# SIZE FUNCTIONS OF SUBGEOMETRY-CLOSED CLASSES OF REPRESENTABLE COMBINATORIAL GEOMETRIES

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ABSTRACT. Let $ex_q(G;n)$ be the maximum number of points in a rank-$n$ geometry (simple matroid) that is representable over $GF(q)$ and that has no restriction isomorphic to the geometry $G$. We find $ex_q(G;n)$ for several infinite families of geometries $G$, and we show that if $G$ is a binary affine geometry, then

$$\lim_{n\to\infty} \frac{ex_2(G;n)}{2^n-1} = 0.$$

## 1. INTRODUCTION.

Let $EX_q(G;n)$ be the class of $GF(q)$-representable geometries (simple matroids) of rank $n$ that have no restriction (subgeometry) isomorphic to the geometry $G$. We refer to $G$ as the *excluded subgeometry* of the class $EX_q(G;n)$. In contrast to most classes considered in [3], while subgeometries of members of $\cup_{n\geq 1} EX_q(G;n)$ are in $\cup_{n\geq 1} EX_q(G;n)$, the class $\cup_{n\geq 1} EX_q(G;n)$ need not be closed under minors. Let $ex_q(G;n)$ be the maximum number of points in a geometry in $EX_q(G;n)$. Thus, $ex_q(G;n)$ is the size function (see [3]) of the intersection of the class of geometries that are representable over $GF(q)$ and the class of geometries formed by excluding $G$ as a restriction. Let $MAX(EX_q(G;n))$ denote the set of (isomorphism types of) geometries in $EX_q(G;n)$ with $ex_q(G;n)$ points. The following theorem of Bose and Burton [1] gives $ex_q(G;n)$ and $MAX(EX_q(G;n))$ when $G$ is a projective geometry of order $q$.

**Theorem 1.** *For all $n \geq m \geq 2$, we have*

$$ex_q(PG(m-1,q);n) = \frac{q^n - q^{n-m+1}}{q-1}.$$

*Furthermore, the deletion $PG(n-1,q)\backslash PG(n-m,q)$ of $PG(n-1,q)$ is the only $GF(q)$-representable geometry of rank $n$ with no restriction isomorphic to $PG(m-1,q)$ and with $(q^n - q^{n-m+1})/(q-1)$ points.*

Several theorems in matroid theory, including results of Oxley [5], have elements in common with Theorem 1. In this paper, we extend Theorem 1 in several new directions. In Section 2, we give exact values for $ex_q(G;n)$ and find $MAX(EX_q(G;n))$ for several infinite families of geometries $G$. Among the geometries $G$ treated are: rank-$m$ geometries that have critical exponent $m-1$ and a hyperplane whose deletion yields a free matroid (Theorem 3; note that the cycle matroid $M(K_5)$ is such a geometry); a large class of geometries containing the Reid geometries (Theorem 11 and Corollary 12); the geometry formed by restricting $PG(m-1,q)$ to the union of the hyperplanes spanned by the single-element deletions of some fixed basis of $PG(m-1,q)$ (Theorem 15 and Corollary 16); and, for $q = 2$, all rank-$m$ binary geometries that have a hyperplane isomorphic to $PG(m-2,2)$ (Theorems 6 and 14). In Section 3, we show that if $G$ is a binary affine geometry, then $ex_2(G;n)$ is relatively small; more precisely,

$$\lim_{n\to\infty} \frac{ex_2(G;n)}{2^n-1} = 0.$$

Our notation and terminology follow [6] with the following additions. We refer to simple matroids as geometries (short for combinatorial geometries). Hence restrictions of geometries are also called subgeometries.

