

MATROIDS WITH DIFFERENT CONFIGURATIONS AND THE SAME \mathcal{G} -INVARIANT

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ABSTRACT. From the configuration of a matroid (which records the size and rank of the cyclic flats and the containments among them, but not the sets), one can compute several much-studied matroid invariants, including the Tutte polynomial and a newer, stronger invariant, the \mathcal{G} -invariant. To gauge how much additional information the configuration contains compared to these invariants, it is of interest to have methods for constructing matroids with different configurations but the same \mathcal{G} -invariant. We offer several such constructions along with tools for developing more.

1. INTRODUCTION

The configuration of a matroid, which Eberhardt [9] introduced, is obtained from its lattice of cyclic flats (that is, flats that are unions of circuits) by recording the abstract lattice structure along with just the size and rank of each cyclic flat, not the set. Eberhardt proved that from the configuration of a matroid M , one can compute its *Tutte polynomial*,

$$T(M; x, y) = \sum_{A \subseteq E(M)} (x-1)^{r(M)-r(A)} (y-1)^{|A|-r(A)}.$$

The data recorded in the Tutte polynomial is the multiset of size-rank pairs, $(|A|, r(A))$, over all $A \subseteq E(M)$. The Tutte polynomial is one of the most extensively studied invariants of a matroid (see, e.g., [6, 10]).

Strengthening Eberhardt's result, Bonin and Kung [2, Theorem 7.3] showed that from the configuration of a matroid M , one can compute its \mathcal{G} -invariant, $\mathcal{G}(M)$. Derksen [7] introduced the \mathcal{G} -invariant and showed that the Tutte polynomial can be computed from it. The perspective on the \mathcal{G} -invariant that we use is the reformulation introduced in [2]: $\mathcal{G}(M)$ records the multiset of sequences of sizes of the sets in flags (maximal chains of flats) of M . (Section 2 gives a more precise formulation.) This information about flags just begins to suggest the wealth of information that $\mathcal{G}(M)$ captures beyond what the Tutte polynomial contains; other examples (from [2, Section 5]) include the number of saturated chains of flats with specified sizes and ranks, the number of circuits and cocircuits of each size (in particular, the number of spanning circuits), and, for each triple (m, k, c) of integers, the number of flats F with $|F| = m$ and $r(F) = k$ for which the restriction $M|_F$ has c coloops. Beyond the multiset of size-rank pairs $(|A|, r(A))$ noted above as equivalent to the Tutte polynomial, as [3, Theorem 5.3] shows, from $\mathcal{G}(M)$, one can find, for each triple (m, k, c) of integers, the number of sets A with $|A| = m$ and $r(A) = k$ for which the restriction $M|_A$ has c coloops. Akin to the universality property of the Tutte polynomial among matroid invariants that satisfy deletion-contraction rules, Derksen and Fink [8] showed that the \mathcal{G} -invariant is a universal valuative invariant for subdivisions of matroid base polytopes.

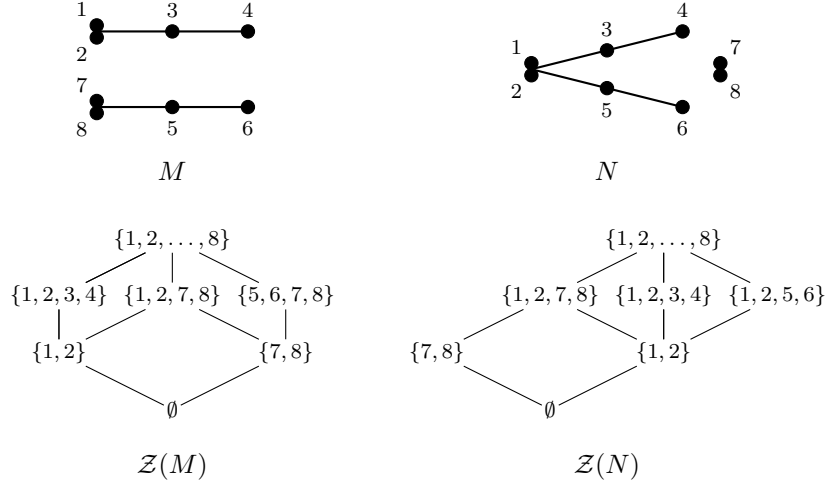


FIGURE 1. Two rank-3 matroids, M and N , and their lattices of cyclic flats.

Reflecting on the proof of [2, Theorem 7.3] reveals that the chains of cyclic flats in the configuration play the key role; the lattice structure is secondary. So if one can construct pairs of non-isomorphic lattices for which one can relate their chains via bijections, one might be able to produce pairs of matroids with different configurations and the same \mathcal{G} -invariant. That is the idea that we develop in this paper and use to shed light on how much stronger the configuration is compared to the \mathcal{G} -invariant. Since the Tutte polynomial can be computed from the \mathcal{G} -invariant, our results also contribute to the theory of Tutte-equivalent matroids (see [5]).

In Section 2, we review the relevant background and previously known examples of matroids with different configurations but the same \mathcal{G} -invariant. The core of the paper is Section 3, where we develop the tools that we apply in Sections 4 and 5 to give the constructions of interest. These tools and the strategy in the proofs of Theorems 4.1 and 5.1 are likely to be useful to obtain more such constructions.

2. NOTATION, BACKGROUND, AND PRIOR RESULTS

Our notation and terminology for matroid theory follow Oxley [11]. Let $[n]$ denote the set $\{1, 2, \dots, n\}$. Let $\hat{0}$ denote the least element of a lattice L , and let $\hat{1}$ denote its greatest element. If we need to clarify in which lattice an interval is formed, we use a subscript, as in $[a, b]_L$. Likewise, we may use $\hat{0}_L$ and $\hat{1}_L$. For a lattice L , we let L° denote L with $\hat{0}$ and $\hat{1}$ removed, so L° is the open interval $(\hat{0}, \hat{1})$.

Given a matroid M , a subset A of $E(M)$ is *cyclic* if A is a (possibly empty) union of circuits, or, equivalently, the restriction $M|_A$ has no coloops. It follows that if A and B are cyclic flats, then so are $\text{cl}(A \cup B)$ and the flat obtained from $A \cap B$ by removing the coloops of $M|_{A \cap B}$; these are the join and meet of A and B in the lattice $\mathcal{Z}(M)$ of cyclic flats of M . Figure 1 shows two matroids and their lattices of cyclic flats. We will use the following elementary result about cyclic flats.

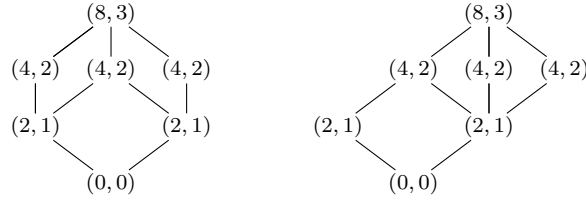


FIGURE 2. The configurations of the matroids M and N in Figure 1. The pairs shown are $(s(x), \rho(x))$, the size and rank of the corresponding cyclic flat.

Lemma 2.1. *Let M be a matroid with neither loops nor coloops. If X is any cyclic flat of M , then $\mathcal{Z}(M|X)$ is the interval $[\emptyset, X]$ in $\mathcal{Z}(M)$, and*

$$\mathcal{Z}(M/X) = \{F - X : F \in \mathcal{Z}(M) \text{ and } X \subseteq F\},$$

so the lattice $\mathcal{Z}(M/X)$ is isomorphic to the interval $[X, E(M)]$ in $\mathcal{Z}(M)$.

We now make the notion of the configuration, introduced informally above, precise. The *configuration* of a matroid M with no coloops is the triple (L, s, ρ) , where L is a lattice and $s : L \rightarrow \mathbb{Z}$ and $\rho : L \rightarrow \mathbb{Z}$ are functions such that there is an isomorphism $\phi : L \rightarrow \mathcal{Z}(M)$ for which $s(x) = |\phi(x)|$ and $\rho(x) = r(\phi(x))$ for all $x \in L$. (Many triples can satisfy these properties, but they all contain the same data, so we view them as the same.) The configurations of the matroids in Figure 1 are shown in Figure 2. Many non-isomorphic matroids can have the same configuration. For instance, consider paving matroids, that is, matroids where all flats that are properly contained in hyperplanes are independent; any two paving matroids of the same rank, on the same number of elements, and with the same number of dependent hyperplanes of each size have the same configuration, and many non-isomorphic paving matroids can share these parameters.

The configuration of M gives the configurations of certain minors of M , as the next lemma states.

Lemma 2.2. *Let M be a matroid with neither loops nor coloops. If $X \in \mathcal{Z}(M)$, then the size and rank of each set in $\mathcal{Z}(M|X)$ and $\mathcal{Z}(M/X)$ can be found from the configuration of M . So, from the configuration of M , we get the configuration of each minor $M|X/Y$ with $X, Y \in \mathcal{Z}(M)$ and $Y \subsetneq X$.*

The following result from [4] explains why we do not impose any conditions on the lattices that we consider other than that they are finite.

