

The Čech cohomology and the spectrum for 1-dimensional tiling systems

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ABSTRACT. This paper discusses the relation between three groups: the dynamical cohomology $H(T)$ of a substitution shift dynamical system T , the first integer Čech cohomology group $\check{H}^1(Y)$ of the suspension space Y over T , and the continuous point-spectrum $\mathbb{E}(F)$ of the suspension flow F^t . In particular, the eigenfunctions of F^t embed as a subgroup of $\check{H}^1(Y)$. We also study a real-valued functional W on $\check{H}^1(Y)$, called the winding number, which assigns each eigenfunction its eigenvalue. In the case that W is injective, we call F^t *cohomological ergodic*, and if the image of W is equal to $\mathbb{E}(F)$, we say F^t has *cohomological pure point spectrum*.

1. Introduction

In this paper we consider 1-dimensional tiling flows F^t corresponding to 1-dimensional tile substitutions S . We also consider substitution shift dynamical systems T corresponding to primitive aperiodic substitutions σ . Our purpose is to explore the relation between two well known topological invariants for these dynamical systems: their point-spectrum, as studied, for example in [Ho-86, Ra-90, FMN-96, So-97, AI-01, CSg-01, SiSo-02, HS-03, FiHR-03, Sg-03, CS-03, R-04, BK-06, BBK-06], and their cohomology, studied, for example in [AP-98, BD-01, FoHK-02, BD-02, CS-03, CS-06, S-08, BD-08, BKeS-12]. The purpose of this paper is to survey what is known about this topic, as well as to add some new observations and results.

We proceed as follows. We let T be a strictly ergodic homeomorphism of a Cantor space X . For a continuous, positive, real-valued function g on X , we let F^t be the corresponding suspension flow, and let Y be the 1-dimensional suspension-space on which F^t acts. We study three abelian groups associated to the dynamical systems T and F^t . The first is the dynamical cohomology $H(T)$ of T , which is defined as the integer-valued continuous functions modulo the coboundaries. The second is the 1st integer Čech cohomology $\check{H}^1(Y)$ of the suspension space Y over T . And the third is the group $E(F)$ of continuous eigenfunctions for F^t , with pointwise multiplication. For each of these groups, there is a natural real-valued functional, and in each case we study its image in \mathbb{R} . The functional L on $H(T)$ is the integral,

2010 *Mathematics Subject Classification.* Primary 37B50; Secondary 37A30.

Key words and phrases. Substitution shift, tiling flow, spectrum, cohomology.

This research was partially supported by a grant from the Simons Foundation (award number 244739) to E. Arthur Robinson, Jr.

and its image, called the “measure group”, is denoted $\mathbb{H}(X) \subseteq \mathbb{R}$. The functional W on $\check{H}^1(Y)$ is called the “Schwartzman winding number” (see Section 3.3). Its image turns out to be the same subgroup $\mathbb{H}(X) \subseteq \mathbb{R}$. The functional on the eigenfunctions $E(F)$ is the map V that assigns the eigenvalue $V(f)$ to an eigenfunction f . For a flow F^t the set of eigenvalues $\mathbb{E}(F)$ is a subgroup of \mathbb{R} .

In the first part of the paper, we investigate the situation in which $\ker(L)$ (equivalently $\ker(W)$) is trivial. We call this “cohomological ergodicity”. In this case, the subgroup $\mathbb{H}(T) \subseteq \mathbb{R}$ is isomorphic to $H(T)$ and to $\check{H}^1(Y)$. We show in Theorem 2.7 that cohomological ergodicity holds for T when the substitution σ is primitive, aperiodic, irreducible, and has a “common prefix”. Examples T that fail to satisfy cohomological ergodicity are also discussed.

The second part of the paper is based on the observation from [Sc-57] that the eigenfunctions $E(F)$ of the flow F^t embed into $\check{H}^1(Y)$. Schwartzman’s paper [Sc-57], published in 1957, was arguably the first serious application of algebraic topology to dynamical systems theory. We ask: how much larger is $\check{H}^1(Y)$ than $E(F)$? In the case that $\check{H}^1(Y)$ is entirely accounted for by eigenfunctions $E(F)$, we say F^t has *cohomological pure point spectrum*. In Theorem 4.2, we show that a primitive aperiodic Pisot substitution with a common prefix has cohomological pure point spectrum. We also give examples of cohomologically ergodic tile substitutions for which the inclusion of $E(F) \subseteq \check{H}^1(Y)$ is strict.

Although much of this paper is a survey, it is also an exposition of some previously unpublished results. Most of these come from the first author’s 2007 Ph.D. dissertation [An-07]. Those parts of the theory needed to support the new results are worked out in more detail, whereas the other parts are relegated to the “Notes” throughout, where references are provided.

The second author wishes to thank Anne Siegel for some helpful suggestions. Both authors thank the referee for many useful comments that vastly improved this paper.

2. Discrete Dynamical Systems

2.1. Cantor Dynamical Systems. A *measured Cantor dynamical system* (X, T, μ) is a homeomorphism of T a Cantor space X (a space that is homeomorphic to the Cantor set), together with a T -invariant Borel probability measure μ on X . We always assume here that T is *strictly ergodic*, meaning T is *minimal* (all orbits are dense) and *uniquely ergodic* (μ is the unique T -invariant Borel probability measure).

Let $C(X, \mathbb{Z})$ denote the set of continuous integer-valued functions on X (non-trivial since X is a Cantor set). The *coboundaries* $B(T)$ are the functions of the form $n(x) = p(Tx) - p(x)$, where $p \in C(X, \mathbb{Z})$. We define the *dynamical cohomology group* of T by $H(T) := C(X, \mathbb{Z})/B(T)$. We define $L : C(X, \mathbb{Z}) \rightarrow \mathbb{R}$ by $L(n) = \int_X n d\mu$. Any $n(x) \in I(T) := \ker(L)$ is called an *infinitesimal*.

Central to this discussion will be the *measure group*, which we denote by $\mathbb{H}(T) := L(H(T))$, namely the image of L . Note that $\mathbb{H}(T)$ is the additive subgroup of \mathbb{R} generated by the measures $\mu(E)$ of the clopen sets (or equivalently, measures of cylinder sets) $E \subseteq X$. Since μ is T -invariant, $B(T) \subseteq I(T)$, which implies $L : H(T) \rightarrow \mathbb{H}(T)$ is well defined and surjective.

DEFINITION 2.1. *If L is injective, we say T is cohomologically ergodic¹.*

Clearly cohomological ergodicity is equivalent to $B(T) = I(T)$, and implies $\mathbb{H}(T)$ is isomorphic to $H(T)$. Later, we will give examples of strictly ergodic T that are, and examples that are not, cohomologically ergodic. Cohomological ergodicity is an instance of the general phenomenon called “stability” in [KR-01]. In particular, it means that $B(T)$ is closed in $C(X, \mathbb{Z})$.

NOTE 1. The dynamical cohomology group $H(T)$ is also sometimes called $K_0(T)$ because it is isomorphic to $K_0(C(X) \rtimes_T \mathbb{Z})$, the 0th K-group of the crossed product C^* -algebra for T (see e.g., [AP-98]). The unique ergodicity of T implies that there is a unique real-valued “trace” τ on this C^* -algebra, and the image of $K_0(C(X) \rtimes_T \mathbb{Z})$ under τ is called the *gap labeling group* because it is related to gaps in the spectrum of a corresponding discrete Schrödinger operator (see [Be-92]). The 1-dimensional *gap labeling theorem* [Be-92] says the image of τ equals $\mathbb{H}(T)$. The higher-dimensional version of this (for \mathbb{Z}^d and \mathbb{R}^d actions), was originally called Bellissard’s *gap labeling conjecture*, but is now a theorem: [BO-03, KaP-03, BeBG-06].

Let (X, T, μ) be a strictly ergodic Cantor dynamical system. Define the *orbit* of $x \in X$ by $O_T(x) = \{T^n x : n \in \mathbb{Z}\}$. Another homeomorphism R on X is said to be *orbit equivalent* to T if $O_T(x) = O_R(x)$ for all $x \in X$. In this case there is a *cocycle* $n : X \rightarrow \mathbb{Z}$ such that $Rx = T^{n(x)}x$. Note that the cocycle $n(x)$ is not necessarily continuous. The *full group* of T , denoted $[T]$ is the set of all R that are orbit equivalent to T . It is easy to see that if $R \in [T]$ then (X, R, μ) is a strictly ergodic Cantor dynamical system. If $R \in [T]$, and the cocycle $n(x)$ is continuous, we say R is in the *topological full group* of T , and write $R \in [[T]]$. Equivalently, R and T are *topologically orbit equivalent*.

Two clopen sets E_1 and E_2 are called *T -equivalent* if $E_2 = RE_1$ for some $R \in [[T]]$. This implies $\mu(E_1) = \mu(E_2)$. Following [BzK-00], we say T is *saturated* if whenever $\mu(E_1) = \mu(E_2)$ for E_1, E_2 clopen, there exists $R \in [[T]]$ so that $E_2 = UE_1$.

