

An overview of recent developments on non-isomorphic matroids that have the same \mathcal{G} -invariant

Joseph E. Bonin

George Washington University

These slides and the related papers are available at

<http://blogs.gwu.edu/jbonin>

A probably-familiar matroid invariant

The **Tutte polynomial** $T(M; x, y)$ of a matroid M on E is

$$T(M; x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)}.$$

Specializations:

the chromatic polynomial of a graph,

the weight enumerator of a linear code,

the Jones polynomial of an alternating knot,

the number of regions in an arrangement of hyperplanes, ...

This is the generating function for the multiset of pairs $(r(A), |A|)$ over all $A \subseteq E$.

See: [Handbook of the Tutte Polynomial and Related Topics](#),
(Joanna A. Ellis-Monaghan, Iain Moffatt, eds.) CRC Press, 2022.

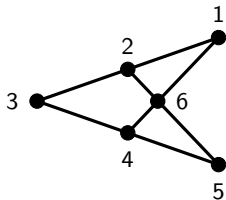
The \mathcal{G} -invariant, first step: rank sequences

Let M be a rank- r matroid on E with $|E| = n$.

The **rank sequence** of a permutation

$\pi = e_1, e_2, \dots, e_n$ of E is $\underline{r}(\pi) = r_1 r_2 \dots r_n$ where

$$r_i = r(\{e_1, e_2, \dots, e_i\}) - r(\{e_1, e_2, \dots, e_{i-1}\}).$$



	5, 2, 6, 4, 1, 3
$r(5) = 1$	1
$r(5, 2) = 2$	11
$r(5, 2, 6) = 2$	110
$r(5, 2, 6, 4) = 3$	1101
$r(5, 2, 6, 4, 1) = 3$	11010
$r(5, 2, 6, 4, 1, 3) = 3$	110100

So $r_i \in \{0, 1\}$. Also, $\{e_i : r_i = 1\}$ is a basis.

The \mathcal{G} -invariant

To a sequence \underline{r} of r 1's and $n - r$ 0's, associate a variable $[\underline{r}]$.
Let the set of these variables be a basis of a vector space, $\mathcal{G}(n, r)$,
over a field of characteristic 0.

Derksen, 2009; recast:

$$\text{The } \mathcal{G}\text{-invariant is } \mathcal{G}(M) = \sum_{\text{permutations } \pi} [\underline{r}(\pi)].$$

In effect, $\mathcal{G}(M)$ is the generating function for rank sequences $r(\pi)$,
using formal symbols $[\underline{r}]$ rather than powers of x .

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This contains the Tutte polynomial since what $T(M; x, y)$ records,
the number of sets of size i and rank j , for each i, j , is

the sum of the coefficients of the terms $[\underline{r}]$ in $\mathcal{G}(M)$ with $r_1 + \cdots + r_i = j$

 $i!(n - i)!$

Some easy observations

The **rank sequence** of $\pi = e_1, e_2, \dots, e_n$: $\underline{r}(\pi) = r_1 r_2 \dots r_n$
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If L is the set of loops of M , we get $\mathcal{G}(M \setminus L)$ by taking just the terms in $\mathcal{G}(M)$ with $|L|$ initial 0s, removing those initial 0s, and dividing each coefficient by $|L|!$.

We get $\mathcal{G}(M)$ from $\mathcal{G}(M \setminus L)$ by “shuffling” $|L|$ zeros into the terms of $\mathcal{G}(M \setminus L)$ and multiplying each coefficient by $|L|!$.

Some easy observations

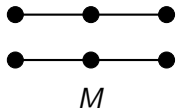
Similar remarks apply to coloops, using the longest final string of 1s in any rank sequence from M .

Indeed, to get $\mathcal{G}(M^*)$, start with $\mathcal{G}(M)$, switch 0 and 1, and reverse the order of each 0, 1-sequence

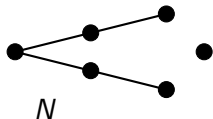
So, if we want, we can assume that M has no loops and no coloops.

Example

The \mathcal{G} -invariant is $\mathcal{G}(M) = \sum_{\text{permutations } \pi} [r(\pi)]$.



The smallest pair of matroids that have the same Tutte polynomial. Each has rank 3.



Two rank sequences:

111000 if $\{e_1, e_2, e_3\}$ is a basis;

there are $\left(\binom{6}{3} - 2\right) \cdot 3! \cdot 3! = 648$ such permutations;

110100 otherwise;

there are $2 \cdot 3! \cdot 3! = 72$ such permutations.

So $\mathcal{G}(M) = \mathcal{G}(N) = 648 [111000] + 72 [110100]$.

Preparing for another perspective — flags and compositions

A **flag** of a rank- r matroid M is a maximal chain of flats

$$\text{cl}(\emptyset) = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_{r-1} \subset X_r = E.$$

Its **composition** is the sequence a_0, a_1, \dots, a_r where $a_i = |X_i - X_{i-1}|$.

The composition is a way to record $|X_0|, |X_1|, \dots, |X_r|$.