Our results are about geometries that are representable over finite fields; in particular, the results are independent of any embeddings of the geometries in projective geometries. However, we often want to use counting results that follow for any embedding of a particular geometry in a projective geometry. To avoid having to refer repeatedly to embeddings, we simply treat such geometries as subgeometries of projective geometries. So that this does not create confusion, we use $\mathrm{cl}_P$ to denote the closure operator of the ambient projective geometry while cl denotes the closure operator on the subgeometry of interest. In cases where ambiguity might otherwise be possible, we use $\mathrm{cl}_M$ to denote the closure operator of $M$. Consistent with [6], hyperplanes of a geometry refer to the hyperplanes of the geometry, rather than to those of the ambient projective geometry. We refer to a subgeometry of $M$ that is isomorphic to $G$ as a $G$-subgeometry of $M$. Since we are concerned with counting the number of points in geometries, we follow the convention of [3] for contractions: in this paper, $M/X$ denotes the simplification of the usual contraction. Thus, $M/X$ is always a geometry. The closure operator of $M/X$ is denoted $\mathrm{cl}_{M/X}$.

To close this introduction, we note that there can be geometries in $EX_q(G;n)$ that have no proper extension in $EX_q(G;n)$ and yet have fewer than $ex_q(G;n)$ points, and so are not in $MAX(EX_q(G;n))$. In other words, not all extremal matroids of the class $EX_q(G;n)$ are in $MAX(EX_q(G;n))$. (A similar phenomenon occurs in the context of the minor-closed classes considered in [3].) For instance, consider excluding the uniform matroid $U_{2,3}$ as a subgeometry of binary geometries; Theorem 1 gives $ex_2(U_{2,3};n) = 2^{n-1}$. However the 5-circuit $U_{4,5}$ is binary, has no $U_{2,3}$-subgeometry, and has no proper extension in $EX_2(U_{2,3};4)$, yet $U_{4,5}$ has only five points.

## 2. Exact values for $ex_q(G;n)$.

In this section, we find $ex_q(G;n)$ and $MAX(EX_q(G;n))$ for several infinite families of geometries $G$. Several of the ideas in the proof of our first result, Theorem 3, can be found in the original proof of Theorem 1. Theorem 3 uses the following lemma [3, Corollary 3.3].

**Lemma 2.** *Let $G$ be a $GF(q)$-representable geometry with rank $m$ and critical exponent $c$ over $GF(q)$. Then*

$$\frac{q^n - q^{n-c+1}}{q-1} \le ex_q(G;n) \le \frac{q^n - q^{n-m+1}}{q-1}.$$

The critical exponent of a $GF(q)$-representable geometry $G$ is the least number of subgeometries into which $G$ can be partitioned, with each subgeometry affine over $GF(q)$. (For background on critical exponents, see, e.g., [4].) The lower bound in Lemma 2 follows since $PG(n-1,q)\backslash PG(n-c,q)$ has critical exponent $c-1$ and so cannot contain $G$ since $G$ has critical exponent $c$. The upper bound follows from Theorem 1.

**Theorem 3.** *Let $m \ge 4$. Let $G$ be a $GF(q)$-representable geometry of rank $m$ having critical exponent $m-1$ over $GF(q)$ and having a hyperplane $H$ for which the deletion $G\backslash H$ is a free matroid. Then, for $n \ge m$, we have*

$$ex_q(G;n) = \frac{q^n - q^{n-m+2}}{q-1}$$

*and*

$$MAX(EX_q(G;n)) = \{PG(n-1,q)\backslash PG(n-m+1,q)\}.$$

*Proof.* By Lemma 2 and the assumption that $G$ has critical exponent $m-1$ over $GF(q)$, we have $ex_q(G;n) \ge (q^n - q^{n-m+2})/(q-1)$. Let $M$ be a $GF(q)$-representable geometry on a set $S$ having rank $n$ and no $G$-subgeometry. If $M$ has no $PG(m-2,q)$-subgeometry, then, by Theorem 1,

$$|S| \le \frac{q^n - q^{n-m+2}}{q-1}. \tag{1}$$

Furthermore, Theorem 1 also gives that $PG(n-1,q)\backslash PG(n-m+1,q)$ (which has critical exponent $m-2$ and so cannot have a subgeometry isomorphic to $G$) is the only rank-$n$ geometry with no subgeometry isomorphic to $PG(m-2,q)$ for which equality holds in (1). Thus to complete the proof, we need only prove strict inequality in the case in which $M$ has a flat $T$ with the restriction $M|T$ isomorphic to $PG(m-2,q)$. There are at most $(q^{n-m+1}-1)/(q-1)$ rank-$m$ flats $X$ of $M$ containing $T$. Since each such $X$ cannot have $m$ independent elements in $X-T$, the restriction $M|(X-T)$ is affine of rank at most $m-1$, so $|X-T| \leq q^{m-2}$. Therefore