LEMMA 2.2 ([BzK-00]). *A strictly ergodic Cantor dynamical system (X, T, μ) is cohomologically ergodic (i.e., satisfies $B(T) = I(T)$) if and only if it is saturated.*

NOTE 2. A strictly ergodic Cantor dynamical system T need not be cohomologically ergodic. It is shown in [BzK-00] that the Chacon transformation T , realized as a strictly ergodic shift on $X \subseteq \{0, 1\}^{\mathbb{Z}}$ is not cohomologically ergodic. Similarly, arguments in [BKes-12] show that the Morse substitution dynamical system T is not cohomologically ergodic. Examples of T that are cohomologically ergodic are discussed in Section 5.

NOTE 3. For T a minimal homeomorphism of a Cantor set, but without the assumption of unique ergodicity, extend the previous definition of $I(T)$ be the set of integer functions with integral zero for all T -invariant measures. Define $G(T) = C(X, \mathbb{T})/I(T)$. Let $G(T)^+ \subseteq G(T)$ (and $H(T)^+ \subseteq H(T)$) be the semi-groups generated by the positive functions. Let $[1]$ denote the class of 1. Then $(G(T), G(T)^+, [1])$ (and $(H(T), H(T)^+, [1])$) are called the *ordered dimension group* (and the *ordered cohomology group*) of T . It is shown in [GPS-95] that, up to order

¹The reason calling this “ergodicity” will be discussed after the definition of the flow version, Definition 3.4.

isomorphism, these ordered groups are complete invariants for *strong orbit equivalence* (and for *orbit equivalence*). In particular, strong orbit equivalence (which lies between orbit equivalence and topological orbit equivalence) allows $n(x)$ to have at most one discontinuity.

NOTE 4. It is shown in [Bo-83] (see also [BoT-98]) that if $Rx = T^{n(x)}x$ for continuous $n(x)$ (topological orbit equivalence) then $R = V^{-1}TV$ or $R = V^{-1}T^{-1}V$ for some homeomorphism V . The latter is called *flip-conjugacy*.

2.2. Shifts. Let $\mathcal{A} = \{0, 1, \dots, d-1\}$, $d > 1$. Let $\mathcal{A}^{\mathbb{Z}}$ be the 2-sided full-shift on \mathcal{A} , provided with the product topology, and let T be the *shift* (homeomorphism), defined $(Tx)_i = x_{i-1}$. A *shift space* $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a closed T -invariant subset of $\mathcal{A}^{\mathbb{Z}}$, and we restrict T to X . We always assume X is uncountable, which implies X is a Cantor space.

Let $\mathcal{A}^* = \cup_{n \geq 0} \mathcal{A}^n$ denote the set of all finite words in \mathcal{A} . The *language* of X is the subset $\mathcal{L} \subseteq \mathcal{A}^*$ of words that appear in some $x \in X$. Each $w = w_0w_1 \dots w_{n-1} \in \mathcal{L}$ defines the *cylinder set* $[w] = \{x \in X : x_j = w_j, 0 \leq j < n\}$. Cylinder sets are clopen and provide a basis for the topology on X .

NOTE 5. Any T -invariant Borel probability measure μ on a subshift X is completely determined by its values $\mu(E)$ on cylinder sets $E = [w]$, $w \in \mathcal{L}$. If T is also uniquely ergodic, then any $w \in \mathcal{L}$ occurs in every $x \in X$ with a common positive frequency, which we denote by $\ell(w) > 0$. These frequencies satisfy $\ell(w) = \mu([w])$ (see [Qu-87]). It follows that $\mathbb{H}(T)$ is the subgroup of \mathbb{R} generated by the frequencies of words in \mathcal{L} . For this reason, $\mathbb{H}(T)$ is sometimes called the *frequency module* (see e.g., [Be-92]).

2.3. Substitutions. A *substitution* is a mapping $\sigma : \mathcal{A} \rightarrow \mathcal{A}^* := \cup_{n \geq 1} \mathcal{A}^n$. The *incidence matrix* Q for σ is the matrix with entries $q_{a,b}$ equal to the number of times a occurs in $\sigma(b)$. We assume $Q^n > 0$ for some $n \geq 1$, and call Q (and σ) *primitive*. For primitive Q , the *Perron-Frobenius Theorem* (see [LM-95]), says Q has a simple positive eigenvalue $\lambda > 1$, called the *Perron-Frobenius eigenvalue*, which satisfies $|\theta| < \lambda$ for all other eigenvalues. We normalize the corresponding *Perron-Frobenius* eigenvectors: (left) $Q^t \mathbf{m} = \lambda \mathbf{m}$ and (right) $Q \mathbf{h} = \lambda \mathbf{h}$, so that $\mathbf{m} \cdot \mathbf{1} = 1$ and $\mathbf{h} \cdot \mathbf{m} = 1$.

Let \mathcal{L} be the language generated by the set of all sub-words of $\{\sigma^n(a) : a \in \mathcal{A}, n \geq 0\}$. There is a unique shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$ whose language is \mathcal{L} (see [Qu-87] or [Fg-02] for details, and various alternative definitions of X). We let T be the shift map restricted to X . We say a primitive substitution σ is (shift) *aperiodic* if T has no periodic points. This implies X is uncountable, and thus a Cantor space. Since σ is a primitive, aperiodic substitution it follows that T is *strictly ergodic* (see [Qu-87]). Also, every eigenfunction f for T (see Section 3.5) can be chosen to be continuous [Ho-86]. We will refer to this property by saying T is *homogeneous* (see [R-04]). We call the strictly ergodic Cantor dynamical system (X, T, μ) the *substitution shift dynamical system* corresponding to σ .

Here are a few additional properties that σ may or may not have:

- (1) σ is *irreducible* if the characteristic polynomial $q(z)$ of Q is irreducible. This implies λ is irrational.
- (2) σ is *unimodular* if $\det(Q) = \pm 1$.

- (3) σ is a *Pisot* substitution if the Perron-Frobenius eigenvalue λ is a Pisot number. A Pisot number is a real algebraic integer $\lambda > 1$ whose conjugates satisfy $|\theta| < 1$ (see e.g., [AI-01, BD-02, HS-03]).
- (4) σ has a *common prefix* $p \in \mathcal{A}$ (or *common suffix* $s \in \mathcal{A}$) if for each $a \in \mathcal{A}$, there exists $u_a \in \mathcal{A}^*$ so that $\sigma(a) = pu_a$ (or $\sigma(a) = v_as$).

NOTE 6. A substitution σ is called *proper* (see [DHS-99]) if it has both a common prefix and a common suffix. For any primitive aperiodic substitution σ with its dynamical system T , there is a proper substitution σ' with its dynamical system T' topologically conjugate to T . However, σ' generally has a much larger alphabet than σ , and is usually not irreducible, even if σ is (see [DHS-99]).

For $\mathbf{v} \in \mathbb{R}^d$, we write $\mathbb{Z}[\mathbf{v}] := \mathbf{v} \cdot \mathbb{Z}^d$, and note that $\mathbb{Z}[\mathbf{v}] \cong \mathbb{Z}^e$ for some $0 \leq e \leq d$. For $\lambda \in \mathbb{R}$, we write $\mathbb{Z}[\lambda]$ for the additive abelian group generated by $\{\lambda^k : k = 0, 1, 2, \dots\}$.

LEMMA 2.3. *Let Q be a primitive matrix with Perron-Frobenius eigenvalue λ and normalized left Perron-Frobenius eigenvector \mathbf{m} . Then $\lambda^{-k}\mathbb{Z}[\mathbf{m}] \subseteq \lambda^{-(k+1)}\mathbb{Z}[\mathbf{m}]$. If Q is irreducible, $\mathbb{Z}[\mathbf{m}] \cong \mathbb{Z}[\lambda] \cong \mathbb{Z}^d$. Moreover, Q is unimodular if and only if λ is a unit, if and only if $\lambda^{-k}\mathbb{Z}[\mathbf{m}] = \mathbb{Z}[\mathbf{m}]$ for all k .*

PROOF. For $\lambda^{-k}\mathbf{m} \cdot \mathbf{n} \in \lambda^{-k}\mathbb{Z}[\mathbf{m}]$, we have $\lambda^{-k}\mathbf{m} \cdot \mathbf{n} = \lambda^{-(k+1)}Q^t\mathbf{m} \cdot \mathbf{n} = \lambda^{-(k+1)}\mathbf{m} \cdot Q\mathbf{n} \in \mathbb{Z}[\lambda^{-(k+1)}\mathbf{m}]$. An elementary calculation shows that Q irreducible implies the entries of \mathbf{m} are rationally independent. In the unimodular case, Q is invertible and $\lambda^{-k}\mathbb{Z}[\mathbf{m}] = \mathbb{Z}[\mathbf{m}]$ for all k . If Q is not unimodular, the inclusions $\lambda^{-k}\mathbb{Z}[\mathbf{m}] \subseteq \lambda^{-(k+1)}\mathbb{Z}[\mathbf{m}]$ are proper. \square

We define the *Perron-Frobenius group* of a primitive matrix Q by

$$(2.1) \quad \text{PF}_Q := \mathbb{Z}[\lambda^{-1}]\mathbb{Z}[\mathbf{m}] = \bigcup_{k=0}^{\infty} \lambda^{-k}\mathbb{Z}[\mathbf{m}].$$

It follows from Lemma 2.3 that PF_Q is finitely generated if and only if Q is unimodular, and in this case $\text{PF}_Q = \mathbb{Z}[\mathbf{m}] \cong \mathbb{Z}^d$.