Lemma 2.3. *For each finite lattice L , there is a matroid M for which $\mathcal{Z}(M)$ is isomorphic to L .*

Given a matroid M on E with rank function r and a permutation π , say e_1, e_2, \dots, e_n , of E , the *rank sequence* $\underline{r}(\pi) = r_1 r_2 \dots r_n$ is given by

$$r_i = r(\{e_1, e_2, \dots, e_i\}) - r(\{e_1, e_2, \dots, e_{i-1}\}).$$

Thus, $\{e_i : r_i = 1\}$ is a basis of M . Let $k = r(M)$. Each rank sequence $\underline{r}(\pi)$ is an (n, k) -*sequence*, that is, a sequence of k ones and $n - k$ zeroes. For each (n, k) -sequence \underline{r} , let $[\underline{r}]$ be a formal symbol, and let $\mathcal{G}(n, k)$ be the vector space over a field of characteristic zero consisting of all formal linear combinations of such symbols. The \mathcal{G} -*invariant* of M

is defined by

$$\mathcal{G}(M) = \sum_{\text{permutations } \pi \text{ of } E} [\underline{r}(\pi)].$$

The matroids M and N in Figure 1 have the same \mathcal{G} -invariant, which is

$$384[10101000] + 2496[10110000] + 1344[11001000] + 7296[11010000] + 28800[11100000]$$

Among the 384 permutations that yield the rank sequence $[10101000]$ are, for instance, $1, 2, 7, 8, 3, 4, 5, 6$, for M and N , and $7, 8, 5, 6, 4, 3, 2, 1$, for M but not N .

In this paper we use a reformulation of $\mathcal{G}(M)$ developed in [2]. An (n, k) -composition is a sequence $\mathbf{a} = (a_0, a_1, \dots, a_k)$ of integers where $a_0 \geq 0$ and $a_i > 0$ for $i \in [k]$ such that $a_0 + a_1 + \dots + a_k = n$. Let M be a matroid as above. A *flag* of M is a sequence $\mathbf{X} = (X_0, X_1, \dots, X_k)$ where X_i is a rank- i flat of M and $X_i \subset X_{i+1}$ for $i < k$. The *composition* of a flag \mathbf{X} is \mathbf{a} where $a_0 = |X_0|$ and $a_i = |X_i - X_{i-1}|$ for $i \in [k]$. Thus, \mathbf{a} is an (n, k) -composition. Let $\nu(M; \mathbf{a})$ be the number of flags of M with composition \mathbf{a} . The *catenary data* of M is the $\binom{n}{k}$ -dimensional vector $(\nu(M; \mathbf{a}))$ that is indexed by (n, k) -compositions.

For example, for the matroid M in Figure 1, we have

$$\begin{aligned} \nu(M; (0, 1, 1, 6)) &= 8, & \nu(M; (0, 1, 2, 5)) &= 4, & \nu(M; (0, 1, 3, 4)) &= 4, \\ \nu(M; (0, 2, 1, 5)) &= 4, & \text{and } \nu(M; (0, 2, 2, 4)) &= 4, \end{aligned}$$

and the same for N . In M , each of the flats $\{1, 2\}$ and $\{7, 8\}$ is in two flags that have composition $(0, 2, 2, 4)$, while in N , the flat $\{1, 2\}$ is in three such flags and $\{7, 8\}$ is in only one.

A special basis of $\mathcal{G}(n, k)$, called the γ -basis, is defined in [2]; its vectors $\gamma(\mathbf{a})$ are indexed by the (n, k) -compositions \mathbf{a} . By the next result [2, Theorem 3.3], the catenary data of M is equivalent to $\mathcal{G}(M)$.

Theorem 2.4. *The \mathcal{G} -invariant of M is determined by its catenary data and conversely. In particular,*

$$\mathcal{G}(M) = \sum_{(n, k)\text{-compositions } \mathbf{a}} \nu(M; \mathbf{a}) \gamma(\mathbf{a}).$$

By [2, Theorem 7.3], if M has no coloops, then $\mathcal{G}(M)$ can be computed from the configuration of M . If M has coloops, then $\mathcal{G}(M)$ can be computed from $\mathcal{G}(M \setminus X)$ and $|X|$ where X is the set of coloops. Likewise, if M has loops, then $\mathcal{G}(M)$ can be computed from $\mathcal{G}(M \setminus Y)$ and $|Y|$ where Y is the set of loops. (More generally, if M is the direct sum $M_1 \oplus M_2$, then $\mathcal{G}(M)$ can be computed from $\mathcal{G}(M_1)$ and $\mathcal{G}(M_2)$; see [2, Section 4.4].) Thus, we focus on matroids M with neither loops nor coloops, and so \emptyset and $E(M)$ are cyclic flats of M . Analogous to the notation L° defined above, we let $\mathcal{Z}^\circ(M)$ denote the set of nonempty, proper cyclic flats of M . Thus, $\mathcal{Z}^\circ(M)$ is empty if and only if M is a uniform matroid.

Prior to this work, the matroids with different configurations but the same \mathcal{G} -invariant that we were aware of were:

- Dowling lattices of a given rank exceeding three, using non-isomorphic groups of the same order [2],
- the free m -cones of non-isomorphic matroids that have the same \mathcal{G} -invariant, along with tipless, baseless, and tipless/baseless variations [3], and
- the duals of any such examples.

(The effect of taking the dual of a matroid on the \mathcal{G} -invariant, in the original formulation, is simple: reverse the order within each rank sequence, and then switch 0s and 1s.)

Another construction is easy to treat using catenary data: given a matroid M and a positive integer t , construct the extension M_t of M by, for each $e \in E(M)$, adding a set A_e of t elements parallel to e , with A_e disjoint from all other such sets and from $E(M)$. (Contracting the tip of a free m -cone of M yields M_m .) Note that all flats of M_t are cyclic, $\mathcal{Z}(M_t)$ is isomorphic to the lattice of flats of M , and each flag (Y_0, Y_1, \dots, Y_k) of M_t corresponds to a flag (X_0, X_1, \dots, X_k) of M where $X_i \subseteq Y_i$ and $|Y_i| = |X_i|(t+1)$ for each i . So if M and N are non-isomorphic and have the same \mathcal{G} -invariant, then M_t and N_t have the same catenary data, and so the same \mathcal{G} -invariant, but different configurations.

For any positive integer n , there is a positive integer m so that there are at least n non-isomorphic groups of order m , so Dowling lattices yield sets of arbitrarily many matroids with different configurations (specifically, non-isomorphic lattices of cyclic flats) but the same \mathcal{G} -invariant. Our constructions yield other large sets of matroids of this type for which the lattices of cyclic flats are simpler and smaller (see Examples 2 and 3).

3. THE TOOLS

The first lemma generalizes an argument from the proof of [2, Theorem 7.3]. The join $F_1 \vee F_2$ mentioned below is the join in the lattice $\mathcal{Z}(M)$ of cyclic flats.

Lemma 3.1. *Assume that the function $g : \mathcal{Z}^\circ(M) \rightarrow 2^{2^{E(M)}}$ has the property that if $g(F_1) \cap g(F_2) \neq \emptyset$, then $F_1 \vee F_2 \in \mathcal{Z}^\circ(M)$ and $g(F_1) \cap g(F_2) \subseteq g(F_1 \vee F_2)$. Then*

$$\left| \bigcup_{F \in \mathcal{Z}^\circ(M)} g(F) \right| = \sum_{\substack{S \subseteq \mathcal{Z}^\circ(M), \\ S \neq \emptyset}} (-1)^{|S|+1} \left| \bigcap_{F \in S} g(F) \right| = \sum_{\substack{\text{nonempty chains} \\ S \subseteq \mathcal{Z}^\circ(M)}} (-1)^{|S|+1} \left| \bigcap_{F \in S} g(F) \right|.$$

Before proving the lemma, we note an example of such a function g . For a matroid M , let $g_M : \mathcal{Z}^\circ(M) \rightarrow 2^{2^{E(M)}}$ be given by

$$(3.1) \quad g_M(F) = \{X : X \subset E(M), |X| = r(M) - 1, \text{ and } \text{cl}_M(X \cap F) = F\}.$$

For example, for the matroid M in Figure 1, the sets in $g_M(\{1, 2, 3, 4\})$ are precisely the five bases of $M|_{\{1, 2, 3, 4\}}$, while those in $g_M(\{1, 2\})$ are the thirteen 2-element sets that contain at least one of 1 and 2. It is routine to verify that, for any matroid M , the function g_M has the property in the hypothesis of the lemma. Later in this section we make heavy use of the function g_M for various matroids M .

Proof of Lemma 3.1. The principle of inclusion/exclusion gives the equality of the first two expressions, so we focus on the two sums. Let F_1 and F_2 be incomparable flats in $\mathcal{Z}^\circ(M)$ with $g(F_1) \cap g(F_2) \neq \emptyset$, so $F_1 \vee F_2$ properly contains both of them. Now $g(F_1) \cap g(F_2) = g(F_1) \cap g(F_2) \cap g(F_1 \vee F_2)$ by the assumed property. If F_1 and F_2 are in a subset S of $\mathcal{Z}^\circ(M)$, and if S' is the symmetric difference $S \Delta \{F_1 \vee F_2\}$, then $(-1)^{|S|} = -(-1)^{|S'|}$ and

$$\bigcap_{F \in S} g(F) = \bigcap_{F \in S'} g(F),$$

so such terms in the first sum could cancel. To cancel all such terms, take a linear extension \leq of the order \subseteq on $\mathcal{Z}^\circ(M)$ and, if a subset S of $\mathcal{Z}^\circ(M)$ contains incomparable cyclic flats, let F_1 and F_2 be such a pair for which (F_1, F_2) is least in the lexicographic order that \leq induces on $\mathcal{Z}^\circ(M) \times \mathcal{Z}^\circ(M)$; in the first sum, cancel the term that arises from S with the one that arises from $S \Delta \{F_1 \vee F_2\}$. Any incomparable cyclic flats F_1 and F_2 in a set

S that remains after cancellation satisfy $g(F_1) \cap g(F_2) = \emptyset$; such terms contribute zero to the first sum and so can be omitted. The result of all such cancellations and omissions is the sum on the right side. \square

Let (X_0, X_1, \dots, X_r) be a flag of a matroid M of rank r that has neither loops nor coloops, so $X_0 = \emptyset$ and $X_r = E(M)$. If, for each flat X_i , we remove the coloops of $M|X_i$ from X_i , we obtain a chain of cyclic flats. The (possibly empty) chain obtained by removing \emptyset and $E(M)$ from this chain is the *reduced cyclic chain* of the flag. For a chain C in $\mathcal{Z}^\circ(M)$, let $\text{flag}(C)$ be the set of all flags of M whose reduced cyclic chain is C . For a set T of chains in $\mathcal{Z}^\circ(M)$, let $\text{flag}(T)$ be the union of all sets $\text{flag}(C)$ where $C \in T$. For a set S of flags, let $\text{comp}(S)$ be the multiset of compositions of the flags in S . Thus, $\text{comp}(\text{flag}(T))$ is the multiset of all compositions of all of the flags of M whose reduced cyclic chain is in the set T of chains in $\mathcal{Z}^\circ(M)$.