$$|S| \leq \frac{q^{n-m+1}-1}{q-1}q^{m-2} + \frac{q^{m-1}-1}{q-1}.$$

Since $n \geq m \geq 4$, we get

$$|S| < \frac{q^n - q^{n-m+2}}{q-1}.$$

□

For $q=2$, the cycle matroid $M(K_5)$ of the complete graph on five vertices satisfies the hypothesis of Theorem 3; the required hyperplanes are those isomorphic to $M(K_4)$. Thus we have the following corollary.

**Corollary 4.** *For $n \geq 4$, we have $ex_2(M(K_5);n) = 2^{n-1}+2^{n-2}$. Furthermore, $PG(n-1,2)\backslash PG(n-3,2)$ is the only binary geometry of rank $n$ with $2^{n-1}+2^{n-2}$ points and no $M(K_5)$-subgeometry.*

Theorem 3 does not apply to $ex_2(M(K_4);n)$. The cycle matroid $M(K_4)$ is the unique single-element deletion $PG(2,2)\backslash x$ of the projective plane of order 2. Theorem 6 finds $ex_2(PG(m-1,2)\backslash x;n)$ and $MAX(EX_2(PG(m-1,2)\backslash x;n))$ for all $n \geq m \geq 3$.

To motivate this, we examine the two geometries Theorem 6 says are in $MAX(EX_2(PG(m-1,2)\backslash x;n))$. Consider $PG(n-1,2)\backslash\big(PG(n-m+1,2)-y\big)$, the geometry formed by deleting all but one point in some fixed flat of rank $n-m+2$ in the rank-$n$ binary projective geometry. Since any rank-$m$ flat of $PG(n-1,2)$ intersects the fixed rank-$(n-m+2)$ flat in at least a line (hence, in at least three points), at least two points have been removed from such a flat in the deletion, so $PG(n-1,2)\backslash\big(PG(n-m+1,2)-y\big)$ has no $(PG(m-1,2)\backslash x)$-subgeometry. Consider $PG(n-1,2)\backslash\big(PG(n-m,2)\oplus PG(n-m,2)\big)$ for $n$ in the range $m+1 \leq n \leq 2m-2$. This geometry is formed by deleting two fixed disjoint flats of rank $n-m+1$ in the rank-$n$ binary projective geometry. (It is clear that there is, up to isomorphism, only one such deletion of $PG(n-1,2)$. Note that $n \leq 2m-2$ guarantees that there are two such disjoint flats in $PG(n-1,2)$.) Any rank-$m$ flat of $PG(n-1,2)$ intersects each of the fixed rank-$(n-m+1)$ flats in at least one point, so at least two points have been removed from such a flat in the deletion, so $PG(n-1,2)\backslash\big(PG(n-m,2)\oplus PG(n-m,2)\big)$ has no $(PG(m-1,2)\backslash x)$-subgeometry.