NOTE 7. The Perron-Frobenius group is closely related to a version of the *dimension group*² Δ_Q that is studied in [LM-95, Ki-98]. Assume that Q is primitive and irreducible. This implies Q is nonsingular, and we have

$$\Delta_Q := \bigcup_{k=0}^{\infty} Q^{-k}\mathbb{Z}^d \subseteq \mathbb{Q}^d,$$

(this is simpler than the definition in [LM-95, Ki-98], which does not assume Q is nonsingular). In the unimodular case $\Delta_Q = \mathbb{Z}^d$. An easy calculation shows that the Perron Frobenius group (2.1) satisfies $\text{PF}_Q = \mathbf{m} \cdot \Delta_Q$.

2.4. Kakutani-Rohlin partitions. A *semi-partition* $\mathcal{P} = \{P_0, \dots, P_{n-1}\}$ on a Cantor space X is a collection of pairwise disjoint clopen sets in X . A *partition* is a semi-partition such that $\cup_{P \in \mathcal{P}} P = X$. A partition \mathcal{Q} *refines* partition \mathcal{P} , denoted $\mathcal{Q} \geq \mathcal{P}$, if for each $Q \in \mathcal{Q}$ there is a $P \in \mathcal{P}$ with $Q \subseteq P$. A sequence \mathcal{P}_k of partitions is *refining* if $\mathcal{P}_{k+1} \geq \mathcal{P}_k$ for all k . A semi-partition of the form $\mathcal{P} = \{B, TB, T^2B, \dots, T^{h-1}B\}$, for B clopen, is called the *height h Rohlin tower*

²A dimension group usually also comes together with an order structure (like in Note 3), but we that ignore that structure here.

with base B for a Cantor dynamical system T . A *Kakutani-Rohlin partition* is a partition \mathcal{P} that is a finite union of disjoint Rohlin towers. A sequence of partitions \mathcal{P}_k on X is said to be *generating* if any clopen set E is the union of elements of \mathcal{P}_k for k sufficiently large. Equivalently, \mathcal{P}_k is generating if and only if for each $n \geq 1$, there is a $K \in \mathbb{N}$, so that for $k \geq K$, the function $x \mapsto x_{[-n,n]} : X \rightarrow \mathcal{A}^{2n+1}$ is constant on each $P \in \mathcal{P}^k$.

Now let T be primitive aperiodic substitution shift dynamical system, for substitution σ . For each $a \in \mathcal{A}$, $B_a^{(k)} := \sigma^k(\lfloor a \rfloor)$, is the base of a height $h_a := |\sigma^k(a)|$ Kakutani-Rohlin tower (see [DHS-99]) and together these towers form a Kakutani-Rohlin partition we denote \mathcal{P}_k . These partitions satisfy $\mathcal{P}_{k+1} \geq \mathcal{P}_k$, and if σ is also proper, then \mathcal{P}_k is a generating sequence (see [DHS-99]). The following proposition generalizes these results.

PROPOSITION 2.4. *Let σ be a primitive, aperiodic substitution on \mathcal{A} , and $r_k \in \mathbb{Z}$ be a sequence. Fix k . Then for each $a \in \mathcal{A}$, $B_a^{(k)} := T^{r_k} \sigma^k(\lfloor a \rfloor)$ is the base of a Rohlin tower of height $\ell_a := |\sigma^k(a)|$. These towers are disjoint and their union, denoted \mathcal{P}_k , is a Kakutani-Rohlin partition. In the case that σ has a common prefix, the sequence r_k can be chosen so that the sequence \mathcal{P}_k of partitions is generating.*

PROOF. It is easy to see (by [DHS-99]) that \mathcal{P}_k is a Kakutani-Rohlin partition. Now suppose σ has a common prefix (the common suffix case is similar). Then for $a \in \mathcal{A}$, $\sigma(a) = pu_a$, and for any $ab \in \mathcal{L}$ (including, possibly aa), $\sigma(ab) = pu_a pu_b$. Thus for $k > 1$,

$$(2.2) \quad \sigma^k(ab) = \sigma^{k-1}(p)\sigma^{k-1}(u_a)\sigma^{k-1}(p)\sigma^{k-1}(u_b).$$

Let $e_k = |\sigma^{k-1}(p)|$ and let $d_k = \lfloor e_k/2 \rfloor$ (the integer part of $e_k/2$), and let $c_{a,k} = |\sigma^{k-1}(u_a)|$.

Fix $n \geq 0$ and take k large enough that $e_k \geq n$. This is possible because σ is primitive. Since $\sigma^k(a) = \sigma^{k-1}(p)\sigma^{k-1}(u_a)$, it follows that $\ell_{a,k} = |\sigma^k(a)| = |\sigma^{k-1}(p)| + |\sigma^{k-1}(u_a)| = e_k + c_{a,k}$, which implies $[0, 2e_k + c_{a,k}] = [0, e_k + \ell_{a,k}]$. Thus by (2.2), any $x \in \lfloor \sigma^k(a) \rfloor$ satisfies

$$(2.3) \quad x_{[0, e_k + \ell_{a,k}]} = \sigma^{k-1}(p)\sigma^{k-1}(u_a)\sigma^{k-1}(p)$$

Now $\sigma^k(\lfloor a \rfloor) \subseteq \lfloor \sigma^k(a) \rfloor$, so (2.3) holds for $x \in \sigma^k(\lfloor a \rfloor)$.

Let $Q \in \mathcal{P}_k$. Then there exists $a \in \mathcal{A}$ and $0 \leq j < \ell_{a,k}$ so that $Q = T^{j+d_k} \sigma^k(\lfloor a \rfloor)$. Any $x \in Q$ satisfies $T^{-j+d_k} x \in \sigma^k(\lfloor a \rfloor)$, so by (2.3),

$$(2.4) \quad x_{[-j-d_k, \ell_{a,k}-j]} = (T^{-j-d_k} x)_{[0, d_k + \ell_{a,k}]} = \sigma^{k-1}(p)\sigma^{k-1}(u_a)\sigma^{k-1}(p).$$

But $n < e_k \leq d_k/2 < d_k + j$, and since $j \leq \ell_{a,k}$, $n < e_k \leq d_k/2 < \ell$. Thus $[-n, n] \subseteq [-j-d_k, d_k + c_{a,k} - j]$, so $x_{[-n,n]}$ is constant on Q . \square

Let σ be primitive aperiodic substitution, $r_k \in \mathbb{Z}$, and \mathcal{P}_k be the sequence of Kakutani-Rohlin partitions from Proposition 2.4. Define the *tower base vectors* $\mathbf{m}^{(k)} = (m_0^{(k)}, m_1^{(k)}, \dots, m_{d-1}^{(k)})$, by $m_a^{(k)} = \mu(B_a^{(k)}) = \mu(\sigma^k(\lfloor a \rfloor))$.

LEMMA 2.5. *The tower base vectors satisfy $\mathbf{m}^{(k)} = \lambda^{-k} \mathbf{m}$, and these vectors generate PF_Q .*

PROOF. First note that $\mathbf{m}^{(0)} = \mathbf{m}$, and also, $Q^t \mathbf{m}^{(k)} = \mathbf{m}^{(k-1)}$ for all k . By induction, $\mathbf{m}^{(k)} = Q^t \mathbf{m}^{(k-1)} = \lambda^{k-1} Q^t \mathbf{m} = \lambda^k \mathbf{m}$. Then

$$\bigcup_{k=1}^{\infty} \mathbb{Z}[\mathbf{m}^{(k)}] = \bigcup_{k=1}^{\infty} \lambda^{-k} \mathbb{Z}[\mathbf{m}] = \text{PF}_Q.$$

□

The next theorem characterizes the measure group $\mathbb{H}(T)$ in many cases. Examples of measure groups are computed in Section 5.1.