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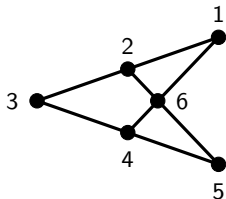
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A permutation of E yields a flag: take the closures of initial intervals.

E.g., 5, 2, 6, 4, 1, 3 gives the flag $\emptyset \subset \{5\} \subset \{2, 5, 6\} \subset E$.



$$\text{cl}(\{5\}) = \{5\}$$

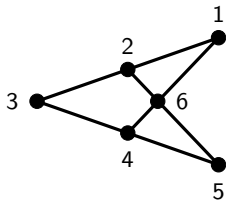
$$\text{cl}(\{5, 2\}) = \{2, 5, 6\} = \text{cl}(\{5, 2, 6\})$$

Each of $\{5, 2, 6, 4\}$, $\{5, 2, 6, 4, 1\}$, and $\{5, 2, 6, 4, 1, 3\}$ has closure E .

Going from flags to permutations and rank sequences

With basic counting, we can deduce how many permutations yield a given flag, and how many of them have a given rank sequence.

We get the flag $\emptyset \subset \{5\} \subset \{2, 5, 6\} \subset E$ from the permutations that:



start with 5;

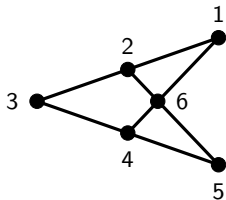
follow that by either 2 or 6;

and then have any permutation of the remaining elements.

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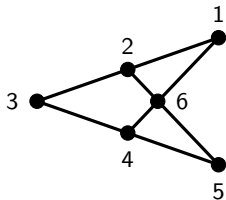
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Those that start 5, 2, 6 or 5, 6, 2 have the rank sequence 110100;
the others have rank sequence 111000.

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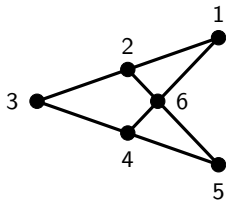
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These deductions use only the sizes of the flats, so just the composition of the flag.

Another basis of $\mathcal{G}(n, r)$

If a flag (X_0, X_1, \dots, X_r) has composition a_0, a_1, \dots, a_r , then

$$\gamma(a_0, a_1, \dots, a_r) = \sum_{\pi \text{ giving } (X_0, X_1, \dots, X_r)} [r(\pi)]$$

depends only on a_0, a_1, \dots, a_r , not M and not (X_0, X_1, \dots, X_r) .

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Theorem

(Bonin and Kung, 2018)

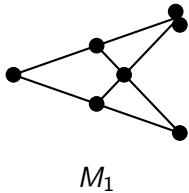
Let $\nu(M; a_0, a_1, \dots, a_r)$ be the number of flags in M with composition a_0, a_1, \dots, a_r . We have

$$\mathcal{G}(M) = \sum_{(a_0, a_1, \dots, a_r)} \nu(M; a_0, a_1, \dots, a_r) \gamma(a_0, a_1, \dots, a_r).$$

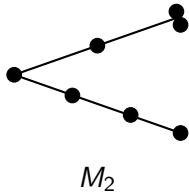
This re-interpretation of $\mathcal{G}(M)$ via flags and compositions provides essential insight into $\mathcal{G}(M)$.

The \mathcal{G} -invariant is stronger than the Tutte polynomial

Two matroids with the same Tutte polynomial but different \mathcal{G} -invariants.



	M_1	M_2
$\nu(M_i; 0, 1, 1, 5)$	4	6
$\nu(M_i; 0, 1, 2, 4)$	7	3
$\nu(M_i; 0, 1, 3, 3)$	4	6
$\nu(M_i; 0, 2, 1, 4)$	1	3
$\nu(M_i; 0, 2, 2, 3)$	2	1



The Tutte polynomial does not detect the fact that the number of rank-2 flats with two elements is different (2 versus 3; the first row in the table).

More broadly, the Tutte polynomial detects the number of flats of a given rank that have the largest size, but not the numbers of smaller flats of that rank.

The \mathcal{G} -invariant is far stronger than the Tutte polynomial

The Tutte polynomial $T(M; x, y)$ is a generating function for the multiset of pairs $(|A|, r(A))$ for $A \subseteq E$.

From $\mathcal{G}(M)$, one can find, for each triple (m, k, c) of integers, the number of sets A with $|A| = m$ and $r(A) = k$ for which the restriction $M|_A$ has c coloops (and more!).

From $\mathcal{G}(M)$, but not $T(M; x, y)$, one can deduce the numbers of circuits and cocircuits of M of each size (including whether there are spanning circuits).

From $\mathcal{G}(M)$, but not $T(M; x, y)$, one can deduce the the number of chains of flats with specified sizes and ranks.

The Tutte polynomial is universal among invariants that satisfy deletion-contraction rules.

Derksen and Fink showed that the \mathcal{G} -invariant is a universal valiative invariant for subdivisions of matroid base polytopes.

A central question

A key question for any invariant: how strong is it?