Example 1. For the reduced cyclic chain $C = (\{1, 2\})$ in the matroid M in Figure 1, $\text{flag}_M(C)$ contains four flags, namely, (omitting \emptyset and $E(M)$ for brevity) $(\{1, 2\}, \{1, 2, i\})$ and $(\{i\}, \{1, 2, i\})$ for $i \in \{5, 6\}$; the compositions in $\text{comp}(\text{flag}_M(C))$ are $(0, 2, 1, 5)$ and $(0, 1, 2, 5)$, and each has multiplicity 2. For the matroid N in that figure, $\text{flag}_N(C)$ is empty. Likewise, for $C' = (\{7, 8\})$, $\text{flag}_M(C')$ contains four flags: $(\{7, 8\}, \{7, 8, i\})$ and $(\{i\}, \{7, 8, i\})$ for $i \in \{3, 4\}$; the compositions in $\text{comp}(\text{flag}_M(C'))$ are $(0, 2, 1, 5)$ and $(0, 1, 2, 5)$, each of which has multiplicity 2. Also, $\text{flag}_N(C')$ contains eight flags: $(\{7, 8\}, \{7, 8, i\})$ and $(\{i\}, \{7, 8, i\})$ for $i \in \{3, 4, 5, 6\}$; the compositions in the multiset $\text{comp}(\text{flag}_N(C'))$ are $(0, 2, 1, 5)$ and $(0, 1, 2, 5)$, and each has multiplicity 4. Thus, the multisets $\text{comp}(\text{flag}_M(C))$ and $\text{comp}(\text{flag}_M(C'))$ differ from $\text{comp}(\text{flag}_N(C))$ and $\text{comp}(\text{flag}_N(C'))$, but $\text{comp}(\text{flag}_M(\{C, C'\})) = \text{comp}(\text{flag}_N(\{C, C'\}))$, so together C and C' make the same contribution to the catenary data of M and N .

To complete that example, note that neither M nor N has flags with reduced cyclic chain $(\{1, 2, 7, 8\})$. For each of M and N , there are eight flags with the empty reduced cyclic chain, and each has composition $(0, 1, 1, 6)$. In each of M and N , each of the four 2-element reduced cyclic chains arises from exactly one flag, and that flag has composition $(0, 2, 2, 4)$. Finally, the reduced cyclic chains $(\{1, 2, 3, 4\})$ and $(\{5, 6, 7, 8\})$ in M , and $(\{1, 2, 3, 4\})$ and $(\{1, 2, 5, 6\})$ in N , each arise from two flags (e.g., $(\{3\}, \{1, 2, 3, 4\})$ and $(\{4\}, \{1, 2, 3, 4\})$ for the first); each of these reduced cyclic chains contributes two copies of $(0, 1, 3, 4)$ to the catenary data.

This example motivates a general technique, stated next, for showing that two matroids have the same \mathcal{G} -invariant. This result, which follows immediately from Theorem 2.4 and the definitions above, is what we use in Sections 4 and 5 to show that the constructions presented there produce matroids with the same \mathcal{G} -invariant.

Theorem 3.2. *Let M and N be matroids. Let $\{P_1, P_2, \dots, P_d\}$ be a partition of the set of reduced cyclic chains of $\mathcal{Z}^\circ(M)$. Let $\{Q_1, Q_2, \dots, Q_d\}$ be a partition of the set of reduced cyclic chains of $\mathcal{Z}^\circ(N)$. If $\text{comp}(\text{flag}_M(P_i)) = \text{comp}(\text{flag}_N(Q_i))$ for each $i \in [d]$, then $\mathcal{G}(M) = \mathcal{G}(N)$.*

In the rest of this section, we treat the tools that we will use to establish equalities of the form $\text{comp}(\text{flag}_M(P_i)) = \text{comp}(\text{flag}_N(Q_i))$.

Part of the proof of [2, Theorem 7.3] establishes the following lemma. In this result and what follows, we use $\iota(N)$ to denote the number of independent hyperplanes (that is, hyperplanes that are independent sets) of a matroid N .

Lemma 3.3. *Let M be a matroid that has no loops and no coloops, so \emptyset and $E(M)$ are cyclic flats of M . Let $C = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_t\}$ be a chain in $\mathcal{Z}^\circ(M)$. Set $F_0 = \emptyset$ and $F_{t+1} = E(M)$. Let \mathcal{L} be the set of all lists of length $r(M)$, with no repeated entries, that are obtained as follows:*

- for each $j \in [t+1]$, pick an independent hyperplane H_j of $M|F_j/F_{j-1}$,
- consider lists in which the entries are the sets F_1, F_2, \dots, F_{t+1} along with the singleton subsets of H_1, H_2, \dots, H_{t+1} ,
- such a list is in \mathcal{L} if and only if, for each $j \in [t]$, F_j occurs before F_{j+1} , and, for each $j \in [t+1]$, all singleton subsets of H_j occur before F_j .

Mapping $L \in \mathcal{L}$ to the sequence $\phi(L)$, the i th entry of which is the union of the first i sets in L , for $0 \leq i \leq r(M)$, defines a bijection $\phi : \mathcal{L} \rightarrow \text{flag}(C)$.

Thus, the multiset $\text{comp}(\text{flag}(C))$ is determined by the sizes and ranks of F_1, F_2, \dots, F_t , and $E(M)$, along with $\iota(M|F_1)$, $\iota(M|F_j/F_{j-1})$ for $2 \leq j \leq t$, and $\iota(M/F_t)$.

The number of independent hyperplanes of a matroid can be deduced from its \mathcal{G} -invariant [2, Proposition 5.5] and so from its configuration.

Corollary 3.4. *Let M and M' be matroids, both having neither loops nor coloops. Let $C = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_t\}$ be a chain in $\mathcal{Z}^\circ(M)$, and $C' = \{F'_1 \subsetneq F'_2 \subsetneq \cdots \subsetneq F'_t\}$ a chain in $\mathcal{Z}^\circ(M')$. Assume that the following pairs of minors have the same configuration:*

- (a) the restrictions $M|F_1$ and $M'|F'_1$,
- (b) for each i with $2 \leq i \leq t$, the minors $M|F_i/F_{i-1}$ and $M'|F'_i/F'_{i-1}$, and
- (c) the contractions M/F_t and M'/F'_t .

Then $\text{comp}(\text{flag}_M(C)) = \text{comp}(\text{flag}_{M'}(C'))$.

Corollary 3.4 and Lemmas 3.6 and 3.7 are the key tools for the results in the rest of the paper. We next prove a result that we use in the proofs of those lemmas.

Lemma 3.5. *Let M and M' be matroids of rank r for which $|E(M)| = |E(M')|$. Assume that M and M' have neither loops nor coloops. Let $\{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_t\}$ be a chain in $\mathcal{Z}^\circ(M)$, and $\{F'_1 \subsetneq F'_2 \subsetneq \cdots \subsetneq F'_t\}$ a chain in $\mathcal{Z}^\circ(M')$, for which the following pairs of minors have the same configuration:*

- (a) the restrictions $M|F_1$ and $M'|F'_1$, and
- (b) for each i with $2 \leq i \leq t$, the minors $M|F_i/F_{i-1}$ and $M'|F'_i/F'_{i-1}$.

Let g_M and $g_{M'}$ be given by equation (3.1). Then

$$(3.2) \quad \left| \bigcap_{i \in [t]} g_M(F_i) \right| = \left| \bigcap_{i \in [t]} g_{M'}(F'_i) \right|.$$

Proof. We get each set in $g_M(F_1) \cap g_M(F_2) \cap \cdots \cap g_M(F_t)$ exactly once by following the steps below, allowing for all possible choices:

- choose integers $a_1, a_2, \dots, a_t, a_{t+1}$ with $a_1 + a_2 + \cdots + a_t + a_{t+1} = r - 1$, and with $a_1 \geq r_M(F_1)$ and $a_i \geq r(F_i) - r(F_{i-1})$ for i with $2 \leq i \leq t$, and $a_{t+1} \geq 0$;
- choose
 - (i) a spanning set W_1 of $M|F_1$ where $|W_1| = a_1$,
 - (ii) for each i with $2 \leq i \leq t$, a spanning set W_i of $M|F_i/F_{i-1}$ where $|W_i| = a_i$, and
 - (iii) any set $W_{t+1} \subseteq E(M) - F_t$ where $|W_{t+1}| = a_{t+1}$;
- then $W_1 \cup W_2 \cup \cdots \cup W_{t+1} \in g_M(F_1) \cap g_M(F_2) \cap \cdots \cap g_M(F_t)$.