THEOREM 2.6. *Let T be the substitution shift for a primitive, aperiodic, irreducible substitution σ with a common prefix. Let Q be its incidence matrix, and \mathbf{m} its left normalized Perron-Frobenius eigenvector. Then $\mathbb{H}(T) = \text{PF}_Q = \mathbb{Z}[\lambda^{-1}] \mathbb{Z}[\mathbf{m}]$, with $\mathbb{H}(T)$ finitely generated if and only if Q is unimodular, in which case $\mathbb{H}(T) = \mathbb{Z}[\mathbf{m}] \cong \mathbb{Z}^d$.*

PROOF. Let \mathcal{P}_k be the sequence of Kakutani-Rohlin towers from Proposition 2.4. It follows from Proposition 2.4 that for any clopen set E , there exists k , so that E is a union of levels of \mathcal{P}_k . Thus by Lemma 2.5, $\mu(E) \in \lambda^{-k} \mathbb{Z}[\mathbf{m}]$. □

NOTE 8. For σ a primitive, aperiodic, but without the common prefix assumption, the following characterization of $\mathbb{H}(T)$ was obtained in [Be-92]. Let X_2 be the 2-block shift over X (see [Qu-87], [LM-95]): $X_2 \subseteq \mathcal{A}_2^{\mathbb{Z}}$ where $\mathcal{A}_2 = \mathcal{L} \cap \mathcal{A}^2$ is the set of 2-blocks in \mathcal{L} . Then X_2 is the substitution shift for a primitive aperiodic substitution σ_2 , with incidence matrix Q_2 having the same Perron-Frobenius eigenvalue as Q (see [Qu-87]). Let $\mathbf{m}^{[2]}$ be the left Perron-Frobenius eigenvector, normalized $\mathbf{m}^{[2]} \cdot \mathbf{1} = 1$. Then $\mathbb{H}(T)$ is the subgroup of \mathbb{R} generated by $\{\lambda^{-k}(\mathbf{n} \cdot \mathbf{m} + \mathbf{k} \cdot \mathbf{m}^{[2]}) : k-1 \in \mathbb{N}, \mathbf{n} \in \mathbb{Z}^d, \mathbf{k} \in \mathbb{Z}^{d_2}\}$. In [Be-92] this is called the $\mathbb{Z}[\lambda^{-1}]$ -module generated by \mathbf{m} and $\mathbf{m}^{[2]}$.

The next result gives a class of substitution dynamical systems that are cohomologically ergodic.

THEOREM 2.7. *If σ is a primitive, aperiodic, irreducible substitution with a common prefix, then the corresponding substitution shift dynamical system T is cohomologically ergodic. In particular, $\mathbb{H}(T) \cong H(T)$.*

PROOF. Let E and F be clopen sets such that $\mu(E) = \mu(F)$. By Proposition 2.4, there exists k so that E and F are both unions of the levels of \mathcal{P}_k . The entries of $\mathbf{m}^{(k)}$ give the measures of the bases of \mathcal{P}_k , and all the levels in each tower have the same measure as the base. By Lemma 2.3, $\mathbf{m}^{(k)}$ has rationally independent entries, since by Lemma 2.5, $\mathbf{m}^{(k)} = \lambda^{-k} \mathbf{m}$. Thus E and F must consist of the same number of levels from each tower in \mathcal{P}_k . Clearly we can define a homeomorphism $U \in [[T]]$ that matches up components of E with components of F within each tower and that is constant on the rest of X . It follows that T is saturated. Lemma 2.2 implies T is cohomologically ergodic. □

3. Flows and tiling systems

3.1. Suspension flows. Let T be a strictly ergodic Cantor dynamical system on X , and let $g(x)$ be a continuous positive real-valued function on X such that $\int_X g d\mu = 1$. Define $\tilde{T} : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$ by $\tilde{T}(x, s) = (Tx, s + g(x))$, and a flow \tilde{F}^t

on $X \times \mathbb{R}$ by $\tilde{F}^t(x, s) = (x, s + t)$. Define $Y = X \times \mathbb{R} / \sim_{\tilde{T}}$, where $(x, s) \sim_{\tilde{T}} (x', s')$ if $(x, s) = \tilde{T}^n(x', s')$ for some n . Then Y is a 1-dimensional compact metric space, called the *suspension space* of T by g . We define continuous flow F^t on Y by $F^t = \tilde{F}^t / \sim_{\tilde{T}}$, called the *suspension flow* (or *flow under the function* g). The strict ergodicity of T implies strict ergodicity for F^t . In particular, the unique F^t invariant probability measure ν on Y is defined by $\nu(E \times [a, b]) = (b - a)\mu(E)$, where $E \subseteq X$ and $0 \leq a < b < \min_{x \in E} g(x)$.

3.2. The Bruschlinsky group. In the suspension space Y , every point y has a neighborhood that is a product of an interval and a Cantor set. Such spaces are sometimes called *laminations* or *matchbox manifolds*. Such spaces have a well defined 1-dimensional integer Čech cohomology $\check{H}^1(Y)$ (see e.g., [S-08] for the definition in this context).

Let $C(Y, \mathbb{T})$ be the group of continuous unimodular complex functions on Y (where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$). Define a “sum” as pointwise multiplication: $(f_0 + f_1)(y) = f_0(y)f_1(y)$ and $(-f)(y) = 1/f(y)$. Let $R(Y) = \{\exp(2\pi ip(y)) : p : Y \rightarrow \mathbb{R} \text{ continuous}\}$. It is easy to see that f_1 is homotopic to f_2 , denoted $f_1 \sim f_2$, if and only if $f_0 - f_1 \in R(Y)$. Define the *Bruschlinsky group* by $Br(Y) := C(Y, \mathbb{T})/R(Y)$. We will need the following old result.

THEOREM 3.1 (Bruschlinsky’s Theorem [Br-34]). *The group $Br(Y)$ is isomorphic to $\check{H}^1(Y)$.*

This theorem is often quoted in the dynamics and tiling literature (see e.g., [Sc-57, PT-82, BoH-96, AP-98, Yi-10]). The standard reference seems to be [Hu-59] (pp. 47-52 and Exercise C, p. 59), but this is only slightly more modern than [Br-34]. In Note 6 (below), we try to offer some insight into the theorem in the case that Y is a substitution tiling space. Following [PT-82]), from now on, we will sometimes write $\check{H}^1(Y)$ to really mean $Br(Y)$.

For $n(x) \in C(X, \mathbb{Z})$ define

$$(3.1) \quad f_n(x, r) = \exp(2\pi i r n(x)/g(x)) \in C(Y, \mathbb{T}).$$

LEMMA 3.2. *For any $f \in C(Y, \mathbb{T})$ there is $n \in C(X, \mathbb{Z})$ so that $f \sim f_n$. Moreover, $n \in B(T)$ if and only if $f_n \in R(Y)$. It follows that $H(T) \cong \check{H}^1(Y)$.*

PROOF. This is proved in the case $g(x) = 1$ in [PT-82]. The general case follows from the observation that the suspension flow F^t on the suspension space Y for an arbitrary continuous return time $g(x)$, and the unit suspension F_1^t on Y_1 , (i.e., the suspension for the return time $g_1(x) = 1$), are topologically conjugate via the homeomorphism $S(x, r) = (x, r/g(x))$ between Y and Y_1 . \square

3.3. Winding numbers. Now suppose (Y, F, ν) is a uniquely ergodic flow on a compact metric space. We say that $f \in C(Y, \mathbb{T})$ is *continuously differentiable* at $y \in Y$ if

$$(3.2) \quad f'(y) := \lim_{t \rightarrow 0} \frac{1}{t}(f(F^t y) - f(y)) \in C(Y, \mathbb{T}).$$

We define the *winding number* of a continuously differentiable f by

$$(3.3) \quad W(f) = \frac{1}{2\pi i} \int_Y \frac{f'(y)}{f(y)} d\nu(y).$$

The winding number was defined by Schwartzman [Sc-57], who proved the following.

LEMMA 3.3. (Schwartzman, [Sc-57]) *Every homotopy class in $C(Y, \mathbb{T})$ contains a continuously differentiable function, and $f_1 \sim f_2$ implies $W(f_1) = W(f_2)$. Moreover, $W(f_1 + f_2) = W(f_1) + W(f_2)$. Thus $W : \check{H}^1(Y) \rightarrow \mathbb{R}$ is a well defined homomorphism (i.e., a real valued functional).*

DEFINITION 3.4. *We say F^t is cohomologically ergodic if W is injective.*

NOTE 9. We think of the functions f with $W(f) = 0$ as being “cohomologically invariant”. Then cohomological ergodicity says that $W(f) = 0$ implies $f = 0$.