In this setting: what can account for non-isomorphic matroids that have the same \mathcal{G} -invariant?

How can we construct such matroids?

Cyclic flats

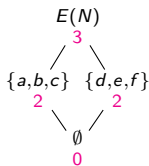
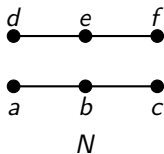
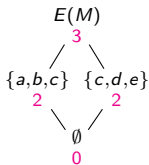
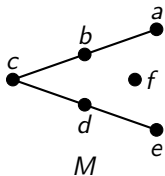
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We are most interested in cyclic **flats**.

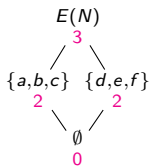
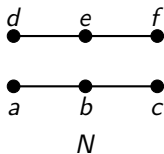
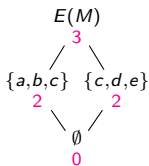
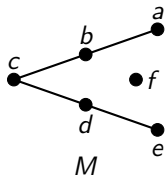


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The set $\mathcal{Z}(M)$ of **cyclic flats** of M , ordered by inclusion, is a lattice.

join: $X \vee Y = \text{cl}(X \cup Y)$.

meet: $X \wedge Y$ is the union of all circuits in $M|X \cap Y$.

The cyclic flats and their rank determine a a matroid

A matroid M is determined by E and the pairs $(X, r(X))$ with $X \in \mathcal{Z}(M)$. (Brylawski, 1975)

One way to see this: show that, for all $X \subseteq E$,
 $r(X) = \min\{r(A) + |X - A| : A \in \mathcal{Z}(M)\}$.

There are axioms for matroids using cyclic flats and their ranks, and that justifies some of the examples we will see later.

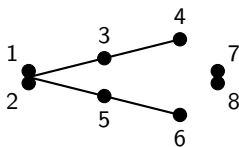
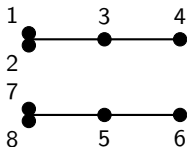
Flags and chains of cyclic flats

Now assume that M has no loops and no coloops, so $\emptyset, E \in \mathcal{Z}(M)$.

Let $(X_0 = \emptyset, X_1, \dots, X_r = E)$ be a flag of M .

From each flat X_i , remove the coloops of $M|X_i$; this gives a chain of cyclic flats.

The (possibly empty) chain we get by removing \emptyset and E from this chain is the **reduced cyclic chain** of the flag.



$$(\emptyset, \{1, 2\}, \{1, 2, 3, 4\}, E) \mapsto (\{1, 2\}, \{1, 2, 3, 4\})$$

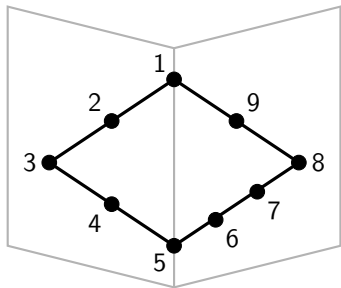
$$(\emptyset, \{4\}, \{1, 2, 3, 4\}, E) \mapsto (\{1, 2, 3, 4\})$$

$$(\emptyset, \{5\}, \{3, 5\}, E) \mapsto ()$$

$$(\emptyset, \{7, 8\}, \{3, 7, 8\}, E) \mapsto (\{7, 8\})$$

Flags and chains of cyclic flats: going the other way

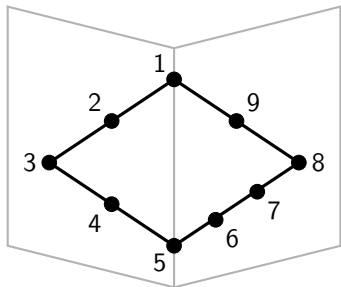
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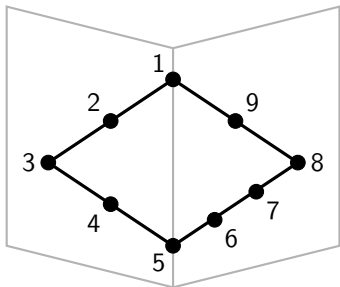
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where $i \in \{1, 2, 3\}$ and $j \in \{6, 7\}$.

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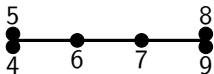
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Why? $\{1\}$, $\{2\}$, and $\{3\}$ are the independent hyperplanes of $M \setminus \{1, 2, 3\}$, and $\{6\}$ and $\{7\}$ are the independent hyperplanes of $M / \{1, 2, 3\}$.



Flags and chains of cyclic flats: going the other way

Let $\mathcal{Z}^\circ(M) = \mathcal{Z}(M) - \{\emptyset, E\}$.