A similar sequences of choices yields all sets in $g_{M'}(F'_1) \cap g_{M'}(F'_2) \cap \cdots \cap g_{M'}(F'_t)$. From this, equality (3.2) follows since, by the hypothesis, the restrictions $M|F_1$ and $M'|F'_1$ have the same configuration and so the same number of spanning sets of size a_1 , and, for each i with $2 \leq i \leq t$, the minors $M|F_i/F_{i-1}$ and $M'|F'_i/F'_{i-1}$ have the same configuration and so the same number of spanning sets of size a_i , and those assumptions along with the hypothesis $|E(M)| = |E(M')|$ also give $|E(M) - F_t| = |E(M') - F'_t|$. \square

For a matroid M , let $\text{ch}(\mathcal{Z}^\circ(M))$ be the set of non-empty chains in $\mathcal{Z}^\circ(M)$.

The next two lemmas and their proofs have a similar flavor and could be merged. We treat them separately for clarity. Note that for the matroids M and N in Figure 1, there are many bijections $\sigma : \text{ch}(\mathcal{Z}^\circ(M)) \rightarrow \text{ch}(\mathcal{Z}^\circ(N))$ that satisfy the conditions below, but none is induced by a bijection from $\mathcal{Z}^\circ(M)$ onto $\mathcal{Z}^\circ(N)$.

Lemma 3.6. *Let the rank- r matroids M and M' on n elements have neither loops nor coloops. Assume that there is a bijection $\sigma : \text{ch}(\mathcal{Z}^\circ(M)) \rightarrow \text{ch}(\mathcal{Z}^\circ(M'))$ such that for each chain $S = \{X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_t\}$ in $\text{ch}(\mathcal{Z}^\circ(M))$,*

- (a) *its image $\sigma(S) = \{X'_1 \subsetneq X'_2 \subsetneq \cdots \subsetneq X'_t\}$ has $|S|$ elements, and*
- (b) *the restrictions $M|X_1$ and $M'|X'_1$ have the same configuration, as do, for each i with $2 \leq i \leq t$, the minors $M|X_i/X_{i-1}$ and $M'|X'_i/X'_{i-1}$.*

Then $\iota(M) = \iota(M')$ and, for the empty chain \emptyset , $\text{comp}(\text{flag}_M(\emptyset)) = \text{comp}(\text{flag}_{M'}(\emptyset))$.

Proof. Let g_M and $g_{M'}$ be given by equation (3.1). Then

$$\iota(M) = \binom{n}{r-1} - \left| \bigcup_{F \in \mathcal{Z}^\circ(M)} g_M(F) \right|,$$

and similarly for M' , so in order to show that $\iota(M) = \iota(M')$, it suffices to show that

$$\left| \bigcup_{F \in \mathcal{Z}^\circ(M)} g_M(F) \right| = \left| \bigcup_{F \in \mathcal{Z}^\circ(M')} g_{M'}(F) \right|.$$

By Lemma 3.1, it suffices to show that

$$\sum_{S \in \text{ch}(\mathcal{Z}^\circ(M))} (-1)^{|S|+1} \left| \bigcap_{F \in S} g_M(F) \right| = \sum_{S \in \text{ch}(\mathcal{Z}^\circ(M'))} (-1)^{|S|+1} \left| \bigcap_{F \in S} g_{M'}(F) \right|.$$

This holds by applying Lemma 3.5 to each pair of chains $(S, \sigma(S))$ for $S \in \text{ch}(\mathcal{Z}^\circ(M))$.

We now get $\text{comp}(\text{flag}_M(\emptyset)) = \text{comp}(\text{flag}_{M'}(\emptyset))$ since the reduced cyclic chain of a flag is empty if and only if the composition of the flag is $(0, 1, 1, \dots, 1, n-r+1)$, and, by Lemma 3.3, the number of such flags in M is $(r-1)\iota(M)$, and likewise for M' . \square

In Example 1, we have $\text{comp}(\text{flag}_M(\{C, C'\})) = \text{comp}(\text{flag}_N(\{C, C'\}))$ for reduced cyclic chains C and C' even though what C and C' contribute individually differs in M versus N . The next lemma identifies conditions that yield the same conclusion.

Lemma 3.7. *Let M and M' be rank- r matroids on n elements with no loops and no coloops. Let C_1, C_2, \dots, C_p be distinct (not necessarily disjoint) nonempty chains in $\mathcal{Z}^\circ(M)$ and let C'_1, C'_2, \dots, C'_p be distinct (not necessarily disjoint) nonempty chains in $\mathcal{Z}^\circ(M')$ with $|C_i| = |C'_j|$ for all $i, j \in [p]$. Write C_i as $\{F_{i,1} \subsetneq F_{i,2} \subsetneq \cdots \subsetneq F_{i,t}\}$ and C'_i as $\{F'_{i,1} \subsetneq F'_{i,2} \subsetneq \cdots \subsetneq F'_{i,t}\}$. Assume that for all $i, j \in [p]$,*

- (a) *$M|F_{i,1}$ and $M'|F'_{j,1}$ have the same configuration, as do the minors $M|F_{i,h}/F_{i,h-1}$ and $M'|F'_{j,h}/F'_{j,h-1}$ for each h with $2 \leq h \leq t$.*

Also assume that there is a bijection

$$\Phi : \bigcup_{i \in [p]} (\text{ch}(\mathcal{Z}^\circ(M/F_{i,t})) \times \{C_i\}) \rightarrow \bigcup_{i \in [p]} (\text{ch}(\mathcal{Z}^\circ(M'/F'_{i,t})) \times \{C'_i\})$$

with the following property:

- (b) if $\Phi(\{\{X_1 \subsetneq X_2 \subsetneq \dots \subsetneq X_k\}, C_i\}) = (\{X'_1 \subsetneq X'_2 \subsetneq \dots \subsetneq X'_{k'}\}, C'_j)$, then $k = k'$ and $M|X_1/F_{i,t}$ and $M'|X'_1/F'_{j,t}$ have the same configuration, as do the minors $M|X_h/X_{h-1}$ and $M'|X'_h/X'_{h-1}$ for each h with $2 \leq h \leq k$.

Then $\text{comp}(\text{flag}_M(\{C_1, C_2, \dots, C_p\})) = \text{comp}(\text{flag}_{M'}(\{C'_1, C'_2, \dots, C'_p\}))$.

This result implies Corollary 3.4, but it applies more broadly, even for $p = 1$, since property (b) may hold for some bijection from $\text{ch}(\mathcal{Z}^\circ(M/F_{1,t}))$ onto $\text{ch}(\mathcal{Z}^\circ(M'/F'_{1,t}))$ even if $M/F_{1,t}$ and $M'/F'_{1,t}$ have different configurations (see the proof of Theorem 4.1). We may have $F_{i,t} = F_{j,t}$ even if $i \neq j$, so chains in $\text{ch}(\mathcal{Z}^\circ(M/F_{i,t}))$ may arise multiple times; the second entry in the Cartesian product distinguishes these occurrences. When $p = 1$, we can omit the second factor in the Cartesian product.

Proof of Lemma 3.7. By Lemma 3.3, we get each flag of M whose reduced cyclic chain is C_i exactly once by the following procedure:

- pick an independent hyperplane of $M|F_{i,1}$ and take a permutation of its singleton subsets; insert the set $F_{i,1}$ as the last entry of this list;
- treat each h with $2 \leq h \leq t$ in order; for h , pick an independent hyperplane of $M|F_{i,h}/F_{i,h-1}$ and insert its singleton subsets anywhere into the list we have so far, and then insert the set $F_{i,h}$ as the last entry of this list;
- pick an independent hyperplane of $M/F_{i,t}$ and insert its singleton subsets anywhere into the list we have so far, and then insert $E(M)$ as the last entry of the list;

the flag that we get from the final list of sets has as its j th flat, for $0 \leq j \leq r$, the union of the first j sets in the list. The conditions above make it clear that the multiset of compositions of flags that result from any particular choice of independent hyperplanes of $M|F_{i,1}$, $M|F_{i,h}/F_{i,h-1}$ for $2 \leq h \leq t$, and $M/F_{i,t}$ does not depend on those sets; only their sizes (which are fixed) and the sizes of $F_{i,1}, F_{i,2}, \dots, F_{i,t}$ and $E(M)$ matter. A similar description applies to the flags of M' whose reduced cyclic chain is C'_j . The numbers of independent hyperplanes of $M|F_{i,1}$ and of $M|F_{i,h}/F_{i,h-1}$ for $2 \leq h \leq t$ are determined by their configurations, so by condition (a) they are equal to the numbers of independent hyperplanes of the corresponding minors of M' using the chain C'_j . The hypotheses do not imply that $\iota(M/F_{i,t})$ and $\iota(M'/F'_{j,t})$ are equal, but for the conclusion in the theorem to hold, all that we need is

$$(3.3) \quad \sum_{i \in [p]} \iota(M/F_{i,t}) = \sum_{i \in [p]} \iota(M'/F'_{i,t}).$$

For each $i \in [p]$, let $g_{M/F_{i,t}}$, which we shorten to g_i , be given by equation (3.1), and likewise for g'_i , the shortened form of $g_{M'/F'_{i,t}}$. Then

$$\iota(M/F_{i,t}) = \binom{n - |F_{i,t}|}{r - r_M(F_{i,t}) - 1} - \left| \bigcup_{F \in \mathcal{Z}^\circ(M/F_{i,t})} g_i(F) \right|,$$