PROPOSITION 3.5. *The image of the winding number functional satisfies $W(\check{H}^1(Y)) = \mathbb{H}(T)$ in \mathbb{R} , and thus does not depend on the return time $g(x)$. Moreover, the suspension flow F^t is cohomologically ergodic if and only if T is cohomologically ergodic.*

PROOF. For $n \in C(X, \mathbb{Z})$ let f_n be defined by (3.1). Then the derivative satisfies $f'_n(x, t) := 2\pi i(n(x)/g(x)) f_n(x, t)$ satisfies $f'_n/f_n = 2\pi i(n(x)/g(x))$. Thus $W(f_n) = \int_Y (n(x)/g(x)) d\nu = \int_X n d\mu$. □

3.4. Eigenfunctions and eigenvalues. We say $\omega \in \mathbb{R}$ is an *eigenvalue* for F^t corresponding to an *eigenfunction* $f \in L^2(Y, \nu)$ if

$$(3.4) \quad f(F^t y) = \exp(2\pi i \omega t) f(y),$$

for all $t \in \mathbb{R}$ and ν a.e., $y \in Y$. We denote the set of all eigenvalues by $\mathbb{E}(F)$. We assume F^t is *homogeneous* (in the sense of [R-04]). This means there is a continuous eigenfunction f for each eigenvalue $\omega \in \mathbb{E}(F)$. Since F^t is strictly ergodic, each eigenvalue ω is simple, each eigenfunction f is \mathbb{T} -valued, and $\mathbb{E}(F)$ is a countable subgroup of \mathbb{R} . Recall that F^t is *weakly mixing* if $\mathbb{E}(F) = \{1\}$, and has *pure point spectrum* if $E(F)$ generates $L^2(Y, \nu)$.

Let $E(F)$ denote the group of all (continuous) eigenfunctions modulo constant multiples. To make it an additive group we define $(f_1 + f_2)(y) = f_1(y)f_2(y)$. For $f \in E(F)$ define $V(f)$ to be the corresponding eigenvalue in \mathbb{R} . Then V is a homomorphism, and $\mathbb{E}(F)$ is the image of $E(F)$ under V . We call V the *eigenvalue functional*.

LEMMA 3.6 (Schwartzman [Sc-57]). *Continuous eigenfunctions are continuously differentiable, and $V(f) = W(f)$ for $f \in E(F)$.*

THEOREM 3.7. *Suppose F^t is a suspension of a strictly ergodic Cantor dynamical system T , for some positive continuous return time function g , (so that F^t is strictly ergodic). Assume F^t is homogeneous. Then $E(F) \subseteq Br(Y) = \check{H}^1(Y)$, and $V = W|_{E(F)}$. Thus the eigenvalue group $\mathbb{E}(F)$ is a subgroup of the measure group $\mathbb{H}(T)$.*

PROOF. We may assume eigenfunctions satisfy $f \in C(Y, \mathbb{T})$. Two eigenfunctions for the same eigenvalue are constant multiples, and thus homotopic. So there is a natural map $E(F) \rightarrow R(Y)$. Now by definition, any distinct $f_1, f_2 \in E(F)$ satisfy $V(f_1) \neq V(f_2)$. By Lemma 3.6, $W(f_1) \neq W(f_2)$, and by Lemma 3.3, $f_1 \not\sim f_2$. So $(f_1 - f_2) \notin R(Y)$. Thus the map $E(F) \rightarrow R(Y)$ is injective. □

Since $W(\check{H}^1(Y)) = \mathbb{H}(T)$, it follows that the image of W does not depend on the return time g . We will see in the Section 5, however, that $\mathbb{E}(F)$ generally does depend on g . We interpret the inclusion $\mathbb{E}(F) \subseteq \mathbb{H}(T)$ to mean that $\mathbb{H}(T)$ provides an “upper bound” on what eigenvalues $\mathbb{E}(F)$ a suspension can have. Recall that $\mathbb{H}(T)$ is the subgroup of \mathbb{R} generated by the measures of clopen (or cylinder) sets, so these measures determine all the possible eigenvalues. But sometimes, not all these eigenvalues occur.

NOTE 10. It is shown in [I-O-07] that $E(F)$ embeds in $K_0(C(X) \rtimes_T \mathbb{Z})$ (which is $\cong \check{H}^1(Y)$). Similarly, in [BKES-12] the “maximal equicontinuous” factor (G, K^t) of (Y, F^t) is considered. The maximal equicontinuous is the unique Kronecker flow K^t (i.e., a rotation action on a metric compact abelian group G) that satisfies $\mathbb{E}(K) = \mathbb{E}(F)$. They show $\check{H}^1(G) \cong \mathbb{E}(K)$ ($= \mathbb{E}(F)$), and that $\pi^* : \check{H}^1(G) \rightarrow \check{H}^1(Y)$ is injective, where π^* is the map induced by the factor map $\pi : Y \rightarrow G$.

DEFINITION 3.8. *We say a strictly ergodic, homogeneous flow F^t has cohomological pure point spectrum if it is cohomologically ergodic and $\check{H}^1(Y) = E(F)$, or equivalently it is cohomologically ergodic and $\mathbb{H}(T) = \mathbb{E}(F)$.*

In dynamical systems, one studies both topological and metric pure point spectrum. Topological pure point spectrum is equivalent to F^t being equicontinuous, or equivalently, topologically conjugate to a strictly ergodic Kronecker flow K^t . Such a flow always has cohomological pure point spectrum. But substitution tiling flows can never be equicontinuous because they always have proximal points. Metric pure point spectrum means F^t is metrically isomorphic to a strictly ergodic Kronecker flow K^t . Many examples of substitution tiling flows with metric pure point spectrum are known, including some discussed in Section 5. All of those examples cohomological pure point spectrum.

It is reasonable to conjecture that metric pure point spectrum is equivalent to cohomological pure point spectrum for a suspension of a strictly ergodic Cantor dynamical system T , but at this time we do not know either implication. We do not even know whether a strictly ergodic Cantor dynamical system T with metric pure point spectrum must be cohomologically ergodic.

3.5. The unit suspension and spectrum of T . In this section we reconsider the case of a strictly ergodic and homogeneous Cantor dynamical system (X, T, μ) . An *eigenvalue* η for T , corresponding to an *eigenfunction* $f \in C(X, \mathbb{C})$, is an $\eta \in \mathbb{T}$ satisfying $f(Tx) = \eta f(x)$ for μ a.e. $x \in X$. We denote the set of all eigenvalues of T by $\mathbb{D}(T)$. Since T is strictly ergodic, each eigenvalue η is simple, each eigenfunction f is \mathbb{T} -valued, and $\mathbb{D}(T)$ is a countable subgroup of \mathbb{T} . We define $\mathbb{E}(T)$ to be the set of all $\omega \in \mathbb{R}$ so that $e^{2\pi i \omega} \in \mathbb{D}(T)$.

The *unit suspension* of T is the suspension flow F_1^t on Y_1 corresponding to the constant return time function $g(x) = 1$. The flow F_1^t is strictly ergodic and homogeneous. It is cohomologically ergodic if and only if T is cohomologically ergodic. It is well known that $\mathbb{E}(T) = \mathbb{E}(F_1)$. We say that T has *cohomological pure point spectrum* the unit suspension F_1^t has cohomological pure point spectrum.

4. Substitution tilings

Let σ be a primitive, aperiodic substitution on $\mathcal{A} = \{0, 1, \dots, d-1\}$ and let (X, T, μ) be the corresponding strictly ergodic substitution shift dynamical system.

Let \mathbf{m} and \mathbf{h} be the normalized left and right Perron-Frobenius eigenvectors of the incidence matrix Q . Let $\mathbf{g} = (g_0, g_1, \dots, g_{d-1})$ be any positive vector satisfying $\mathbf{g} \cdot \mathbf{m} = 1$. Define $g : X \rightarrow \mathbb{R}$ by $g(x) = \sum \mathbf{g}_{x_0}$ where $x = \dots x_{-1}.x_0x_1x_2\dots$, and let F^t on Y be the corresponding suspension flow. Then F^t is strictly ergodic and homogeneous (see [CS-03]). We call the pair $S = (\sigma, \mathbf{g})$ a *tile substitution*. We call Y a 1-dimensional *substitution tiling space*, and we call F^t a *substitution tiling flow*.

Tile substitutions have natural a geometric interpretation (see also [CS-03, R-04, KSS-05, FR-08]). Let $\mathcal{T} = \{I_0, I_1, \dots, I_{d-1}\}$ be a set of labeled, half-closed intervals $I_a = [0, g_a)_a$, called *prototiles*. For any $u = u_0u_2\dots u_{n-1} \in \mathcal{A}^*$, let $t_0 = 0$ and for $j \geq 1$, $t_j = g_{u_0} + \dots + g_{u_{j-1}}$, and define a labeled *tiling*

$$I_u = [t_0, t_1)_{u_0}[t_1, t_2)_{u_1} \dots [t_{n-1}, t_n)_{u_n}$$

of the interval $[0, t_n)$ by (translations of the tiles in) \mathcal{T} .

