

*MIXING PROPERTIES
OF NEARLY MAXIMAL ENTROPY MEASURES
FOR \mathbb{Z}^d SHIFTS OF FINITE TYPE*

BY

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Dedicated to the memory of Professor Anzelm Iwanik

Abstract. We prove that for a certain class of \mathbb{Z}^d shifts of finite type with positive topological entropy there is always an invariant measure, with entropy arbitrarily close to the topological entropy, that has strong metric mixing properties. With the additional assumption that there are dense periodic orbits, one can ensure that this measure is Bernoulli.

1. Introduction. It is well known that a topologically mixing shift of finite type (SFT) in one dimension has positive topological entropy and a unique measure of maximal entropy. Moreover, with respect to this measure, the shift is metrically isomorphic to a Bernoulli shift. For a \mathbb{Z}^d SFT with $d > 1$, topological mixing is a much weaker condition. For example, such a shift may have topological entropy zero (see [6], [8]). Positive topological entropy can be achieved by imposing a stronger topological mixing property such as Burton and Steif's *strong irreducibility* (see [1]) or the slightly weaker *uniform filling property* (UFP) studied in [9] and [10]. However, even though these properties lead to a more manageable theory (see [9] and [10]), their behavior is still different from the one-dimensional case. A multi-dimensional strongly irreducible SFT may have nonunique measures of maximal entropy (see [1]). Moreover, with respect to an ergodic measure of maximal entropy, the shift need not be metrically weakly mixing (see [2]).

In this paper we show that for a SFT satisfying the UFP, weak mixing can always be achieved provided one is willing to accept a measure of slightly less than maximal entropy (i.e., a measure of *nearly maximal* entropy). In

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fact, we show that nearly maximal entropy measures with much stronger mixing properties can always be found.

To state our results, let $h(Y, S)$ denote the topological entropy of a SFT (Y, S) , let $\mathcal{M}(Y, S)$ denote the S -invariant measures on Y and for $\nu \in \mathcal{M}(Y, S)$, let $h(Y, \nu, S)$ denote the metric entropy of (Y, ν, S) . Our main result is:

THEOREM 1.1. *Let (Y, S) be a \mathbb{Z}^d shift of finite type satisfying the UFP. Then for any $\varepsilon > 0$ there exists $\nu' \in \mathcal{M}(Y, S)$ such that (Y, ν', S) has the K-property and $h(Y, \nu', S) > h(Y, S) - \varepsilon$.*

Now suppose that (Y, S) is a SFT as in the statement of Theorem 1.1 which also has dense periodic orbits. In [10] we use Theorem 1.1 to prove a modeling theorem for such a SFT. We show that if (X, μ, T) is an ergodic, measure preserving \mathbb{Z}^d action with $h(X, \mu, T) < h(Y, S)$, then there is a measure $\nu \in \mathcal{M}(Y, S)$ such that (X, μ, T) is isomorphic to (Y, ν, S) . The corollary stated below now follows easily by applying this modeling theorem to a Bernoulli \mathbb{Z}^d action (X, μ, T) with entropy $h(Y, S) > h(X, \mu, T) > h(Y, S) - \varepsilon$.

COROLLARY 1.2. *Let (Y, S) be a \mathbb{Z}^d shift of finite type which satisfies the UFP and has dense periodic orbits. Then for any $\varepsilon > 0$ there exists $\nu' \in \mathcal{M}(Y, S)$ such that (Y, ν', S) is a Bernoulli action and $h(Y, \nu', S) > h(Y, S) - \varepsilon$.*

We note that in the case $d = 2$ a SFT with the UFP always has dense periodic orbits (see [11]), so in this case we obtain the conclusion of Corollary 1.2 assuming only the hypotheses of Theorem 1.1. It is not known whether the UFP implies the existence of dense periodic orbits for $d > 2$.

Our proof of Theorem 1.1 uses a result of Fieldsteel and Friedman [3], which says that every positive entropy \mathbb{Z}^d action is evenly Kakutani equivalent to an action with the K-property. We will show how to apply this result to (Y, ν, S) , where ν is a measure of maximal entropy, to obtain a new action that will be denoted by $(Y, \nu, S^{\alpha^{-1}})$ and that will have the K-property. Although $S^{\alpha^{-1}}$ will not be the shift on Y , we will show that it is possible to use the UFP to find a factor of $S^{\alpha^{-1}}$ modeled by a measure in $\mathcal{M}(Y, S)$ that will have entropy close to (Y, ν, S) . Theorem 1.1 will follow since the K-property is closed under factorization.