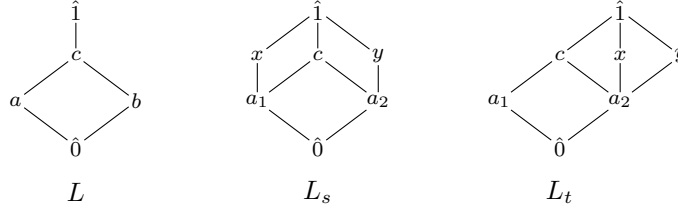


FIGURE 3. A lattice L and two extension of the type in the construction treated in Section 4, where L_s is isomorphic to $\mathcal{Z}(M)$ and where L_t is isomorphic to $\mathcal{Z}(N)$ for the matroids M and N in Figure 1.

and similarly for $\iota(M'/F'_{j,t})$. We get $|F_{i,t}| = |F'_{j,t}|$ and $r_M(F_{i,t}) = r_{M'}(F'_{j,t})$ for all $i, j \in [p]$ by condition (a), so equation (3.3) will follow by showing that

$$\sum_{i \in [p]} \left| \bigcup_{F \in \mathcal{Z}^\circ(M/F_{i,t})} g_i(F) \right| = \sum_{i \in [p]} \left| \bigcup_{F \in \mathcal{Z}^\circ(M'/F'_{i,t})} g'_i(F) \right|.$$

By Lemma 3.1, it suffices to show that

$$\sum_{i \in [p]} \sum_{S \in \text{ch}(\mathcal{Z}^\circ(M/F_{i,t}))} (-1)^{|S|+1} \left| \bigcap_{F \in S} g_i(F) \right| = \sum_{i \in [p]} \sum_{S \in \text{ch}(\mathcal{Z}^\circ(M'/F'_{i,t}))} (-1)^{|S|+1} \left| \bigcap_{F \in S} g'_i(F) \right|.$$

Lemma 3.5 applies to the chains S and S' whenever $\Phi((S, C_i)) = (S', C'_j)$ with $i, j \in [p]$, so this equality holds. \square

4. APPLICATION: A CONSTRUCTION MODIFYING A LATTICE

In this section we show how to extend a finite lattice in different ways to produce lattices in different configurations that yield the same \mathcal{G} -invariant.

We first construct the lattices that appear in Theorem 4.1. Let a_1, a_2, \dots, a_m be distinct elements of a finite lattice L for which

- there is a $b \in L$ with $a_i \wedge a_j = b$ for all $i, j \in [m]$ with $i \neq j$, and
- for each $i \in [m-1]$, there is a lattice isomorphism $\tau_{m,i} : [\hat{0}, a_m] \rightarrow [\hat{0}, a_i]$ with $\tau_{m,i}(y) = y$ for all $y \in [\hat{0}, b]$.

Set $\tau_{i,m} = \tau_{m,i}^{-1}$ and, for distinct $i, j \in [m-1]$, set $\tau_{i,j} = \tau_{m,j} \circ \tau_{i,m}$. Therefore $\tau_{i,j} : [\hat{0}, a_i] \rightarrow [\hat{0}, a_j]$ is a lattice isomorphism and $\tau_{i,j}(y) = y$ for all $y \in [\hat{0}, b]$. Let L_1, L_2, \dots, L_n be finite lattices that are disjoint from each other and from L . Fix two functions $s : [n] \rightarrow [m]$ and $t : [n] \rightarrow [m]$. Form a lattice L_s as follows:

- for $i \in [n]$, form L'_i from L_i by replacing $\hat{1}_{L_i}$ by $\hat{1}_L$ and $\hat{0}_{L_i}$ by $a_{s(i)}$,
- viewing lattices as relations, let L_s be the transitive closure of $L \cup L'_1 \cup L'_2 \cup \dots \cup L'_n$.

It is routine to check that L_s is indeed a lattice. In terms of Hasse diagrams, L_s is obtained by, for each $i \in [n]$, inserting L_i into the interval $[a_{s(i)}, \hat{1}_L]$ of L , where $\hat{0}_{L_i}$ is identified with $a_{s(i)}$, and $\hat{1}_{L_i}$ is identified with $\hat{1}_L$. Define L_t similarly. Thus, L_s and L_t have the same elements. The lattices L_s and L_t may be isomorphic, but in many examples they are not. Figure 3 shows how the lattices of cyclic flats of the two matroids in Figure 1 are obtained by extending a lattice using this construction. We give more examples after the proof of the next result.

Theorem 4.1. *Let L_s and L_t be as defined above. Let M_s and M_t be matroids, with rank functions r_s and r_t , respectively, neither having loops nor coloops, for which, for some lattice isomorphisms $\phi_s : L_s \rightarrow \mathcal{Z}(M_s)$ and $\phi_t : L_t \rightarrow \mathcal{Z}(M_t)$,*

- (1) $|\phi_s(y)| = |\phi_t(y)|$ and $r_s(\phi_s(y)) = r_t(\phi_t(y))$ for all y in L_s , and
- (2) if $y \in [\hat{0}, a_m]$, then $|\phi_s(y)| = |\phi_s(\tau_{m,i}(y))|$ and $r_s(\phi_s(y)) = r_s(\phi_s(\tau_{m,i}(y)))$ for all $i \in [m-1]$.

Then $\mathcal{G}(M_s) = \mathcal{G}(M_t)$.

From the \mathcal{G} -invariant of a matroid, we can deduce the number of cyclic flats of each size and rank [2, Proposition 5.5], so condition (1) just says how this data lines up. Condition (2) is the only restrictive assumption on this data.

Proof of Theorem 4.1. By Theorem 3.2, it suffices produce a partition $\{P_1, P_2, \dots, P_d\}$ of the set of chains in $\mathcal{Z}^\circ(M_s)$ and a partition $\{Q_1, Q_2, \dots, Q_d\}$ of the set of chains in $\mathcal{Z}^\circ(M_t)$ so that, for each $h \in [d]$, one of Corollary 3.4, Lemma 3.6, or Lemma 3.7 gives $\text{comp}(\text{flag}_{M_s}(P_h)) = \text{comp}(\text{flag}_{M_t}(Q_h))$. We specify chains in $\mathcal{Z}^\circ(M_s)$ as chains in L_s° ; to get the corresponding chains in $\mathcal{Z}^\circ(M_s)$, apply ϕ_s ; we handle chains in $\mathcal{Z}^\circ(M_t)$ similarly. Let $\text{ch}(L_s^\circ)$ denote the set of nonempty chains in L_s° ; define $\text{ch}(L_t^\circ)$ similarly.

We start with some useful observations for verifying that certain minors have the same configuration, as required by Corollary 3.4 and Lemmas 3.6 and 3.7. These observations are cast in terms of an interval $[x, z]$ in L_s . The corresponding minor $M_s|_{\phi_s(z)/\phi_s(x)}$ of M_s can be, if $x = \hat{0}$, the restriction to the first cyclic flat in a chain, or, if $z = \hat{1}$, the contraction by the last cyclic flat in a chain, or the minor determined by consecutive cyclic flats in a chain. Only applications of Corollary 3.4 use intervals of the form $[x, \hat{1}]$. Properties (1) and (2) imply that, for all distinct integers $i, j \in [m]$ and all $y \in [\hat{0}, a_i]$,

- (S) $|\phi_s(y)| = |\phi_s(\tau_{i,j}(y))| = |\phi_t(\tau_{i,j}(y))|$, and
- (R) $r_s(\phi_s(y)) = r_s(\phi_s(\tau_{i,j}(y))) = r_t(\phi_t(\tau_{i,j}(y)))$.

Consider an interval $[x, z]_{L_s}$ with $x \neq z$. The next three items treat all intervals that arise below when we apply Corollary 3.4 and Lemmas 3.6 and 3.7.

- If there is no $i \in [m]$ with $x \leq a_i < z$, then $M_s|_{\phi_s(z)/\phi_s(x)}$ and $M_t|_{\phi_t(z)/\phi_t(x)}$ have the same configuration.

This holds since $[x, z]_{L_s}$ is also an interval of L_t and the equalities in property (1) hold for all y in $[x, z]_{L_s}$. This next statement gives additional information for some of the intervals to which the last statement applies.

- If $z \leq a_i$ for some $i \in [m]$, then, for all $j \in [m]$ with $j \neq i$, the minors $M_s|_{\phi_s(z)/\phi_s(x)}$ and $M_t|_{\phi_t(\tau_{i,j}(z))/\phi_t(\tau_{i,j}(x))}$ have the same configuration.

This holds since properties (S) and (R) apply to all $y \in [x, z]$.

- If $x \leq a_i$ and $z \in L_h^\circ$ where $s(h) = i$ and $t(h) = j$, then
 - if $i = j$, then the minors $M_s|_{\phi_s(z)/\phi_s(x)}$ and $M_t|_{\phi_t(z)/\phi_t(x)}$ have the same configuration, while
 - if $i \neq j$, then the minors $M_s|_{\phi_s(z)/\phi_s(x)}$ and $M_t|_{\phi_t(z)/\phi_t(\tau_{i,j}(x))}$ have the same configuration.

This holds by property (1) and, for the second part, properties (S) and (R) applied to all $y \in [x, a_i]$.

To treat the empty chain via Lemma 3.6, consider $\sigma : \text{ch}(L_s^\circ) \rightarrow \text{ch}(L_t^\circ)$ given by, for $C \in \text{ch}(L_s^\circ)$,

$$\sigma(C) = \begin{cases} (C - L) \cup \tau_{s(h), t(h)}(C \cap L), & \text{if } C \cap L_h^\circ \neq \emptyset \text{ and } s(h) \neq t(h), \\ C, & \text{otherwise.} \end{cases}$$

It is routine to check that σ is a bijection. Condition (a) clearly holds. The observations in the second paragraph show that condition (b) holds, so Lemma 3.6 applies.