Lemma

(Bonin and Kung, 2018)

Let $C = \{F_1 \subset F_2 \subset \dots \subset F_t\}$ be a chain in $\mathcal{Z}^\circ(M)$. Set $F_0 = \emptyset$ and $F_{t+1} = E$. Let \mathcal{L} be the set of all lists of $r(M)$ entries, all different, obtained this way:

- ▶ for $j \in [t + 1]$, pick an independent hyperplane H_j of $M|_{F_j/F_{j-1}}$,
- ▶ consider lists in which the entries are the sets F_1, F_2, \dots, F_{t+1} along with the singleton subsets of H_1, H_2, \dots, H_{t+1} ,
- ▶ such a list is in \mathcal{L} **iff**, for all $j \in [t]$, F_{j+1} occurs after (a) all smaller F_i and (b) all singleton subsets of H_{j+1} .

Map $L \in \mathcal{L}$ to the flag $\phi(L)$, the i th entry of which is the union of the first i sets in L , for $0 \leq i \leq r(M)$. The map ϕ is a bijection from \mathcal{L} onto the set of flags for which C is the reduced cyclic flag.

Flags and chains of cyclic flats: going the other way

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Let $\iota(M)$ be the number of independent hyperplanes of M .

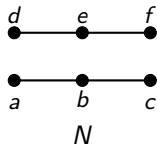
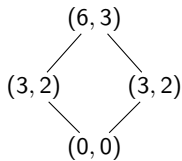
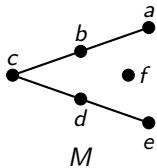
Corollary

The multiset of compositions we get from flags whose reduced cyclic chain is C is determined by the sizes and ranks of F_1, F_2, \dots, F_t , and E , along with $\iota(M|F_1)$, $\iota(M|F_j/F_{j-1})$ for $2 \leq j \leq t$, and $\iota(M/F_t)$.

The configuration of a matroid

The multiset of compositions we get from flags whose reduced cyclic chain is $\{F_1 \subset F_2 \subset \dots \subset F_t\}$ is determined by the sizes and ranks of F_1, F_2, \dots, F_t , and E , along with $\iota(M|F_1)$, $\iota(M|F_j/F_{j-1})$ for $2 \leq j \leq t$, and $\iota(M/F_t)$.

The **configuration** of M is the triple (L, s, ρ) , where L is a lattice, $s : L \rightarrow \mathbb{Z}$, $\rho : L \rightarrow \mathbb{Z}$, and there is an isomorphism $\phi : L \rightarrow \mathcal{Z}(M)$ with $s(x) = |\phi(x)|$ and $\rho(x) = r(\phi(x))$ for all $x \in L$. (Eberhardt, 2014)



The configuration of a matroid

The multiset of compositions we get from flags whose reduced cyclic chain is $\{F_1 \subset F_2 \subset \dots \subset F_t\}$ is determined by the sizes and ranks of F_1, F_2, \dots, F_t , and E , along with $\iota(M|F_1)$, $\iota(M|F_j/F_{j-1})$ for $2 \leq j \leq t$, and $\iota(M/F_t)$.

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From the configuration of a matroid with no coloops, we can get its Tutte polynomial.

Theorem (Bonin and Kung, 2018)

From the configuration of a matroid with no coloops, we can get its \mathcal{G} -invariant.

The key is to show how to compute $\iota(M)$ from the configuration.

The configuration and minors

The multiset of compositions we get from flags whose reduced cyclic chain is $\{F_1 \subset F_2 \subset \cdots \subset F_t\}$ is determined by the sizes and ranks of F_1, F_2, \dots, F_t , and E , along with $\iota(M|F_1)$, $\iota(M|F_j/F_{j-1})$ for $2 \leq j \leq t$, and $\iota(M/F_t)$.

If $F \in \mathcal{Z}(M)$, then $\mathcal{Z}(M|F)$ is the interval $[\text{cl}(\emptyset), F]$ in $\mathcal{Z}(M)$, and $\mathcal{Z}(M/F)$ is $\{A - F : A \in \mathcal{Z}(M) \text{ and } F \subseteq A\}$, which is isomorphic to the interval $[F, E]$ in $\mathcal{Z}(M)$.

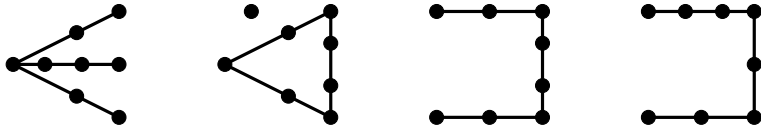
So, from the configuration of M , we get the configuration of any minor $M|F/G$ with $F, G \in \mathcal{Z}(M)$ and $G \subset F$.

Huge collections of \mathcal{G} -equivalent matroids

Huge collections of non-isomorphic matroids can share the same configuration, and so have the same \mathcal{G} -invariant and Tutte polynomial.

A **paving matroid** is a matroid M in which each flat of rank $r(M) - 2$ or less is independent.

Two paving matroids of rank r on n elements where, for each i , both have the same number of hyperplanes of size i have the same configuration.



It is conjectured that, asymptotically, almost all matroids are paving.

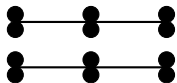
Going further

How can we get matroids with **different configurations** but the same \mathcal{G} -invariant?

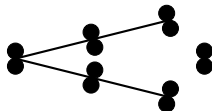
Going further

How can we get matroids with **different configurations** but the same \mathcal{G} -invariant?