2. Shifts of finite type and the uniform filling property. Let \mathcal{A} be finite. For $y \in \mathcal{A}^{\mathbb{Z}^d}$ and $\vec{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$, let $y[\vec{m}]$ denote the element of \mathcal{A} occurring in the \vec{m} th position of y . $\mathcal{A}^{\mathbb{Z}^d}$ is compact and metrizable in the product topology, and the *shift* $(S^{\vec{m}}y)[\vec{m}] = y[\vec{m} + \vec{n}]$ is a continuous

\mathbb{Z}^d action. A *subshift* (Y, S) is the restriction of S to a closed S -invariant subspace $Y \subseteq \mathcal{A}^{\mathbb{Z}^d}$. We call \mathcal{A} the *alphabet* of the subshift.

If $R \subseteq \mathbb{Z}^d$ is finite, we call $\mathbf{b} \in \mathcal{A}^R$ a *finite block* with *shape* R . We let R^c denote the complement of R . The block obtained by restricting $y \in \mathcal{A}^{\mathbb{Z}^d}$ to R is denoted by $y[R]$. For $\vec{v}, \vec{w} \in \mathbb{Z}^d$, with $w_i > v_i$ for all i , let $[\vec{v}, \vec{w}] = \prod_{i=1}^d [v_i, w_i] \subseteq \mathbb{Z}^d$. In this context, we regard a scalar r to be equivalent to a vector with all entries equal to r . Thus, for example, $[-r, r]$ denotes a cube with sides of length $2r + 1$ and $[\vec{v} - r, \vec{v} + r]$ denotes a box thickened by r .

Given a finite collection of finite blocks $\mathcal{F} = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ with shapes $\{R_1, \dots, R_n\}$, we define a subshift, called a *shift of finite type* (SFT), by $Y_{\mathcal{F}} = \{y \in \mathcal{A}^{\mathbb{Z}^d} : (S^{\vec{n}}y)[R_j] \neq \mathbf{f}_j \text{ for any } \mathbf{f}_j \in \mathcal{F}, \vec{n} \in \mathbb{Z}^d\}$. Without loss of generality we can assume there is an $s \geq 0$, called the *step size*, so that all the blocks in \mathcal{F} have shape $R = [-s, s]$.

DEFINITION 2.1. A SFT (Y, S) satisfies the *uniform filling property* (UFP) with *filling length* $l > 0$ if

- (i) $\text{card}(Y) > 1$, and
- (ii) for any $y_1, y_2 \in Y$, and any $[\vec{v}, \vec{w}]$, there exists $y \in Y$ such that

$$y[[\vec{v}, \vec{w}]] = y_1[[\vec{v}, \vec{w}]] \quad \text{and} \quad y[[\vec{v} - l, \vec{w} + l]^c] = y_2[[\vec{v} - l, \vec{w} + l]^c].$$

3. Cocycles and orbit equivalences. Let (X, μ, T) be a measure preserving \mathbb{Z}^d action. A measurable mapping $\alpha : X \times \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ is called a *cocycle* for T if it satisfies the *cocycle condition*: $\alpha(x, \vec{n} + \vec{m}) = \alpha(x, \vec{n}) + \alpha(T^{\vec{n}}x, \vec{m})$. Fixing $x \in X$, we have a mapping $\alpha_x : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ defined by $\alpha_x(\vec{n}) = \alpha(x, \vec{n})$. We call α a *bijective cocycle* if α_x is a bijection for μ -a.e. $x \in X$. Note that $\alpha_x(\vec{0}) = \vec{0}$ and

$$(3.1) \quad \alpha_{T^{\vec{n}}x} = \tau_{-\alpha(x, \vec{n})} \circ \alpha_x \circ \tau_{\vec{n}},$$

where $\tau_{\vec{v}}$ denotes translation by \vec{v} .