A chain C is in the symmetric difference $\text{ch}(L_s^\circ) \Delta \text{ch}(L_t^\circ)$ if and only if (i) $C \not\subseteq L$, (ii) $C \cap L \not\subseteq [\hat{0}, b]$, and (iii) $s(h) \neq t(h)$ for the h with $C - L \subseteq L_h^\circ$. Restricting σ gives a bijection $\sigma : \text{ch}(L_s^\circ) - \text{ch}(L_t^\circ) \rightarrow \text{ch}(L_t^\circ) - \text{ch}(L_s^\circ)$. Corollary 3.4 applies to the chains $\phi_s(C)$ and $\phi_t(\sigma(C))$ for each $C \in \text{ch}(L_s^\circ) - \text{ch}(L_t^\circ)$; the observations in the second paragraph show that conditions (a)–(c) hold.

We now focus on chains in $\text{ch}(L_s^\circ) \cap \text{ch}(L_t^\circ)$. Let $\max(C)$ denote the greatest element in a chain C .

Let C be a nonempty chain of both L_s° and L_t° . If no $i \in [m]$ has $C \subseteq [\hat{0}, a_i]$, then Corollary 3.4 applies to $\phi_s(C)$ and $\phi_t(C)$, with the observations in the second paragraph showing that conditions (a)–(c) hold. If $C \subseteq [\hat{0}, b]$, then Lemma 3.7 applies to $\phi_s(C)$ and $\phi_t(C)$ (so $p = 1$): restrict the map σ defined above to $\text{ch}((\max(C), \hat{1})_{L_s^\circ})$ to get a bijection $\Phi : \text{ch}((\max(C), \hat{1})_{L_s^\circ}) \rightarrow \text{ch}((\max(C), \hat{1})_{L_t^\circ})$; conditions (a) and (b) hold by the observations in the second paragraph. Now consider a chain C for which there is exactly one $i \in [m]$ with $C \subseteq [\hat{0}, a_i]$. The images of C under the maps $\tau_{i,j}$, along with C itself, give m chains C_1, C_2, \dots, C_m with $C_j \subseteq [\hat{0}, a_j]$ and $\tau_{j,h}(C_j) = C_h$ for all distinct $j, h \in [m]$. We apply Lemma 3.7 to the chains $\phi_s(C_1), \phi_s(C_2), \dots, \phi_s(C_m)$ in $\mathcal{Z}^\circ(M_s)$, and $\phi_t(C_1), \phi_t(C_2), \dots, \phi_t(C_m)$ in $\mathcal{Z}^\circ(M_t)$. By the observations in the second paragraph, condition (a) in that lemma holds. Note that if $D \in \text{ch}((\max(C_h), \hat{1})_{L_s})$ and $D \cap L_j^\circ \neq \emptyset$, then $s(j) = h$. The map

$$\Phi : \bigcup_{h \in [m]} (\text{ch}((\max(C_h), \hat{1})_{L_s}) \times \{C_h\}) \rightarrow \bigcup_{h \in [m]} (\text{ch}((\max(C_h), \hat{1})_{L_t}) \times \{C_h\})$$

given by

$$\Phi((D, C_h)) = \begin{cases} ((D - L) \cup \tau_{h, t(j)}(D \cap L), C_{t(j)}), & \text{if } D \cap L_j^\circ \neq \emptyset \text{ and } t(j) \neq h, \\ (D, C_h), & \text{otherwise,} \end{cases}$$

is easily seen to be a bijection that, by the observations in the second paragraph, when we apply ϕ_s and ϕ_t , satisfies property (b) in Lemma 3.7. \square

Example 2. If the lattice L in the construction above is isomorphic to the lattice of flats of an m -point line and each L_i is a three-element chain, then L_s has the form illustrated in Figure 4, where some open intervals $(a_h, \hat{1})$ may be empty. The number of non-isomorphic lattices of this type is the number of integer partitions of n with at most m parts. One type of matroid for which its lattice of cyclic flats has this form is a rank-4 matroid with m three-point lines A_1, A_2, \dots, A_m , each pair of which spans the matroid, and where each cyclic plane contains just one cyclic line, namely, some A_i . If the sizes of the n cyclic planes are fixed and distinct, then, up to isomorphism, $\sum_{i \in [m]} S(n, i)$ matroids satisfy these conditions, and their configurations are different. (Here, $S(n, i)$ is the Stirling number of the second kind. For some pairs of configurations, the lattices are isomorphic but the assignments of sizes differ.) This produces huge sets of matroids with very simple but different configurations and the same \mathcal{G} -invariant. For instance, for $m = n$, the cyclic

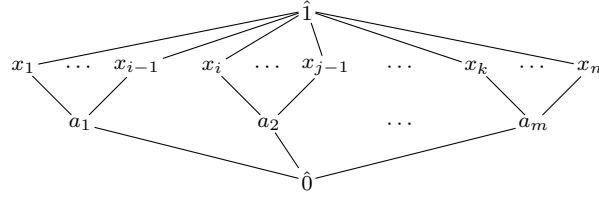


FIGURE 4. The lattice in Example 2.

planes can have distinct sizes with only $n(n + 9)/2$ elements in the matroids, and the size of the set of matroids produced this way, with different configurations and the same \mathcal{G} -invariant, is the n th Bell number. More such examples result by altering the sizes and ranks.

Example 3. Since a cyclic flat is both cyclic (a union of circuits) and a flat (an intersection of hyperplanes, the complements of which are circuits of the dual matroid), it follows that X is a cyclic flat of a matroid M if and only if $E(M) - X$ is a cyclic flat of the dual matroid, M^* . Thus, $\mathcal{Z}(M^*)$ is isomorphic to the order dual of $\mathcal{Z}(M)$. So a counterpart of Theorem 4.1 holds for the order duals of L_s and L_t . An example of a matroid whose lattice of cyclic flats is isomorphic to the order dual of the lattice considered in the previous example is a rank- r matroid, with $r \geq 4$, in which the restriction to each of the cyclic hyperplanes A_1, A_2, \dots, A_m , which are pairwise disjoint and have the same cardinality, is a paving matroid, where any two cyclic flats of rank $r - 2$ in distinct cyclic hyperplanes span the matroid. It is conjectured that asymptotically almost all matroids are paving, and, while that is still unresolved (see [1] for recent results), the number of paving matroids is known to be enormous, so this construction produces huge sets of matroids with different configurations but the same \mathcal{G} -invariant. Also, as in the previous example, as r grows, we have more options for the ranks of the cyclic flats A_i and for the cyclic flats X_j that they contain.

5. APPLICATION: A CONSTRUCTION USING PAVING MATROIDS

In Theorem 5.1 we show how to use the lattices of flats of paving matroids as the lattices in different configurations that yield the same \mathcal{G} -invariant. Examples are given after the proof.

Theorem 5.1. *Let N_1 and N_2 be non-isomorphic rank- r paving matroids on a set E . For $i \in [2]$, let \mathcal{H}_i be the set of hyperplanes of N_i and let L_i be the lattice of flats of N_i . Assume that there are partitions $\{A_1, A_2, \dots, A_p\}$ of $\mathcal{H}_1 - \mathcal{H}_2$ and $\{B_1, B_2, \dots, B_p\}$ of $\mathcal{H}_2 - \mathcal{H}_1$, and, for each $j \in [p]$, a bijection $\alpha_j : E \rightarrow E$ such that $X \in A_j$ if and only if $\alpha_j(X) \in B_j$. Let M_1 and M_2 be matroids, with rank functions r_1 and r_2 , respectively, for which there are lattice isomorphisms $\phi_i : L_i \rightarrow \mathcal{Z}(M_i)$, for $i \in [2]$, such that*

- if $X \in L_1 \cap L_2$, then $|\phi_1(X)| = |\phi_2(X)|$ and $r_1(\phi_1(X)) = r_2(\phi_2(X))$, and
- if $j \in [p]$ and $Y \in A_j$, then
 - $|\phi_1(Y)| = |\phi_2(\alpha_j(Y))|$ and $r_1(\phi_1(Y)) = r_2(\phi_2(\alpha_j(Y)))$, and
 - $|\phi_1(X)| = |\phi_1(\alpha_j(X))|$ and $r_1(\phi_1(X)) = r_1(\phi_1(\alpha_j(X)))$ for all $X \subseteq Y$ with $|X| < r - 1$.

Then $\mathcal{G}(M_1) = \mathcal{G}(M_2)$.

The hypotheses imply that N_1 and N_2 have the same configuration. Also, the conditions on ϕ_1 and ϕ_2 give the following equalities: if $Y \in A_j$ and $X \subseteq Y$ with $|X| < r - 1$, then

- $|\phi_2(X)| = |\phi_1(X)| = |\phi_1(\alpha_j(X))| = |\phi_2(\alpha_j(X))|$ and
- $r_2(\phi_2(X)) = r_1(\phi_1(X)) = r_1(\phi_1(\alpha_j(X))) = r_2(\phi_2(\alpha_j(X)))$.