A simple construction: take non-isomorphic matroids with the same \mathcal{G} -invariant (perhaps with the same configuration) and replace each element by m parallel elements, for a fixed $m > 1$.



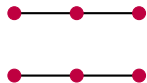
An example with $m = 2$.



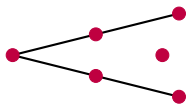
Why this works:

- ▶ this takes each flag of each matroid and multiplies all terms in the corresponding composition by m , and
- ▶ the lattices of cyclic flats of the new matroids are isomorphic to the lattice of flats (not cyclic flats!) of the original matroids.

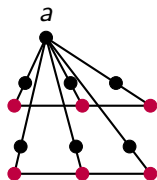
Free cones



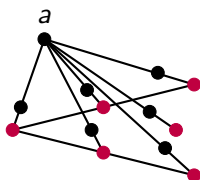
M



N



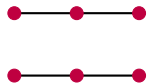
$Q_1(M)$



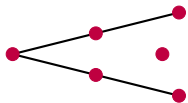
$Q_1(N)$

The **free m -cone** of a matroid M is formed from the direct sum of M and $U_{1,1}$ on a (the **tip**) by adding m point freely to each line $\{a, e\}$ with $e \in E(M)$ (via principal extension).

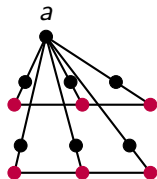
Free cones



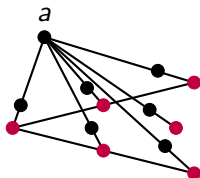
M



N



$Q_1(M)$



$Q_1(N)$

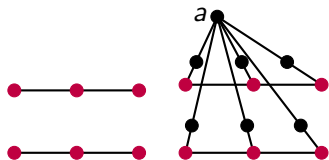
The **free m -cone** of a matroid M is formed from the direct sum of M and $U_{1,1}$ on a (the **tip**) by adding m point freely to each line $\{a, e\}$ with $e \in E(M)$ (via principal extension).

Theorem

(Bonin and Long, 2022)

Let M and N be nonisomorphic loopless matroids with $\mathcal{G}(M) = \mathcal{G}(N)$. For all $m \geq 1$, $Q_m(M)$ and $Q_m(N)$ have the same \mathcal{G} -invariant and different configurations.

Free cones



If M and N are nonisomorphic loopless matroids with $\mathcal{G}(M) = \mathcal{G}(N)$, then $Q_m(M)$ and $Q_m(N)$ have the same \mathcal{G} -invariant and different configurations.

The bird's-eye view of the proof:

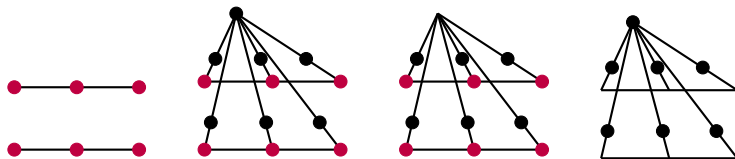
Why the \mathcal{G} -invariants are the same: we can write the flags and compositions of a cone using the flags and compositions of the original matroid (a combination of geometry and counting).

Why the configurations are different: flats F with $a \in F$ and $|F| > 1$ are cyclic; in the lattice of cyclic flats of the cone, in effect these flats give a copy of the lattice of flats of the original matroid. That sublattice can be picked out via lattice properties.

For both parts, working with the lattice of cyclic flats of the free m -cone is very useful.

Free cones

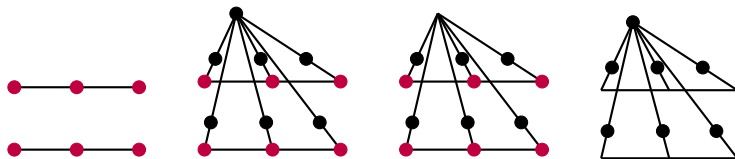
Let M and N be nonisomorphic loopless matroids with $\mathcal{G}(M) = \mathcal{G}(N)$. For all $m \geq 1$, $Q_m(M)$ and $Q_m(N)$ have the same \mathcal{G} -invariant and different configurations.



With the basic result, we adapt the ideas to get counterparts for the tipless (if $m > 1$), baseless (if $m > 1$), and tipless and baseless (if $m > 2$) variations on the construction.

Free cones

Let M and N be nonisomorphic loopless matroids with $\mathcal{G}(M) = \mathcal{G}(N)$. For all $m \geq 1$, $Q_m(M)$ and $Q_m(N)$ have the same \mathcal{G} -invariant and different configurations.



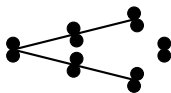
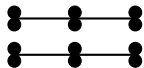
With the basic result, we adapt the ideas to get counterparts for the tipless (if $m > 1$), baseless (if $m > 1$), and tipless and baseless (if $m > 2$) variations on the construction.

This construction assigns fairly restricted size and rank data to the lattice of cyclic flat of $Q_m(M)$ to get the configuration.