The proofs of Theorems 6 and 7 use the characterization of $PG(n-1,q)\backslash\big(PG(n-m,q)\oplus PG(n-m,q)\big)$ in Lemma 5. To motivate the lemma, consider the following. Let $S_1$ and $S_2$ be disjoint rank-$(n-m+1)$ flats of $PG(n-1,q)$ where $n-m+1$ is at least 2. Thus, $m+1 \leq n \leq 2m-2$. Let $B_1 = \{p_1, p_2, \ldots, p_{n-m+1}\}$ and $B_2 = \{p'_1, p'_2, \ldots, p'_{n-m+1}\}$ be bases of $S_1$ and $S_2$ respectively. For each $i = 1, 2, \ldots, n-m+1$, let $q_i$ be a point of $\mathrm{cl}_P(\{p_i, p'_i\}) - \{p_i, p'_i\}$. Let $B = \{q_1, q_2, \ldots, q_{n-m+1}\}$ and let $S_3 = \mathrm{cl}_P(B)$. Since $\mathrm{cl}_P(B_1 \cup B) = \mathrm{cl}_P(B_1 \cup B_2)$, and this is a flat of rank $2(n-m+1)$, it follows that $S_3$ is a flat of $PG(n-1,q)$ of rank $n-m+1$. Extend $B_1 \cup B$ to a basis $B'$ of $PG(n-1,q)$ and let $T$ be the rank-$(m-1)$ flat $\mathrm{cl}_P(B'-B_1)$. By the modular law in $PG(n-1,q)$, each rank-$m$ flat of $PG(n-1,q)$ containing $T$ contains precisely one element of each of $S_1$ and $S_2$. Likewise, for each rank-$(m+1)$ flat $F$ of $PG(n-1,q)$ containing $T$, both $F \cap S_1$ and $F \cap S_2$ are lines of $PG(n-1,q)$; thus for such a flat $F$, the restriction $PG(n-1,q)|(F \cap (S_1 \cup S_2))$ is isomorphic to $U_{2,q+1} \oplus U_{2,q+1}$. In Lemma 5, the last two properties are used to characterize the geometry $PG(n-1,q)\backslash\big(PG(n-m,q)\oplus PG(n-m,q)\big)$.

**Lemma 5.** *Assume $m+1 \leq n \leq 2m-2$. A subset $S$ of $PG(n-1,q)$ is the union of two disjoint flats of $PG(n-1,q)$, each having rank $n-m+1$, if and only if there is a rank-$(m-1)$ flat $T$ of $PG(n-1,q)$ satisfying these conditions:*

(1) *each rank-$m$ flat of $PG(n-1,q)$ containing $T$ contains precisely two elements of $S$, and*

(2) *for each rank-*$(m+1)$ *flat* $F$ *of* $PG(n-1,q)$ *containing* $T$*, the restriction* $PG(n-1,q)|(F \cap S)$ *is isomorphic to* $U_{2,q+1} \oplus U_{2,q+1}$.

*Proof.* Half of this has been shown, so assume there is a rank-$(m-1)$ flat $T$ of $PG(n-1,q)$ satisfying (1) and (2). It follows from (1) and (2) that $S$ and $T$ are disjoint. From (1), we get $|S| = 2(q^{n-m+1}-1)/(q-1)$. We want to show that $S$ can be partitioned into two subsets, $S_1$ and $S_2$, each with $(q^{n-m+1}-1)/(q-1)$ points, and each of which is a flat of $PG(n-1,q)$.

The following labeling of the rank-$m$ flats of $PG(n-1,q)$ containing $T$ will be useful. By (1), each such flat $X$ contains two elements, say $x$ and $x'$, of $S$. The line $\mathrm{cl}_P(\{x,x'\})$ contains a unique point, say $a$, of $T$; label the flat $X$ with $a$. Let $T'$ be the set of labels used. Thus $T' \subseteq T$.

By (2), it follows that no two rank-$m$ flats of $PG(n-1,q)$ containing $T$ receive the same label. Let $F$ be a rank-$(m+1)$ flat of $PG(n-1,q)$ containing $T$, and let $X_1, X_2, \ldots, X_{q+1}$ be the rank-$m$ flats of $PG(n-1,q)$ with $T \subset X_i \subset F$. Note that $r(F \cap S) = 4$ by (2), so the modular law in $PG(n-1,q)$ gives $r(\mathrm{cl}_P(F \cap S) \cap T) = 2$. Thus, the labels assigned to the flats $X_1, X_2, \ldots, X_{q+1}$ are precisely the points on a line of $PG(n-1,q)$. Therefore $T'$ is a flat of $PG(n-1,q)$. Furthermore, in this way there is a bijection between the lines of $PG(n-1,q)$ contained in $T'$ and the rank-$(m+1)$ flats of $PG(n-1,q)$ containing $T$.