Let \mathcal{T}^* denote the set of all finite tilings of intervals by \mathcal{T} . We can interpret the tile substitution $S = (\sigma, \mathbf{g})$ as a mapping $S : \mathcal{T} \rightarrow \mathcal{T}^*$ defined by $S(I_a) = I_{\sigma(a)}$.³ For example, consider the *golden mean substitution* σ defined by $\sigma(0) = 01$, $\sigma(1) = 0$. Take $\mathbf{g} = \mathbf{h} = (\lambda + 1)^{-1}(1, \lambda)$, the normalized right Perron-Frobenius eigenvector for Perron-Frobenius eigenvalue $\lambda = (1 + \sqrt{5})/2$. Then the tile substitution S , realized as a map on prototiles, looks like this:

$$\begin{array}{c} \text{---} 0 \text{---} \\ \text{---} 1 \text{---} \end{array} \rightarrow \begin{array}{c} \text{---} 0 \text{---} \\ \text{---} 0 \text{---} \end{array}$$

Let Z be the set of all tilings z of \mathbb{R} by (translations of the tiles in) \mathcal{T} satisfying the additional requirement that every finite subtiling of z is a translate of a subtiling of some $S^k(I_a)$, $a \in \mathcal{A}$, $k \geq 0$. We call Z the *substitution tiling space* corresponding to S . This space has a natural compact metric topology (see e.g., [R-04]), and there is a *tiling flow* E^t , which acts on Z by translation. One easily shows that (Z, E^t) is topologically conjugate to (Y, F^t) , and from now on we identify them. A portion of a tiling z for the Fibonacci substitution S is shown below (where the dot indicates the position of $0 \in \mathbb{R}$):

$$\dots \text{---} 0 \text{---} \bullet \text{---} 0 \text{---} 1 \text{---} 0 \text{---} 1 \text{---} \dots$$

We say that S is *primitive*, *aperiodic*, *irreducible*, *unimodular*, *Pisot*, or has a *common prefix* according to σ . For a given σ there are two special tile substitutions corresponding to two special tile length vectors \mathbf{g} . We call $S_1 = (\sigma, \mathbf{1})$, $\mathbf{g} = \mathbf{1} = (1, 1, \dots, 1)$, the *unit* tile substitution (all the tiles have length 1). The corresponding substitution tiling flow F_1^t is the unit suspension of T . We call $S_{\text{PF}} = (\sigma, \mathbf{h})$, where $\mathbf{g} = \mathbf{h}$ is the normalized right Perron-Frobenius eigenvector, a *self-similar* tile substitution. It is *self-similar* in the sense of ([So-97]), which means for each $a \in \mathcal{A}$, the length of $S_{\text{PF}}(I_a)$ is λ times the length of I_a , where λ is the Perron-Frobenius eigenvalue. We sometimes write F_1^t and F_{PF}^t for the corresponding substitution tiling flows.

THEOREM 4.1. *Let F^t be the substitution tiling flow corresponding to a primitive, aperiodic tile substitution S . Then F^t is strictly ergodic and homogeneous. If*

³The substitution S also evidently defines a mapping on Y which, as a result of σ being aperiodic, is a homeomorphism (see [R-04]).

S is also irreducible, and has a common prefix, then F^t is cohomologically ergodic and $\mathbb{E}(F) \subseteq \mathbb{H}(T)$.

PROOF. As previously noted, strict ergodicity and homogeneity are proved in [CS-03]. Cohomological ergodicity follows from Proposition 3.5, and the inclusion $\mathbb{E}(F) \subseteq \mathbb{H}(T)$ follows from Theorem 3.7. \square

Note that if Q is the incidence matrix and \mathbf{m} the normalized left Perron-Frobenius eigenvector, $\mathbb{H}(T) = \text{PF}_Q = \mathbb{Z}[\lambda^{-1}]\mathbb{Z}[\mathbf{m}]$ which is $\cong \check{H}^1(Y)$ in the cohomologically ergodic case. If S is also unimodular, then $\text{PF}_Q = \mathbb{Z}[\mathbf{m}] \cong \mathbb{Z}^d$. Now we turn to the point spectrum.

THEOREM 4.2. *Let F^t be the substitution tiling flow corresponding to a primitive, aperiodic, irreducible, unimodular, Pisot tile substitution $S = (\sigma, \mathbf{g})$ with a common prefix. Then F^t has cohomological pure point spectrum. In particular, F^t is cohomologically ergodic, and $\mathbb{E}(F) = \mathbb{H}(T) = \text{PF}_Q \cong \mathbb{Z}[\mathbf{m}]$.*

COROLLARY 4.3. *If σ is a primitive, aperiodic, unimodular, Pisot substitution with a common prefix, then T has cohomological pure point spectrum. In particular, T is cohomologically ergodic and $\mathbb{D}(T) = \exp(2\pi i \text{PF}_Q)$.*

Cohomological ergodicity and the inclusion $\mathbb{E}(F) \supseteq \mathbb{H}(T)$ follow from Theorem 4.1. The proof of \subseteq takes up the remainder of this section. We prove it first in the self-similar case $\mathbf{g} = \mathbf{h}$, then in general.

For a word $u = u_0 u_1 \dots u_{n-1} \in \mathcal{L}$, the *population vector* is defined $\mathbf{p}_u = (p_0, p_1, \dots, p_{d-1}) \in \mathbb{Z}^d$, where $p_a = |\{j = 0, \dots, n-1 : u_j = a\}|$. We say $u \in \mathcal{L}$ is a *recurrence word* if $u = x_{[r,s]}$, $r \leq s$, for some $x \in X$ such that $x_r = x_{s+1}$. For a real number t define $\{t\} = t - \lfloor t \rfloor$. We begin with a general lemma from [CS-03] (see also [Ho-86] and [So-98]).

LEMMA 4.4. *If S is a primitive, aperiodic tile substitution, then $\omega \in \mathbb{E}(F)$ if and only if for every recurrence word u ,*

$$(4.1) \quad \lim_{n \rightarrow \infty} \{\omega \mathbf{g} \cdot Q^n \mathbf{p}_u\} = 0.$$

For a Perron-Frobenius substitution $S_{\mathbf{PF}}$ (corresponding to $\mathbf{g} = \mathbf{h}$, the Perron-Frobenius eigenvector), we have the following.

LEMMA 4.5. *Suppose $S_{\mathbf{PF}}$ is a primitive, aperiodic Perron-Frobenius tile substitution. If for each $a \in \mathcal{A}$, $\omega \in \mathbb{R}$ satisfies*

$$(4.2) \quad \lim_{n \rightarrow \infty} \{\omega h_a \lambda^n\} = 0,$$

where λ is the Perron-Frobenius eigenvalue, then $\omega \in \mathbb{E}(F)$.

PROOF. It suffices to prove (4.1) for \mathbf{p}_a . But $\omega \mathbf{h} \cdot Q^n \mathbf{p}_a = \omega (Q^t)^n \mathbf{h} \cdot \mathbf{p}_a = \omega \lambda^n \mathbf{h} \cdot \mathbf{p}_a = \omega \lambda^n h_a$, since \mathbf{p}_a is the a th standard basis vector. \square

LEMMA 4.6. *Suppose σ is primitive, irreducible, Pisot substitution. Then Q has 1-dimensional expanding subspace $L^u(Q)$ and a $d-1$ dimensional contracting subspace $L^s(Q)$, and the same is true for Q^t . Moreover, $L^u(Q) \perp L^s(Q^t)$ and $L^u(Q^t) \perp L^s(Q)$. If we define $P\mathbf{v} := \mathbf{v} - (\mathbf{h} \cdot \mathbf{v})\mathbf{m}$ for $\mathbf{v} \in \mathbb{R}^d$, then $P\mathbf{v} \in L^s(Q^t)$.*

PROOF. Since Q is irreducible, it has a left $\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_d$ and right $\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_d$ eigenbases. Let $\mathbf{m}_1 = \mathbf{m}$, $\mathbf{h}_1 = \mathbf{h}$, which are positive real. But other eigenvectors \mathbf{m}_j and \mathbf{h}_j , $j > 1$ may be complex. Then $\mathbf{h}_i \cdot \mathbf{m}_j = 0$, for $i \neq j$, because $\lambda_i \mathbf{h}_i \cdot \mathbf{m}_j = (Q^t \mathbf{h}_i) \cdot \mathbf{m}_j = \mathbf{h}_i \cdot (Q \mathbf{m}_j) = \lambda_j \mathbf{h}_i \cdot \mathbf{m}_j$ for $\lambda_i \neq \lambda_j$ (note that here: $\mathbf{v} \cdot \mathbf{w} := v_1 \overline{w_1} + v_2 \overline{w_2} + \dots + v_d \overline{w_d}$). Define $L^u(Q)$ and $L^u(Q^t)$ to be the spans of \mathbf{m} and \mathbf{h} . The complex eigenvalues, if they occur at all, occur in conjugate pairs. We replace such a pair with the new real pair consisting of the real and imaginary parts of the conjugate pair. Both of these lie in the complex span of the original pair. We then define $L^s(Q)$ and $L^s(Q^t)$ to be the real parts of the spans of $\mathbf{m}_2, \dots, \mathbf{m}_d$ and $\mathbf{h}_2, \dots, \mathbf{h}_d$. For the second assertion, $\mathbf{h} \cdot P\mathbf{v} = \mathbf{h} \cdot (\mathbf{v} - (\mathbf{h} \cdot \mathbf{v})\mathbf{m}) = \mathbf{h} \cdot \mathbf{v} - (\mathbf{h} \cdot \mathbf{v})(\mathbf{h} \cdot \mathbf{m}) = 0$ since $\mathbf{h} \cdot \mathbf{m} = 1$. \square

The next result is one direction of Theorem 4.2 in the case of $\mathbf{g} = \mathbf{h}$.