Proof of Theorem 5.1. We use the technique outlined in Theorem 3.2: we give partitions $\{P_1, P_2, \dots, P_d\}$ of the set of chains in $\mathcal{Z}^\circ(M_1)$ and $\{Q_1, Q_2, \dots, Q_d\}$ of the set of chains in $\mathcal{Z}^\circ(M_2)$ so that, for each $h \in [d]$, we get $\text{comp}(\text{flag}_{M_1}(P_h)) = \text{comp}(\text{flag}_{M_2}(Q_h))$ from Corollary 3.4, Lemma 3.6, or Lemma 3.7. We can specify chains in $\mathcal{Z}^\circ(M_i)$, for $i \in [2]$, as chains in L_i° ; to get the corresponding chain in $\mathcal{Z}^\circ(M_i)$, apply ϕ_i . Let $\text{ch}(L_i^\circ)$ denote the set of nonempty chains in L_i° .

To treat the empty chain via Lemma 3.6, we need a bijection $\sigma : \text{ch}(L_1^\circ) \rightarrow \text{ch}(L_2^\circ)$ so that the composition $\phi_2 \circ \sigma \circ \phi_1^{-1}$ satisfies the conditions in Lemma 3.6. For a chain $C = \{Y_1 \subsetneq Y_2 \subsetneq \dots \subsetneq Y_t\}$ in $\text{ch}(L_1^\circ)$, let

$$\sigma(C) = \begin{cases} \alpha_j(C), & \text{if } Y_t \in A_j, \\ C, & \text{otherwise,} \end{cases}$$

where $\alpha_j(C) = \{\alpha_j(Y_h) : h \in [t]\}$. The hypotheses imply that σ is a bijection. Condition (a) clearly holds. To see that $\phi_2 \circ \sigma \circ \phi_1^{-1}$ satisfies condition (b), set $Y_0 = \emptyset$. If $\sigma(C) = C$, then $|\phi_1(X)| = |\phi_2(X)|$ and $r_1(\phi_1(X)) = r_2(\phi_2(X))$ for all $X \in [Y_{h-1}, Y_h]$ with $h \in [t]$, which gives condition (b) in this case. If $\sigma(C) = \alpha_j(C)$, then $|\phi_1(X)| = |\phi_2(\alpha_j(X))|$ and $r_1(\phi_1(X)) = r_2(\phi_2(\alpha_j(X)))$ for all $X \in [Y_{h-1}, Y_h]$ with $h \in [t]$, which gives condition (b) in this case.

Now $\text{ch}(L_{i_1}^\circ) - \text{ch}(L_{i_2}^\circ) = \{\{Y_1 \subsetneq Y_2 \subsetneq \dots \subsetneq Y_t\} : Y_t \in \mathcal{H}_{i_1} - \mathcal{H}_{i_2}\}$ for (i_1, i_2) in $\{(1, 2), (2, 1)\}$. For each $C = \{Y_1 \subsetneq Y_2 \subsetneq \dots \subsetneq Y_t\} \in \text{ch}(L_1^\circ) - \text{ch}(L_2^\circ)$, there is a unique $j \in [p]$ with $Y_t \in A_j$, and Corollary 3.4 applies to the chains $\phi_1(C)$ of M_1 and $\phi_2(\alpha_j(C))$ of M_2 . Properties (a)–(c) hold as in the previous paragraph.

We now treat the nonempty chains in $\text{ch}(L_1^\circ) \cap \text{ch}(L_2^\circ)$. For any chain C that contains a set in $\mathcal{H}_1 \cap \mathcal{H}_2$, Corollary 3.4 applies to the chains $\phi_1(C)$ in M_1 and $\phi_2(C)$ in M_2 . To treat the remaining chains, we define a graph G whose vertices are the nonempty chains in $\text{ch}(L_1^\circ) \cap \text{ch}(L_2^\circ)$ that contain no hyperplane. Two such chains $C = \{Y_1 \subsetneq Y_2 \subsetneq \dots \subsetneq Y_t\}$ and $C' = \{Y'_1 \subsetneq Y'_2 \subsetneq \dots \subsetneq Y'_t\}$ are adjacent in G if $t = t'$ and there is a $j \in [p]$ and an $H \in A_j$ such that either (i) $Y_t \subseteq H$ and $C' = \alpha_j(C)$ or (ii) $Y'_t \subseteq H$ and $C = \alpha_j(C')$. Thus, if C and C' are adjacent, then, setting $Y_0 = Y'_0 = \emptyset$, there is a bijection $\beta_h : [Y_{h-1}, Y_h] \rightarrow [Y'_{h-1}, Y'_h]$ for each $h \in [t]$ such that for each $X \in [Y_{h-1}, Y_h]$,

- $|\phi_2(X)| = |\phi_1(X)| = |\phi_1(\beta_h(X))| = |\phi_2(\beta_h(X))|$, and
- $r_2(\phi_2(X)) = r_1(\phi_1(X)) = r_1(\phi_1(\beta_h(X))) = r_2(\phi_2(\beta_h(X)))$.

(The bijection β_h is a restriction of α_j if $Y_t \subseteq H$ and $C' = \alpha_j(C)$, otherwise β_h is a restriction of α_j^{-1} .) So the following minors of M_1 and M_2 have the same configuration:

- the restrictions $M_2|_{\phi_2(Y_1)}$, $M_1|_{\phi_1(Y_1)}$, $M_1|_{\phi_1(Y'_1)}$, and $M_2|_{\phi_2(Y'_1)}$, and
- the minors $M_2|_{\phi_2(Y_h)/\phi_2(Y_{h-1})}$, $M_1|_{\phi_1(Y_h)/\phi_1(Y_{h-1})}$, $M_1|_{\phi_1(Y'_h)/\phi_1(Y'_{h-1})}$, and $M_2|_{\phi_2(Y'_h)/\phi_2(Y'_{h-1})}$ for each h with $2 \leq h \leq t$.

It follows that the same conclusions hold for any two chains in the same component of G . Let C_1, C_2, \dots, C_s be the vertices in a connected component of G . We complete the proof by showing that Lemma 3.7 applies to $\phi_1(C_1), \phi_1(C_2), \dots, \phi_1(C_s)$ in M_1 and $\phi_2(C_1), \phi_2(C_2), \dots, \phi_2(C_s)$ in M_2 . Property (a) in that lemma follows from what we just deduced. For $k \in [s]$, let Z_k be the largest set in the chain C_k . The sets Z_1, Z_2, \dots, Z_s

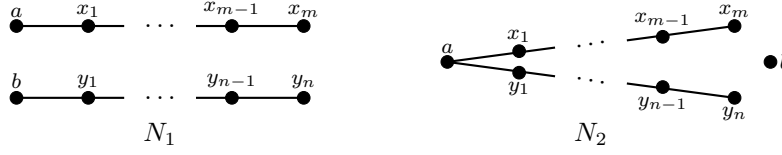


FIGURE 5. The rank-3 paving matroids in Example 4.

need not be distinct. Consider the map

$$\Phi : \bigcup_{k \in [s]} (\text{ch}((Z_k, E)_{L_1}) \times \{C_k\}) \rightarrow \bigcup_{k \in [s]} (\text{ch}((Z_k, E)_{L_2}) \times \{C_k\})$$

given as follows: for a chain $D = \{W_1 \subsetneq W_2 \subsetneq \dots \subsetneq W_s\}$ in the interval $(Z_k, E)_{L_1}$,

$$\Phi((D, C_k)) = \begin{cases} (\alpha_j(D), \alpha_j(C_k)), & \text{if } W_s \in A_j, \\ (D, C_k), & \text{otherwise.} \end{cases}$$

It is easy to see that Φ is a bijection. When we apply ϕ_1 and ϕ_2 , property (b) in Lemma 3.7 follows by the same type of argument as in the second paragraph. \square

When two paving matroids N_1 and N_2 on E have the same configuration, partitions $\{A_1, A_2, \dots, A_p\}$ and $\{B_1, B_2, \dots, B_p\}$ and bijections $\alpha_j : E \rightarrow E$, for $j \in [p]$, that satisfy the hypothesis of Theorem 5.1 exist since the blocks can be singletons and the multiset of cardinalities of the hyperplanes will be the same for N_1 as for N_2 . However, poorly-chosen partitions and bijections can restrict the size-rank data in $\mathcal{Z}(M_1)$ and $\mathcal{Z}(M_2)$ more than necessary. We illustrate two efficient choices of partitions and bijections, as well as how the matroids M_1 and M_2 can be constructed.

Example 4. Fix integers $m \geq 2$ and $n \geq 2$. Let N_1 and N_2 be the rank-3 paving matroids on the set $\{a, b, x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n\}$ where

- the dependent hyperplanes of N_1 are $\{a, x_1, x_2, \dots, x_m\}$ and $\{b, y_1, y_2, \dots, y_n\}$, and
- the dependent hyperplanes of N_2 are $\{a, x_1, x_2, \dots, x_m\}$ and $\{a, y_1, y_2, \dots, y_n\}$.

These paving matroids are shown in Figure 5. Now

- $\mathcal{H}_1 - \mathcal{H}_2 = \{\{b, y_1, y_2, \dots, y_n\}, \{a, y_1\}, \{a, y_2\}, \dots, \{a, y_n\}\}$, and
- $\mathcal{H}_2 - \mathcal{H}_1 = \{\{a, y_1, y_2, \dots, y_n\}, \{b, y_1\}, \{b, y_2\}, \dots, \{b, y_n\}\}$.

We partition each of these into a single block. A bijection α that has the properties in Theorem 5.1 is the 2-cycle (a, b) .

With this pair of rank-3 paving matroids N_1 and N_2 , we illustrate how straightforward it is to construct matroids M_1 and M_2 that satisfy the hypotheses of Theorem 5.1 and so have different configurations and the same \mathcal{G} -invariant. It is well known that a matroid M is determined by the set $\{(X, r(X)) : X \in \mathcal{Z}(M)\}$. We construct M_1 and M_2 by giving their cyclic flats and the ranks of these sets; this approach is justified by the following result [4, Theorem 3.2]; see also [12].