Given a bijective cocycle α we define a mapping $\alpha^{-1} : X \times \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ by $\alpha^{-1}(x, \vec{v}) = \alpha_x^{-1}(\vec{v})$ (see [3]) and a new measure preserving \mathbb{Z}^d action $(X, \mu, T^{\alpha^{-1}})$ by $(T^{\alpha^{-1}})^{\vec{n}}x = T^{\alpha^{-1}(x, \vec{n})}x$. This new action has the same orbits as T . Heuristically, the cocycle α rearranges the points in the orbit of x ; their new order determines the new action. We say the two actions T and $T^{\alpha^{-1}}$ are *orbit equivalent*. Conversely, given any measure preserving \mathbb{Z}^d action (X, μ, R) with the same orbits as (X, μ, T) , there exists a bijective cocycle α such that $R = T^{\alpha^{-1}}$.

In this paper, we will make use of bijective cocycles (or equivalently, orbit equivalences) that satisfy two additional properties. The first property is that α is a cocycle that implements an *even Kakutani equivalence* (see [5] or [3]). Such a cocycle is called a *Kakutani cocycle* and is characterized by

the property that for almost every x there is a full density subset $I(x) \subseteq \mathbb{Z}^d$ such that

$$\lim_{\substack{\|\vec{v}\| \rightarrow \infty \\ \vec{v} \in I(x)}} \frac{\|\alpha(x, \vec{v}) - \vec{v}\|}{\|\vec{v}\|} = 0,$$

where $\|\cdot\|$ denotes the box metric (see [3]). The following result is due to Nadler, and a proof can be found in [4], Corollary 3.

THEOREM 3.1. *For a Kakutani cocycle α , $h(X, \mu, T) = h(X, \mu, T^{\alpha^{-1}})$.*

The second property the cocycle α will satisfy is more technical and requires a preliminary discussion. The Kakutani cocycle α constructed in [3] has some additional geometric structure, called the *sequential blocking* property, that leaves the relative ordering on large blocks of orbits intact. We now describe a minor modification of this sequential blocking property, using exactly the same notation as in [3]. In the discussion that follows s and l are fixed positive integers. Later they will play the role of the step size and filling length of a SFT (Y, S) .

Let (J, K, L) be a triple of positive integers with

$$(3.2) \quad K + J + s + l \leq L < J + 4K.$$

We say a permutation π of the cube $[-L, L] \subseteq \mathbb{Z}^d$ is a (J, K, L) -permutation if there exists $\vec{v} \in [-K, K]$ such that

$$(3.3) \quad \pi|_{[-J, J]}(\vec{w}) = \vec{v} + \vec{w},$$

and

$$(3.4) \quad \pi|_{[-L, L] \setminus [-L+s+l, L-s-l]} = \text{id}.$$

More generally, a permutation π of $[-L, L] + \vec{t}$ is called a (J, K, L) -permutation if it is of the form $\pi = \tau_{\vec{t}} \circ \pi' \circ \tau_{-\vec{t}}$, where π' is a (J, K, L) -permutation of $[-L, L]$.

We say a bijection $\gamma : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ is (J, K, L) -blocked if there is a disjoint collection $\{[-L, L] + \vec{t}_j\}$ of translates of $[-L, L]$ in \mathbb{Z}^d satisfying the following:

- (i) on each set $[-L, L] + \vec{t}_j$, γ acts as a (J, K, L) -permutation, and
- (ii) on $(\bigcup_j [-L, L] + \vec{t}_j)^c$, γ acts as the identity.

In particular, the definition of a blocked bijection implicitly includes a choice of the sets $[-L, L] + \vec{t}_j$ and the vectors \vec{v}_j from the definition of a (J, K, L) -blocked permutation on $[-L, L] + \vec{t}_j$.