PROPOSITION 4.7. *Suppose S_{PF} is a primitive, aperiodic, irreducible Pisot, Perron-Frobenius tile substitution. Then the corresponding substitution tiling flow F^t satisfies $\mathbb{H}(T) \subseteq \mathbb{E}(F)$.*

PROOF. For $\mathbf{v}, \mathbf{k} \in \mathbb{R}^n$, let $t = \mathbf{v} \cdot \mathbf{h}$, and $\alpha = \mathbf{k} \cdot \mathbf{m}$. Then

$$\begin{aligned} t\alpha\lambda^n &= \lambda^n(\mathbf{v} \cdot \mathbf{h})(\mathbf{k} \cdot \mathbf{m}) = \mathbf{k} \cdot (\mathbf{v} \cdot \mathbf{h})\lambda^n \mathbf{m} \\ &= \mathbf{k} \cdot (\mathbf{v} \cdot \mathbf{h})(Q^t)^n \mathbf{m} = Q^n \mathbf{k} \cdot (\mathbf{v} \cdot \mathbf{h})\mathbf{m} \\ &= Q^n \mathbf{k} \cdot (\mathbf{v} - P\mathbf{v}) = Q^n \mathbf{k} \cdot \mathbf{v} - \mathbf{k} \cdot (Q^t)^n P\mathbf{v}. \end{aligned}$$

If $\mathbf{k} = \mathbf{p}_a$ and $\mathbf{v} = \mathbf{p}_b$, then

$$(4.3) \quad \{m_b h_a \lambda^n\} = \{\mathbf{p}_a \cdot Q^n \mathbf{p}_b - \mathbf{p}_a \cdot (Q^t)^n P\mathbf{p}_b\} = \{-\mathbf{p}_a \cdot (Q^t)^n P\mathbf{p}_b\},$$

since $Q^n \mathbf{p}_a \cdot \mathbf{p}_b \in \mathbb{Z}$, and thus $\{m_b h_a \lambda^n\} \rightarrow 0$ since $P\mathbf{p}_b \in L^s(Q)$. This shows $\mathbb{Z}[\mathbf{m}] \subseteq \mathbb{E}(F)$. \square

NOTE 11. In [BK-06], S_{PF} is assumed to be unimodular, but without necessarily, the assumption of a common prefix. The main result there is that $\mathbb{Z}[\mathbf{m}] = \mathbb{E}(F)$. The inclusion $\mathbb{Z}[\mathbf{m}] \subseteq \mathbb{E}(F)$ is said to be well known, and attributed to [BT-86]. But a direct proof is also given, which constructs a Kronecker factor (G, K^t) of (Y, F^t) with $\mathbb{E}(K) = \mathbb{Z}[\mathbf{m}] \subseteq \mathbb{E}(F)$ (but note that [BK-06] uses a different normalization: $\mathbf{m} \cdot \mathbf{m} = 1$ and $\mathbf{h} \cdot \mathbf{m} = 1$). The opposite inclusion, $\mathbb{E}(F) \subseteq \mathbb{Z}[\mathbf{m}]$, is the main result of [BK-06]. For us, this follows from Theorem 4.1 (using the assumption that S has a common prefix).

The next result, which completes the proof of Theorem 4.2, shows that for a primitive, aperiodic, irreducible, Pisot tile substitutions, the Perron-Frobenius eigenvector \mathbf{h} can be replaced by any \mathbf{g} satisfying $\mathbf{g} \cdot \mathbf{m} = 1$. This result appears in [CS-03], but we include a proof for completeness.

PROPOSITION 4.8 ([CS-03]). *Suppose $S_{\text{PF}} = (\sigma, \mathbf{h})$ is a primitive, aperiodic, irreducible, Pisot, Perron-Frobenius tile substitution. Let $S = (\sigma, \mathbf{g})$ be any other tile substitution based on the same discrete substitution σ . Then $\mathbb{E}(F) = \mathbb{E}(F_{\text{PF}})$.*

PROOF. Lemma 4.4 implies $\{\omega \mathbf{h} \cdot Q^n \mathbf{p}_u\} \rightarrow 0$ for every $\omega \in \mathbb{E}(F_{\text{PF}})$ and every return word u . Then

$$\begin{aligned} \{\omega \mathbf{h} \cdot Q^n \mathbf{p}_u\} &= \{\omega(\mathbf{g} - (\mathbf{g} - \mathbf{h})) \cdot Q^n \mathbf{p}_u\} \\ &= \{\omega \mathbf{g} \cdot Q^n \mathbf{p}_u - \omega(Q^t)^n (\mathbf{g} - \mathbf{h}) \cdot \mathbf{p}_u\}. \end{aligned}$$

Since $(\mathbf{g} - \mathbf{h}) \cdot \mathbf{m} = \mathbf{g} \cdot \mathbf{m} - \mathbf{h} \cdot \mathbf{m} = 0$, it follows that $\mathbf{g} - \mathbf{h} \in L^s(Q^t)$ and $(Q^t)^n(\mathbf{g} - \mathbf{h}) \rightarrow \mathbf{0}$. Thus $\{\omega \mathbf{g} \cdot Q^n \mathbf{p}_u\} \rightarrow 0$ and $\omega \in \mathbb{E}(F)$. \square

5. Examples

5.1. Metallic and alloy substitutions. For $n \geq 1$ we define the n th *metallic* substitution $\sigma : 0 \rightarrow 0^n 1, 1 \rightarrow 0$. Also for $m \geq 3$, we define the m th *alloy* substitution $\sigma : 0 \rightarrow 0^{m-1} 1^{m-2}, 1 \rightarrow 01$.

PROPOSITION 5.1. *Each metallic or alloy substitution is primitive, aperiodic, irreducible, unimodular, Pisot, and has a common prefix. The corresponding substitution shift T and, for any \mathbf{g} , the substitution tiling flow F^t corresponding to $S = (\sigma, \mathbf{g})$, is cohomologically ergodic, with $H(T) \cong \mathbb{H}(T) = \mathbb{Z}[\mathbf{m}] \cong \check{H}^1(Y)$.*

PROOF. The characteristic polynomials of Q in the metallic case are $p_n(z) = z^2 - nz - 1$, and in the alloy case are $q_m(z) = z^2 - mz + 1$. \square

The positive roots λ_n of $p_n(z)$ are sometimes called *metallic numbers*: $\lambda_1 = (1/2)(1 + \sqrt{5})$ is called the *golden mean*, $\lambda_2 = 1 + \sqrt{2}$ is called the *silver mean*, etc. We call the roots λ'_m of $q_m(z)$ *alloy numbers* because the incidence matrices satisfy $Q'_{n+2} = Q_1 Q_n$ (gold, together with a base metal). Any monic quadratic polynomial over \mathbb{Z} with constant term ± 1 is either $p_n(\pm z)$, $q_n(\pm z)$ or $r(z) = z^2 \pm 1$. Thus the metallic and alloy numbers make up all the quadratic units. All quadratic units are Pisot.

THEOREM 5.2. *Any metallic or alloy substitution tiling flow F^t has cohomological pure point spectrum: $\mathbb{E}(F) = \mathbb{H}(T)$. In the metallic case, $\mathbb{E}(F) = (\lambda + 1)^{-1} \mathbb{Z}[\lambda] \cong \mathbb{Z}^2$ (with $\mathbb{E}(F) = \mathbb{Z}[\lambda]$ in the Golden mean case, where $\lambda + 1$ is a unit). In the alloy case, $\mathbb{E}(F) = (\lambda - (m - 2))^{-1} \mathbb{Z}[\lambda] \cong \mathbb{Z}^2$. For the substitution shifts T , $\mathbb{D}(T) = \exp(2\pi i \mathbb{E}(F))$, and every such T has pure point spectrum.*

PROOF. In the metallic case, $\mathbf{m} = (\lambda + 1)^{-1}(1, \lambda)$, so $\mathbb{E}(F) = \mathbb{Z}[\mathbf{m}] = (\lambda + 1)^{-1} \mathbb{Z}[\lambda]$. In the alloy case, $m \geq 3$, $\mathbf{m} = (\lambda - m + 2)^{-1}(1, \lambda - m + 1)$, and $\mathbb{Z}[\lambda - m + 1] = \mathbb{Z}[\lambda]$. \square

5.2. A unimodular non-Pisot substitution. A *non-Pisot* substitution is a primitive, aperiodic, irreducible substitution σ , such that λ is not a Pisot number. The non-Pisot substitution $\sigma : 0 \rightarrow 0313, 1 \rightarrow 031313, 2 \rightarrow 03223, 3 \rightarrow 0323$, studied in [FiHR-03], is unimodular and has a common prefix. It is shown in [FiHR-03] that F_{PF}^t is weakly mixing, which means $\mathbb{E}(F_{\text{PF}}) = \{0\}$. Clearly $\mathbb{E}(F_1) = \mathbb{Z}$. However, in both cases, $\check{H}^1(Y) \cong \mathbb{H}(T) = \mathbb{Z}[\mathbf{m}] \cong \mathbb{Z}^4$, so neither has cohomological pure point spectrum.

