with quasi-discrete spectrum.

Proof: By [Pa], (i) and (ii) have coutable Lebesgue spectrum. Cases (iii) and (iv) have countable Lebesgue spectrum in the L^2 orthocomplement to the eigenfunctions by [AGH] and [P1], and case (v) reduces to (iv) [P1]. □

Although rank 1 and funny rank 1 transformations are AT(p) for all $1 \le p < \infty$ [CW] and [H], there exist ergodic rank r transformations with spectral multiplicity r[R]; furthermore, there exist interval exchange maps with non-simple spectrum, cf. [R]. Thus we have the following:

Corollary 4.3

(i) For each $r > 1$ there exists an ergodic transformation of rank r which is not AT(2);

(ii) There exist measure-preserving AT(2) transformations with non-AT(2) two point extensions;

(iii) There exist measure-preserving AT(2) transformations which are not loosely Bernoulli.

Proof: (i) follows from A. Katok's observation [K] that Cartesian powers never have simple spectrum, and (ii) follows from [HP]; (iii) follows from [F]. □

We point out that weak mixing AT(2) transformations and flows do exist and therefore give ergodic, but non-AT(2) Cartesian products.

Finally, contrasting the fact [CW] that the suspension of an AT transformation is an AT flow as well as Lemma 3.2 of this paper, we have:

Corollary 4.5.

(i) Every Kakutani equivalence class of measure-preserving transformations (flows) contains a non-AT(2) transformation (flow).

(ii) For every measure-preserving AT(2) transformation T there is a special flow built over T which is not AT(2).

Proof. (i) follows from equivalence theory presented in [ORW]; every Kakutani equivalence class has an element with a horocycle flow as a factor. (This depends on the fact that the horocycle flow is loosely Bernuoulli [Ra]). The countable Lebesgue spectrum in this factor lifts to an invariant subspace in L^2 for the action.
(ii) follows from Ambrose-Kakutani theorem [AK] and (i). □

These examples contrast with a recent result of Choksi and Nadkarni which states that AT(2) transformations are generic (contain a dense G_δ set) in the space of nonsingular transformations of a Lebesgue space with the coarse topology [CN].

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