Using the same terminology as [3], we call the sets $[-J, J] + \vec{t}_j$ *rigid blocks*, and denote them by $B_j(\gamma)$. We let $A_j(\gamma) = B_j(\gamma) + \vec{v}_j$. We call these blocks *translated rigid blocks* and denote their union by $A(\gamma)$. We let $E(\gamma) = (\bigcup_j ([-L+s, L-s] + \vec{t}_j))^c$. We let $C_j(\gamma) = ([-L+s, L-s] \setminus [-L+s+l, L-s-l]) + \vec{v}_j$ and $D_j(\gamma) = ([-L+s+l, L-s-l] + \vec{t}_j) \setminus A_j(\gamma)$. We

put $C(\gamma) = \bigcup_j (C_j(\gamma) \cup D_j(\gamma))$, and we call $C(\gamma)$ the *collar set* for γ . Note that the disjoint union of $A(\gamma)$, $C(\gamma)$, and $E(\gamma)$ is \mathbb{Z}^d .

LEMMA 3.2 ([9]). *Let γ be a (J, K, L) -blocked bijection of \mathbb{Z}^d , with $A(\gamma)$, $C(\gamma)$, and $E(\gamma)$ as above. Suppose that there is $y \in \mathcal{A}^{A(\gamma) \cup E(\gamma)}$ such that*

(i) *for all j , there exists $y'_j \in Y$ with*

$$y[A_j(\gamma)] = y'_j[A_j(\gamma)],$$

(ii) *there exists $y' \in Y$ with*

$$y[E(\gamma)] = y'[E(\gamma)].$$

Then there exists $\bar{y} \in Y$ such that

$$\bar{y}[A(\gamma) \cup E(\gamma)] = y[A(\gamma) \cup E(\gamma)].$$

This result follows from the UFP since each component $C_j(\gamma) \cup D_j(\gamma)$ of the collar set $C(\gamma)$ contains a collar $C_j(\gamma)$ of thickness l .

A bijective cocycle β is called a (J, K, L) -*blocked cocycle* if for μ -a.e. x , β_x is a (J, K, L) -blocked bijection for a particular specified measurable choice of the sets $[-L, L] + \vec{t}_j$ and vectors \vec{v}_j . In particular, the set $C_\beta = \{x : \vec{0} \in C(\beta_x)\} \subseteq X$ is measurable, as are the sets $A_\beta, E_\beta \subseteq X$ defined similarly. Given $\varepsilon > 0$ we say a (J, K, L) -blocked cocycle is (J, K, L, ε) -*blocked* if $\mu(A_\beta) > 1 - \varepsilon$. In this case, we have $\mu(C_\beta) < \varepsilon$.

A sequence $\alpha_1, \alpha_2, \dots$ of cocycles *converges* to a cocycle α , in symbols $\lim \alpha_i = \alpha$, if for μ -a.e. $x \in X$ and $\vec{n} \in \mathbb{Z}^d$,

$$(3.5) \quad \alpha(x, \vec{n}) = \lim_{j \rightarrow \infty} \alpha_j(x, \vec{n}).$$

It is easy to see that α is a cocycle. For two cocycles β_1 and β_2 we define $\beta_2 \circ \beta_1$ by $(\beta_2 \circ \beta_1)(x, \vec{n}) = \beta_2(x, \beta_1(x, \vec{n}))$. Given a sequence β_1, β_2, \dots we put $\alpha_1 = \beta_1$, and for $j > 1$, $\alpha_j = \beta_j \circ \alpha_{j-1}$. If $\alpha = \lim \alpha_j$, where α_j is as defined above, we write $\alpha = \prod_j \beta_j$.

Given a sequence $\{(J_i, K_i, L_i, \varepsilon_i)\}$ as above, we say that a cocycle α is $\{(J_i, K_i, L_i, \varepsilon_i)\}$ -*sequentially blocked* if $\alpha = \prod_i \beta_i$, where β_i is a $(J_i, K_i, L_i, \varepsilon_i)$ -blocked cocycle. This is the second property (in addition to being a Kakutani cocycle) that we require the cocycle α to satisfy.

The following theorem is the main result of [3] in the positive entropy case, adapted to include (3.2), (3.3) and (3.4).

THEOREM 3.3 (Fieldsteel and Friedman, [3]). *Let (X, μ, T) be a measure preserving \mathbb{Z}^d action with $h(X, \mu, T) > 0$. Let ε_i be an arbitrary summable sequence of positive numbers, and let l and s be positive integers. Then there exists*

(i) *a sequence $\{(J_i, K_i, L_i)\}$ of triples satisfying (3.2) for each i , and*

(ii) a sequence β_i of $\{(J_i, K_i, L_i)\}$ -blocked cocycles such that $\alpha = \prod_i \beta_i$ is a $\{(J_i, K_i, L_i, \varepsilon_i)\}$ -sequentially blocked bijective Kakutani cocycle and $(X, \mu, T^{\alpha^{-1}})$ has the K-property.