Lemma 5.2. For a set \mathcal{Z} of subsets of a set E and a function $r : \mathcal{Z} \rightarrow \mathbb{Z}$, there is a matroid M on E with $\mathcal{Z}(M) = \mathcal{Z}$ and $r_M(X) = r(X)$ for all $X \in \mathcal{Z}$ if and only if

- (Z0) (\mathcal{Z}, \subseteq) is a lattice,
- (Z1) $r(0_{\mathcal{Z}}) = 0$, where $0_{\mathcal{Z}}$ is the least set in \mathcal{Z} ,
- (Z2) $0 < r(Y) - r(X) < |Y - X|$ for all $X, Y \in \mathcal{Z}$ with $X \subsetneq Y$, and

(Z3) $r(X \vee Y) + r(X \wedge Y) + |(X \cap Y) - (X \wedge Y)| \leq r(X) + r(Y)$ for $X, Y \in \mathcal{Z}$.

Let $S = \{A, B, X_1, X_2, \dots, X_m, Y_1, Y_2, \dots, Y_n\}$ be a set of pairwise disjoint sets, each with at least seven elements, and with $|A| = |B|$. Let $a \in E(N_1)$ correspond to $A \in S$, and likewise for the other elements and sets. Let E be the union of the sets in S . Let \mathcal{Z}_1 be the set of unions of sets in S that correspond to flats of N_1 , and let \mathcal{Z}_2 be the set of unions of sets in S that correspond to flats of N_2 . Thus, \mathcal{Z}_1 and \mathcal{Z}_2 are lattices. Define $r_1 : \mathcal{Z}_1 \rightarrow \mathbb{Z}$ and $r_2 : \mathcal{Z}_2 \rightarrow \mathbb{Z}$ as follows.

- Set $r_1(\emptyset) = r_2(\emptyset) = 0$.
- Set $r_1(E) = r_2(E) = 10$.
- For each set $Z \in S$, pick an $i \in \{5, 6\}$ and set $r_1(Z) = r_2(Z) = i$, where, in addition, $r_1(A) = r_2(A) = r_1(B) = r_2(B)$.
- Pick an $i \in \{8, 9\}$ and set $r_1(A \cup B) = r_2(A \cup B) = i$.
- For each $j \in [m]$, pick an $i \in \{8, 9\}$ and set $r_1(B \cup X_j) = r_2(B \cup X_j) = i$.
- For each $j \in [n]$, pick an $i \in \{8, 9\}$ and set $r_1(A \cup Y_j) = r_2(A \cup Y_j) = i$.
- For each $j \in [m]$ and $k \in [n]$, pick an $i \in \{8, 9\}$ and set $r_1(X_j \cup Y_k) = r_2(X_j \cup Y_k) = i$.
- Pick an $i \in \{8, 9\}$ and set $r_1(A \cup X_1 \cup \dots \cup X_m) = r_2(A \cup X_1 \cup \dots \cup X_m) = i$.
- Pick an $i \in \{8, 9\}$ and set $r_1(B \cup Y_1 \cup \dots \cup Y_n) = r_2(B \cup Y_1 \cup \dots \cup Y_n) = i$.

It is easy to see that properties (Z0)–(Z3) hold for the pairs (\mathcal{Z}_1, r_1) and (\mathcal{Z}_2, r_2) , so this yields matroids M_1 and M_2 to which Theorem 5.1 applies.

Even within the narrow numerical range in this example, there can be more flexibility than presented above. For instance,

- if a set in S has rank five, it might have only six elements;
- a set in S can have rank seven (in which case it must have at least eight elements) provided that it is not the intersection of two cyclic flats that each have rank eight;
- sets in S could have rank four provided that, for any two such sets, sets that contain their union and correspond to lines of N_1 or N_2 have rank eight;
- if proper care is taken when assigning ranks (so that condition (Z3) will hold), the sets in S do not have to be pairwise disjoint (for a smaller example of this type, compare N_1 and N_2);
- while we considered just certain unions of the sets in S , we could allow certain supersets of these unions.

Combining these factors with allowing other ranks yields an abundance of matroid M_1 and M_2 to which Theorem 5.1 applies, and the same can be done for any paving matroid N_1 and N_2 .

In the concluding example, we focus exclusively on an efficient choice of the partitions $\{A_1, A_2, \dots, A_p\}$ and $\{B_1, B_2, \dots, B_p\}$ and the bijections $\alpha_j : E \rightarrow E$, for $j \in [p]$.

Example 5. Let N_1 and N_2 be the rank-4 paving matroids on $\{a, b, c, d, e, f, g, p, r, s, t, u\}$ where

- the dependent hyperplanes of N_1 are $\{a, b, c, d\}$, $\{a, b, e, f\}$, and $\{c, d, e, f\}$, and
- the dependent hyperplanes of N_2 are $\{a, b, p, q\}$, $\{c, d, r, s\}$, and $\{e, f, t, u\}$.

These paving matroids are shown in Figure 6. We partition $\mathcal{H}_1 - \mathcal{H}_2$ into

- $A_1 = \{\{a, b, e, f\}, \{a, b, p\}, \{a, b, q\}, \{a, p, q\}, \{b, p, q\}\}$,
- $A_2 = \{\{a, b, c, d\}, \{c, d, r\}, \{c, d, s\}, \{c, r, s\}, \{d, r, s\}\}$,
- $A_3 = \{\{c, d, e, f\}, \{e, f, t\}, \{e, f, u\}, \{e, t, u\}, \{f, t, u\}\}$,

and $\mathcal{H}_1 - \mathcal{H}_2$ into

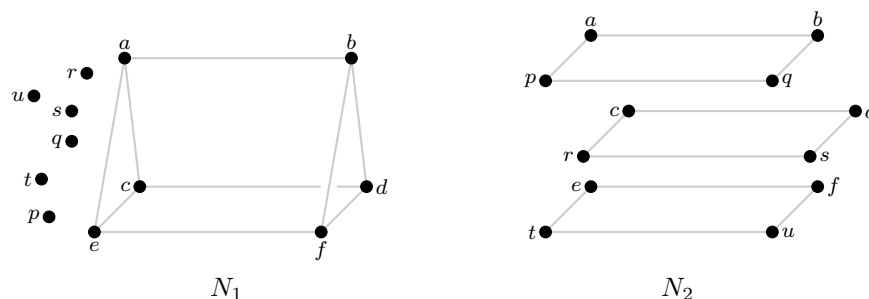


FIGURE 6. The rank-4 paving matroids in Example 5.

- $B_1 = \{\{a, b, p, q\}, \{a, b, e\}, \{a, b, f\}, \{a, e, f\}, \{b, e, f\}\},$
- $B_2 = \{\{c, d, r, s\}, \{c, d, a\}, \{c, d, b\}, \{c, a, b\}, \{d, a, b\}\},$
- $B_3 = \{\{e, f, t, u\}, \{e, f, c\}, \{e, f, d\}, \{e, c, d\}, \{f, c, d\}\}.$

Three bijections that have the properties in Theorem 5.1, given as products of cycles, are $\alpha_1 = (e, p)(f, q)$, $\alpha_2 = (a, r)(b, s)$, and $\alpha_3 = (c, t)(d, u)$.

The lattices of flats, L_1 and L_2 , of N_1 and N_2 have 291 elements, and the isomorphisms $\phi_1 : L_1 \rightarrow \mathcal{Z}(M_1)$ and $\phi_2 : L_2 \rightarrow \mathcal{Z}(M_2)$ must, for instance, assign to $\{e\}$ and $\{p\}$ cyclic flats of the same size and rank, and likewise for $\{a, e\}$ and $\{a, p\}$.

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REFERENCES

- [1] N. Bansal, R.A. Pendavingh, and J. van der Pol, On the number of matroids, *Combinatorica* **35** (2015) 253–277.
- [2] J. Bonin and J.P.S. Kung, The \mathcal{G} -invariant and catenary data of a matroid, *Adv. in Appl. Math.* **94** (2018) 39–70.
- [3] J. Bonin and K. Long, The free m -cone of a matroid and its \mathcal{G} -invariant, submitted.
- [4] J. Bonin and A. de Mier, The lattice of cyclic flats of a matroid, *Ann. Comb.* **12** (2008) 155–170.
- [5] J. Bonin and A. de Mier, Tutte uniqueness and Tutte equivalence, In [10] (in press; 2022).
- [6] T. Brylawski and J. Oxley, The Tutte Polynomial and Its Applications, In N. White (ed), *Matroid Applications* (Encyclopedia of Mathematics and its Applications, pp. 123–225). Cambridge: Cambridge University Press (1992).
- [7] H. Derksen, Symmetric and quasi-symmetric functions associated to polymatroids, *J. Algebraic Combin.* **30** (2009) 43–86.
- [8] H. Derksen and A. Fink, Valuative invariants for polymatroids, *Adv. Math.* **225** (2010) 1840–1892
- [9] J. Eberhardt, Computing the Tutte polynomial of a matroid from its lattice of cyclic flats, *Electron. J. Combin.* **21** (2014) Paper 3.47, 12 pp.
- [10] J.A. Ellis-Monaghan and I. Moffatt (eds), *Handbook of the Tutte Polynomial and Related Topics*. (in press) CRC Press/Taylor & Francis (2022).
- [11] J. Oxley, *Matroid Theory*, second edition (Oxford University Press, Oxford, 2011).
- [12] J.A. Sims, *Some Problems in Matroid Theory*, (Ph.D. Dissertation, Linacre College, Oxford University, Oxford, 1980).

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