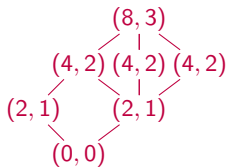
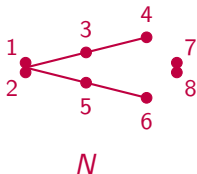
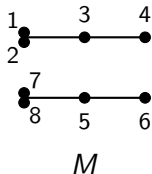
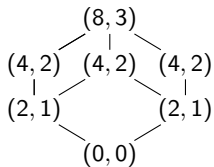
Can we find **large** families of lattices where **mild restrictions** on the size and rank data give different configurations but the same \mathcal{G} -invariant?

Revisit an earlier construction

We turned our first example of matroids with the same \mathcal{G} -invariant into an example with different configurations by replacing each element by m parallel elements.

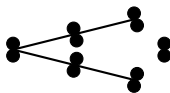
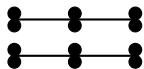


What if we add parallel elements more selectively?

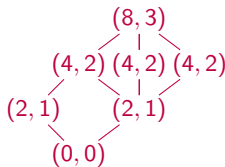
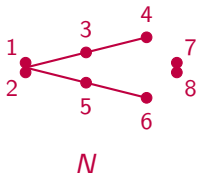
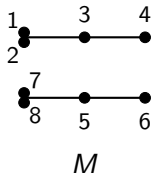
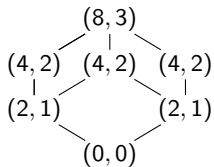


Revisit an earlier construction

We turned our first example of matroids with the same \mathcal{G} -invariant into an example with different configurations by replacing each element by m parallel elements.



What if we add parallel elements more selectively?



The lattices are different, but the chains of cyclic flats are the same.

Recall:

The multiset of compositions we get from flags whose reduced cyclic chain is $\{F_1 \subset F_2 \subset \dots \subset F_t\}$ is determined by the sizes and ranks of F_1, F_2, \dots, F_t , and E , along with $\iota(M|F_1)$, $\iota(M|F_j/F_{j-1})$ for $2 \leq j \leq t$, and $\iota(M/F_t)$.

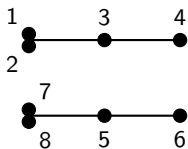
Sorting flags by their reduced cyclic chains

For a reduced cyclic chain C , let $\text{flag}(C)$ be the set of all flags of M whose reduced cyclic chain is C .

For a set T of reduced cyclic chains, let $\text{flag}(T)$ be $\bigcup_{C \in T} \text{flag}(C)$.

For a set S of flags, let $\text{comp}(S)$ be the multiset of compositions of the flags in S .

So, $\text{comp}(\text{flag}(T))$ is the multiset of all compositions of all flags whose reduced cyclic chain is in the set T of reduced cyclic chains.



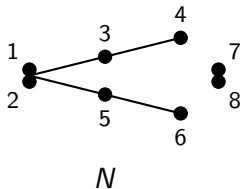
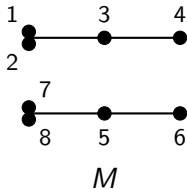
$\text{flag}(\{7, 8\})$ contains

$(\emptyset, \{3\}, \{3, 7, 8\}, E)$ $(\emptyset, \{7, 8\}, \{3, 7, 8\}, E)$

$(\emptyset, \{4\}, \{4, 7, 8\}, E)$ $(\emptyset, \{7, 8\}, \{4, 7, 8\}, E)$

$$\text{comp}(\text{flag}(\{7, 8\})) = \{(0, 1, 2, 5)^2, (0, 2, 1, 5)^2\}$$

An example



(Below, we omit \emptyset and E from flags for brevity.)

$C = (\{1, 2\})$: $\text{flag}_M(C)$ contains $(\{1, 2\}, \{1, 2, i\})$ and $(\{i\}, \{1, 2, i\})$ for $i \in \{5, 6\}$;
 $\text{comp}(\text{flag}_M(C)) = \{(0, 2, 1, 5)^2, (0, 1, 2, 5)^2\}$.

For N , $\text{flag}_N(C) = \emptyset$.

$C' = (\{7, 8\})$: $\text{flag}_M(C')$ contains $(\{7, 8\}, \{7, 8, i\})$ and $(\{i\}, \{7, 8, i\})$ for $i \in \{3, 4\}$;
 $\text{comp}(\text{flag}_M(C')) = \{(0, 2, 1, 5)^2, (0, 1, 2, 5)^2\}$.

For N , $\text{flag}_N(C')$ contains $(\{7, 8\}, \{7, 8, i\})$ and $(\{i\}, \{7, 8, i\})$ for $i \in \{3, 4, 5, 6\}$;
 $\text{comp}(\text{flag}_N(C')) = \{(0, 2, 1, 5)^4, (0, 1, 2, 5)^4\}$.