5.3. The completely non-Pisot case. We call σ *completely non-Pisot* if all the eigenvalues θ of Q satisfy $|\theta| \geq 1$. Primitive, aperiodic, completely non-Pisot substitutions σ , are studied in [CS-03] with the additional assumption that there is a *full recurrence word*: $u \in \mathcal{L}$ such that $\mathbf{p}_u, Q\mathbf{p}_u, \dots, Q^{d-1}\mathbf{p}_u$ are linearly independent. An example is the substitution $0 \rightarrow 0111, 1 \rightarrow 0$ from [FR-08]. If $\mathbf{g} = (g_0, g_1, \dots, g_{d-1})$ has rationally independent entries, then F^t is weakly mixing, [CS-03]. This provides many examples of substitutions without cohomological pure point spectrum, since $\mathbb{E}(F) = \{0\}$ is always a proper subgroup of $\mathbb{H}(T)$, (which, by Theorem 2.6, is not finitely generated). It is also shown in [CS-03] that if $g_a/g_b \in \mathbb{Q} \forall a, b \in \mathcal{A}$, for a primitive, aperiodic, completely non-Pisot tile

substitution then $\mathbb{E}(F) \subseteq g_0\mathbb{Q}$. In the two-letter primitive, aperiodic, completely non-Pisot substitutions σ , it is shown in [KSS-05] that F^t is topologically mixing (which implies weak mixing), if and only if $g_0/g_1 \notin \mathbb{Q}$.

6. Appendix: Remarks on Bruschi's theorem

Let Y be a substitution tiling space for a primitive, aperiodic tile substitution S . Our goal in this section is to provide some insight why the Bruschi group $Br(Y)$ is isomorphic to the integer Čech cohomology $\check{H}^1(Y)$. As we will see, this follows from the fact that Y can be expressed as an inverse limit of an endomorphism $S : \Gamma \rightarrow \Gamma$ of a directed graph Γ . The integer Čech cohomology of a graph Γ is the same as its integer simplicial cohomology, and we will show below that the simplicial cohomology is isomorphic to $Br(\Gamma)$. The desired result, that $\check{H}^1(Y) \cong Br(Y)$, then follows from the fact that the Čech cohomology of an inverse limit is the direct limit of the Čech cohomologies.

The graph Γ mentioned above is the *Anderson-Putnam complex*. The edge set \mathcal{E} of Γ is equal to the (directed) prototiles. The vertices \mathcal{V} are defined by the condition that the front end of one tile meets the back end of another at a vertex $v \in \mathcal{V}$ if and only if that pair also meets somewhere in some tiling in Γ . The substitution S then acts as an endomorphism of this graph; a piecewise linear, one-dimensional dynamical system. For technical reasons one needs to first replace⁴ the substitution with a “collared” 3-higher block substitution S , and take Γ to be the complex with the “collared” prototiles as its edges (see [AP-98] for details). Using this setup, the inverse limit characterization

$$Y = \varprojlim (\Gamma, S)$$

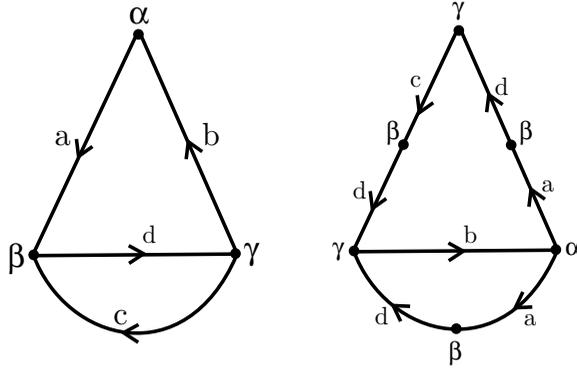
is obtained in [AP-98]. The corresponding direct limit

$$\check{H}^1(Y) = \varinjlim (\check{H}^1(\Gamma), S^*)$$

is a standard property of Čech cohomology.

Here is an example of S and Γ from [AP-98]. Start with the Fibonacci substitution $\sigma: 0 \rightarrow 01, 1 \rightarrow 0$, and let $\mathcal{A}_3 = \{a, b, c, d\}$ where $a = 0.01, b = 1.00, c = 1.01$ and $d = 0.10$. Then the collared substitution σ_3 is given by $a = 0.01 \rightarrow 01.010 = cd, b \rightarrow dc, c \rightarrow d$ and $d \rightarrow db$. Without loss of generality, we take $S = S_1$, the unit tile substitution (where all tiles have length 1). Then $\mathcal{E} = \{a, b, c, d\}$, viewed as (labeled) prototiles. Here \mathcal{V} is a three element set, corresponding to pairs of adjacent prototiles, which we write as $\mathcal{V} = \{\alpha, \beta, \gamma\}$. This complex and the action of S on it are shown below.

⁴If the original substitution S “forces its border”, see [AP-98], then S itself be used.



In general, the the sets of 0- and 1-simplicial cochains are, respectively, $\mathbb{Z}^{\mathcal{E}}$ and $\mathbb{Z}^{\mathcal{V}}$. So in the Fibonacci case they are \mathbb{Z}^3 and \mathbb{Z}^4 . The coboundary operator is a $|\mathcal{V}| \times |\mathcal{E}|$ matrix δ with entries $d_{v,e} = \pm 1$, depending on whether v enters or leaves e , or $d_{v,e} = 0$ if v is not incident on e . Then $\check{H}^1(\Gamma) = H^1(\Gamma) := \mathbb{Z}^{\mathcal{V}}/\delta(\mathbb{Z}^{\mathcal{E}})$.

The coboundary matrix for the Fibonacci case is

$$\delta = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

which has rank 2, and the columns are primitive vectors. Thus $\check{H}^1(\Gamma) = H^1(\Gamma) := \mathbb{Z}^4/\delta(\mathbb{Z}^3) \cong \mathbb{Z}^2$. This calculation concludes with

$$\check{H}^1(Y) = \lim_{\rightarrow} (\check{H}^1(\Gamma), S^*) \cong \mathbb{Z}^2$$

(see [AP-98]), which follows from the fact that the incidence matrix Q for the collared tile substitution S has rank 2, and its projection S^* to $\mathbb{Z}^4/\delta(\mathbb{Z}^3) \cong \mathbb{Z}^2$ is an abelian group isomorphism.

View each edge in Γ is a labeled unit interval. Thus if $y \in \Gamma$ is not at a vertex, it corresponds to $t = t(y) \in (0, 1)$ and to a label $\ell(y) \in \mathcal{A}_3$, where the function $\ell(y)$ is constant on the interiors of the edges. Given $\mathbf{n} \in \mathbb{Z}^{\mathcal{E}}$ define $f_{\mathbf{n}} \in C(\Gamma, \mathbb{T})$ by $f_{\mathbf{c}}(y) = 1$ for $y \in \mathcal{V}$ and $f_{\mathbf{n}}(y) = \exp(2\pi i \mathbf{n}_{\ell(y)} t(y))$ for $y \in \Gamma \setminus \mathcal{V}$. Conversely, if $f \in C(\Gamma, \mathbb{T})$ it is clear that f is homotopic to some $f_1 \in C(\Gamma, \mathbb{T})$ with $f_1(v) = 1$ for each $v \in \mathcal{V}$. And then f_1 is homotopic to $f_{\mathbf{n}}$ for some $\mathbf{n} \in \mathbb{Z}^{\mathcal{E}}$. In particular, if $\mathbf{n} = (n_1, \dots, n_{|\mathcal{E}|})$, then n_e is the degree of $f_1 : e = [0, 1] \rightarrow \mathbb{T}$, viewed as a circle homeomorphism. And if $f \in R(\Gamma)$ then f is homotopic to $1 = f_{\mathbf{0}}$, and $\mathbf{0} \in \delta(\mathbb{Z}^{\mathcal{V}})$.

LEMMA 6.1. *If \mathbf{n} is one of the columns of δ , then $r_{\mathbf{n}}(y) = \mathbf{n}_{\ell(y)} t(y)$ is a continuous real valued function, so $f_{\mathbf{n}}(y) = \exp(2\pi i r_{\mathbf{n}}(y)) \in R(\Gamma)$.*

To see this, suppose \mathbf{n} is column v of δ . Then the function $r_{\mathbf{n}}$ defines a “height function”, where the height of v is 0, and every other vertex has height 1. Going from v to one of its adjacent vertices, $r_{\mathbf{n}}$ increases linearly. Between any other two vertices, it is constant. It follows that $f_{\mathbf{n}} \in R(\Gamma)$.

COROLLARY 6.2. $Br(\Gamma) = \check{H}^1(\Gamma)$.

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