One can check that the proof in [3] is not affected by the minor modifications we have made in the geometry of the blocked cocycle.

4. Proof of Theorem 1.1. Let (Y, S) be as given in the statement of Theorem 1.1. Let s be the step size, l the filling length and \mathcal{A} the alphabet of Y .

Let (X, μ, T) be a measure preserving \mathbb{Z}^d action. A measurable partition Q on X is a measurable mapping $Q : X \rightarrow \mathcal{A}$. The Q -name of $x \in X$ is the element $y \in \mathcal{A}^{\mathbb{Z}^d}$ satisfying $y[\vec{v}] = Q(T^{\vec{v}}x)$ for all $\vec{v} \in \mathbb{Z}^d$. We define a map $\phi_Q : \mathcal{M}(X, T) \rightarrow \mathcal{M}(\mathcal{A}^{\mathbb{Z}^d}, S)$ by $\phi_Q(\mu)(E) = \mu(\{x \in X : Q(x) \in E\})$. Since $Y \subset \mathcal{A}^{\mathbb{Z}^d}$ we have $\mathcal{M}(Y, S) \subset \mathcal{M}(\mathcal{A}^{\mathbb{Z}^d}, S)$, and we call Q a *type* (Y, S) Markov partition for (X, μ, T) if $\phi_Q(\mu) \in \mathcal{M}(Y, S)$. Equivalently, Q is a type (Y, S) Markov partition for (X, μ, T) if the Q -name y of x satisfies $y \in Y$ for μ -a.e. $x \in X$.

We need the following elementary lemma on the entropy of partitions:

LEMMA 4.1. *For any $\varepsilon > 0$ there exists δ such that if Q_1 and Q_2 are two partitions on (X, μ, T) with $\mu(Q_1 \Delta Q_2) < \delta$, then*

$$|h(X, \mu, T, Q_1) - h(X, \mu, T, Q_2)| < \varepsilon.$$

Let P denote the time zero partition of (Y, S) and let ν be a measure of maximal entropy. Given $\varepsilon > 0$, we choose $\delta > 0$ according to Lemma 4.1. Let $\varepsilon_i > 0$ be a sequence so that $\sum_i \varepsilon_i < \delta$. We apply Theorem 3.3 to obtain a $\{(J_i, K_i, L_i, \varepsilon_i)\}$ -sequentially blocked bijective Kakutani cocycle $\alpha = \prod_i \beta_i$ such that the corresponding action $(Y, \nu, S^{\alpha^{-1}})$ has the K-property.

Since P is still a generating partition for $(Y, \nu, S^{\alpha^{-1}})$, the map $\phi_P : \mathcal{M}(Y, S^{\alpha^{-1}}) \rightarrow \mathcal{M}(\mathcal{A}^{\mathbb{Z}^d}, S)$ models $(Y, \nu, S^{\alpha^{-1}})$ on the full shift on \mathcal{A} . Namely, $(\mathcal{A}^{\mathbb{Z}^d}, \phi_P(\mu), S)$ is measurably isomorphic to $(Y, \nu, S^{\alpha^{-1}})$. However, since $S^{\alpha^{-1}}$ is not the shift on Y , P is generally not a type (Y, S) Markov partition for $(Y, \nu, S^{\alpha^{-1}})$, and thus $\phi_P(\mu)$ is generally not an element of $\mathcal{M}(Y, S)$.