$\{\text{comp}(\text{flag}_M(C)), \text{comp}(\text{flag}_M(C'))\} \neq \{\text{comp}(\text{flag}_N(C)), \text{comp}(\text{flag}_N(C'))\}$,
 but $\text{comp}(\text{flag}_M(\{C, C'\})) = \text{comp}(\text{flag}_N(\{C, C'\}))$

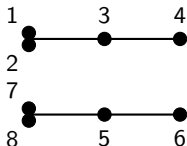
An example

Neither has flags with reduced cyclic chain $(\{1, 2, 7, 8\})$.

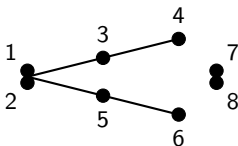
Each has eight flags with the empty reduced cyclic chain; each has composition $(0, 1, 1, 6)$.

Each of M and N has four 2-element reduced cyclic chains; each arises from exactly one flag; that flag has composition $(0, 2, 2, 4)$.

Each of the reduced cyclic chains $(\{1, 2, 3, 4\})$ and $(\{5, 6, 7, 8\})$ in M , and $(\{1, 2, 3, 4\})$ and $(\{1, 2, 5, 6\})$ in N , arise from two flags; each such flag has composition $(0, 1, 3, 4)$.



M



N

So $\mathcal{G}(M) = \mathcal{G}(N)$.

Theorem

(Bonin, 2022)

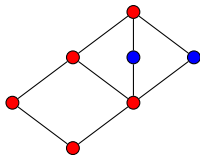
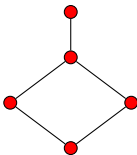
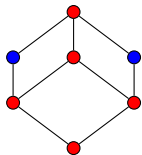
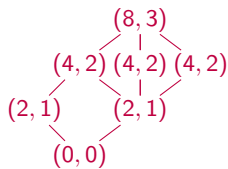
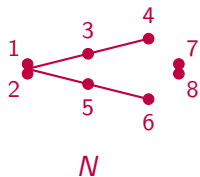
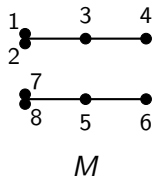
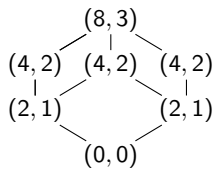
Let M and N have neither loops nor coloops. Let $\{P_1, P_2, \dots, P_d\}$ and $\{Q_1, Q_2, \dots, Q_d\}$ be a partition of the set of reduced cyclic chains of M and of N , resp. If $\text{comp}(\text{flag}_M(P_i)) = \text{comp}(\text{flag}_N(Q_i))$ for each $i \in [d]$, then $\mathcal{G}(M) = \mathcal{G}(N)$.

So we want results that let us deduce that partitions have the key property $\text{comp}(\text{flag}_M(P_i)) = \text{comp}(\text{flag}_N(Q_i))$ for all $i \in [d]$.

The paper has a number of technical lemmas of that type.

We skip them here so we can sample the applications.

Motivating the next construction: gluing lattices together



Applications

Let a_1, a_2, \dots, a_m be distinct elements of a finite lattice L where

- ▶ there is a $b \in L$ with $a_i \wedge a_j = b$ for all $i, j \in [m]$ with $i \neq j$,
- ▶ for all distinct $i, j \in [m]$, there is a lattice isomorphism $\tau_{i,j} : [\hat{0}, a_i] \rightarrow [\hat{0}, a_j]$ with $\tau_{i,j}(y) = y$ for all $y \in [\hat{0}, b]$.

Having $\tau_{1,m}, \tau_{2,m}, \dots, \tau_{m-1,m}$ gives the rest via inverses and compositions.

Let L_1, L_2, \dots, L_n be finite lattices that are disjoint from each other and from L .

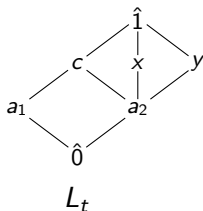
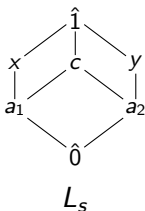
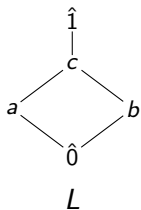
Fix functions $s : [n] \rightarrow [m]$ and $t : [n] \rightarrow [m]$.

Form a lattice L_s as follows:

- ▶ for $i \in [n]$, to get L'_i , in L_i replace $\hat{1}_{L_i}$ by $\hat{1}_L$ and $\hat{0}_{L_i}$ by $a_{s(i)}$,
- ▶ let L_s be the transitive closure of $L \cup L'_1 \cup L'_2 \cup \dots \cup L'_n$ (viewed as a relation).

Define L_t similarly.

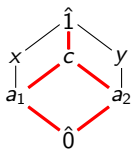
Applications



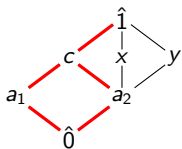
In terms of Hasse diagrams, L_s is obtained by, for each $i \in [n]$, inserting L_i into the interval $[a_{s(i)}, \hat{1}_L]$ of L , where $\hat{0}_{L_i}$ is identified with $a_{s(i)}$, and $\hat{1}_{L_i}$ is identified with $\hat{1}_L$.

So, L_s and L_t have the same elements.

Applications



As Hasse diagrams, L_S is obtained by, for each $i \in [n]$, inserting L_i into the interval $[a_{S(i)}, \hat{1}_L]$ of L , where $\hat{0}_{L_i}$ is identified with $a_{S(i)}$, and $\hat{1}_{L_i}$ is identified with $\hat{1}_L$.



Theorem

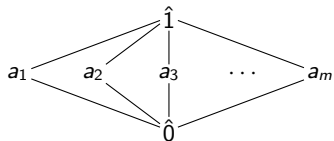
(Bonin, 2022)

Let L_S and L_T be as defined above. Let M_S and M_T be matroids, with rank functions r_S and r_T , respectively, neither having loops nor coloops, for which, for some lattice isomorphisms $\phi_S : L_S \rightarrow \mathcal{Z}(M_S)$ and $\phi_T : L_T \rightarrow \mathcal{Z}(M_T)$,

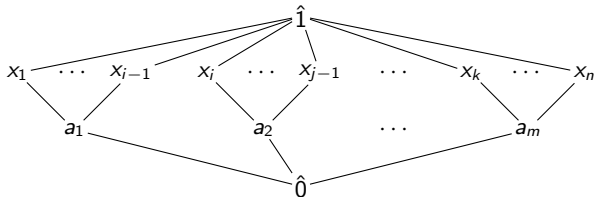
- ▶ $|\phi_S(y)| = |\phi_T(y)|$ and $r_S(\phi_S(y)) = r_T(\phi_T(y))$ for all y in L_S , and
- ▶ if $y \in [\hat{0}, a_m]$, then $|\phi_S(y)| = |\phi_S(\tau_{m,i}(y))|$ and $r_S(\phi_S(y)) = r_S(\phi_S(\tau_{m,i}(y)))$ for all $i \in [m-1]$.

Then $\mathcal{G}(M_S) = \mathcal{G}(M_T)$.

An illustration

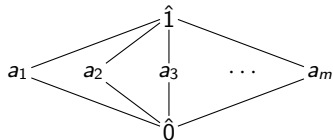


If we attach n copies of a 3-element chain into the lattice of flats of an m -point line, we get lattices of the form

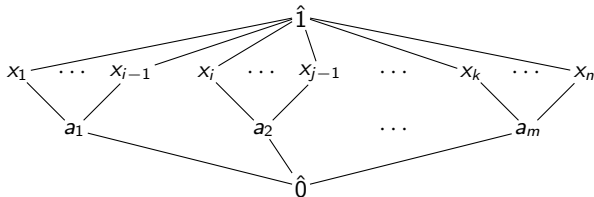


(some intervals $(a_h, \hat{1})$ may be empty).

An illustration



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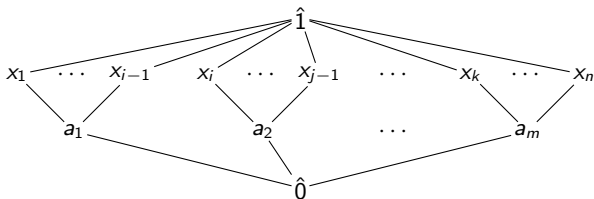


(some intervals $(a_h, \hat{1})$ may be empty).

In two such lattices, if all a_i s are assigned the same rank/size data, and each x_j is assigned the same rank/size data in each lattice (perhaps different from other x_k s), then the \mathcal{G} -invariants are the same.

The number of nonisomorphic lattices of this type is the number of integer partitions of n with at most m parts.

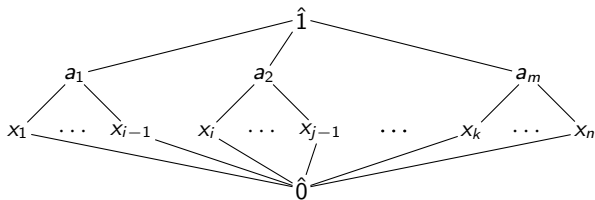
An illustration



E.g., in rank-4 with $m = n$ and $|E| = n(n + 9)/2$, we can let each a_i correspond to a 3-point line, and each x_i correspond to a cyclic plane with a single dependent line (the 3-point line), where all planes have different sizes.

The number of such configurations with non-isomorphic lattices is the n th Bell number, the number of set partitions of $\{1, 2, \dots, n\}$.

An illustration



Similarly, the order dual produces a plethora of \mathcal{G} -equivalent, configuration-distinct matroids since the restrictions to the flats corresponding to the a_i 's can be paving matroids.

Wrap up

The paper has a result of a similar flavor using the lattices of flats of paving matroids that have the same configuration as the lattices for different configurations that yield matroids with the same \mathcal{G} -invariant.

Wrap up

The paper has a result of a similar flavor using the lattices of flats of paving matroids that have the same configuration as the lattices for different configurations that yield matroids with the same \mathcal{G} -invariant.

Open problem

Find good upper bounds on the number of distinct \mathcal{G} -invariants among rank- r matroids on n elements.

Thank you for listening.

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