We now construct a new partition Q of Y that will be a type (Y, S) Markov partition for $S^{\alpha^{-1}}$ and that will be close to P in the sense of Lemma 4.1. Note that if C_{β_i} is the collar set for the bijection β_i in step i , then by the construction of α , we have $\nu(C_{\beta_i}) < \varepsilon_i$ for all i . Letting $C_\alpha = \bigcup_i C_{\beta_i}$, we have

$$(4.1) \quad \mu(C_\alpha) \leq \sum_i \mu(C_{\beta_i}) \leq \sum_i \varepsilon_i < \delta.$$

The partition Q will have the property that

$$(4.2) \quad Q(x) = P(x) \quad \text{for } x \in C_\alpha^c.$$

Thus, using (4.1) and (4.2) with Lemma 4.1, followed by Theorem 3.1 and Theorem 3.3, we will have

$$(4.3) \quad \begin{aligned} h(Y, \nu, S^{\alpha^{-1}}, Q) &\geq h(Y, \nu, S^{\alpha^{-1}}, P) - \varepsilon \\ &= h(Y, \nu, S) - \varepsilon = h(Y, S) - \varepsilon. \end{aligned}$$

The construction of Q proceeds as follows. For each $x \in Y$ (thinking of x as a point for the dynamical system $(Y, \nu, S^{\alpha^{-1}})$) we construct a sequence $y_i(x) \in Y$ for $i = 0, 1, \dots$ by induction, beginning with $y_0(x) = x$.

Given $y_{i-1}(x) \in Y$ we construct $y_i(x) \in Y$ as follows. We look at the i th cocycle β_i in the product $\alpha = \prod_i \beta_i$ and the corresponding bijection $\beta_{i,x}$ on the orbit of x . Recall that the sets $A(\beta_{i,x})$, $E(\beta_{i,x})$ and $C(\beta_{i,x})$ (defined before Lemma 3.2) partition \mathbb{Z}^d . We put

$$(4.4) \quad y_i(x)[E(\beta_{i,x})] = y_{i-1}(x)[E(\beta_{i,x})].$$

Next, recall that $A(\beta_{i,x})$ is a disjoint union of sets $A_j(\beta_{i,x})$, each of which is a translation of a set $B_j(\beta_{i,x})$ by a vector \vec{v}_j . We define

$$(4.5) \quad y_i(x)[A_j(\beta_{i,x})] = (S^{-\vec{v}_j} y_{i-1}(x))[B_j(\beta_{i,x})].$$

By (4.4), (4.5) and the induction hypothesis, the block obtained by restricting $y_i(x)$ to one of the sets $A_j(\beta_{i,x})$ or to $E(\beta_{i,x})$ can be extended to a point $y' \in Y$ (namely, some shift of $y_{i-1}(x)$). Thus we can use Lemma 3.2 to fill in the collar set $C(\beta_{i,x})$ in $y_i(x)$, to obtain an extension $y_i(x) \in Y$. For the sake of uniqueness, we choose the lexicographically smallest filling for each $C_j(\beta_{i,x}) \cup D_j(\beta_{i,x})$.

It follows from (3.5) that for μ -a.e. $x \in X$ and every $M \in \mathbb{N}$, there exists a positive integer $N = N(x, M)$ so that $\alpha_x = \prod_{j=1}^N \beta_{j,x}$ on $[-M, M] \subseteq \mathbb{Z}^d$. Thus if we define $y(x)[-M, M] = y_{N(x, M)}(x)[-M, M]$ for all M , we have $y(x) \in Y$. It follows from (3.1), (4.4) and (4.5) that $y((S^{\alpha^{-1}})^{\vec{v}} x) = S^{\vec{v}} y(x)$. Putting $Q(x) = y(x)[\vec{0}]$, we find that Q is a type (Y, S) Markov partition for $(Y, \nu, S^{\alpha^{-1}})$. To see that (4.2) holds, we note that $x \notin C_\alpha$ implies $Q(x) = y(x)[\vec{0}] = x[\vec{0}] = P(x)$.

To complete the proof, we let $\nu' = \phi_Q(\nu) \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}^d}, S)$. Since Q is a type (Y, S) Markov partition we actually have $\nu' \in \mathcal{M}(Y, S)$. Also, (Y, ν', S) is the factor of $(Y, \nu, S^{\alpha^{-1}})$ corresponding to the partition Q , so by definition, Q is a generating partition for this factor. It follows from (4.3) that $h(Y, \nu', S) = h(Y, \nu, S^{\alpha^{-1}}, Q) \geq h(Y, S) - \varepsilon$. Finally, we note that (Y, ν', S) has the K-property since $(Y, \nu, S^{\alpha^{-1}})$ has the K-property and the K-property is inherited by factors.

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