By (2), the $2q+2$ points of $S$ in a rank-$(m+1)$ flat of $PG(n-1,q)$ containing $T$ are partitioned into two $(q+1)$-point lines. With this, we partition $S$ as follows. Fix a rank-$m$ flat $X$ containing $T$ and let $S \cap X = \{a_1, a_2\}$. For every point $b$ in $S - \{a_1, a_2\}$, the set $\mathrm{cl}_P(T \cup \{a_1, a_2, b\})$ is a rank-$(m+1)$ flat containing $T$, so $b$ is on a $(q+1)$-point line in $S$ with precisely one of $a_1$ or $a_2$. Thus, for $k = 1, 2$, let

$$S_k = \{a_k\} \cup \{x \mid x \in S \text{ and } \mathrm{cl}_P(\{a_k, x\}) \text{ is a } (q+1)\text{-point line in } S\}.$$

It follows that $\{S_1, S_2\}$ is a partition of $S$ with $|S_k| = (q^{n-m+1}-1)/(q-1)$. It remains to show that both $S_1$ and $S_2$ are flats of $PG(n-1,q)$.

By symmetry, it suffices to treat $S_1$. For this, it suffices to show that for each pair $y, z \in S_1$, the line $\mathrm{cl}_P(\{y,z\})$ is in $S_1$. By the construction of $S_1$, the case with $a_1 \in \mathrm{cl}_P(\{y,z\})$ holds, so we focus on the case with $a_1 \notin \mathrm{cl}_P(\{y,z\})$.

The following labeling of the elements of $S$ will be useful. For each point $b \in T'$, there are precisely two points of $S$ with $b$ in the line of $PG(n-1,q)$ spanned by these two points; one of these points is in $S_1$ while the other is in $S_2$; the former will be denoted by $b_1$ and the latter by $b_2$. With this labeling, we have the following immediate consequence of (2).

> **(5.1)** A plane in $PG(n-1,q)|S$ that contains some pair $b_1, b_2$ has precisely one line with more than two points.

Assume that the elements $b_1$ and $d_1$ of $S_1$ are not collinear with $a_1$. Note that the corresponding points $a, b, d$ of $T'$ span a plane $\pi$ of $PG(n-1,q)$. To show that $\mathrm{cl}_P(\{b_1, d_1\})$ is contained in $S_1$, it suffices to show that for each line $\ell$ of $\pi$ containing $a$, the corresponding line $\ell'$ through $a_1$ contains a point of $\mathrm{cl}_P(\{b_1, d_1\})$. For this, it suffices to show that the three lines $\mathrm{cl}_P(\{a_1, b_1\})$, $\mathrm{cl}_P(\{a_1, d_1\})$, and $\ell'$ are coplanar. This is what we turn to now.

Assume that $\mathrm{cl}_P(\{a_1, b_1\})$, $\mathrm{cl}_P(\{a_1, d_1\})$, and $\ell'$ are not coplanar. Assume that in $\pi$, the points $b$ and $d$ are collinear with $g$ on $\ell$. Therefore precisely one of the following four pairs contains collinear points of $S$: $\{b_1, d_1, g_1\}$ and $\{b_2, d_2, g_2\}$; $\{b_1, d_1, g_2\}$ and $\{b_2, d_2, g_1\}$; $\{b_1, d_2, g_1\}$ and $\{b_2, d_1, g_2\}$; $\{b_2, d_1, g_1\}$ and $\{b_1, d_2, g_2\}$. Hence precisely one of the four sets $\{b_1, d_1, g_1\}$, $\{b_1, d_1, g_2\}$, $\{b_1, d_2, g_1\}$, and $\{b_2, d_1, g_1\}$ has rank 2. That $\mathrm{cl}_P(\{a_1, b_1\})$, $\mathrm{cl}_P(\{a_1, d_1\})$, and $\ell'$ are not coplanar yields $r(\{b_1, d_1, g_1\}) > 2$. By the symmetry of the remaining cases, we may assume $r(\{b_1, d_1, g_2\}) = 2$. Therefore $r(\{b_2, d_2, g_1\}) = 2$. Let $c$ be in $\mathrm{cl}_P(\{a,b\}) - \{a,b\}$, and assume $c$ and $d$ are collinear with $f$ on $\ell$. As above, precisely one of $\{c_1, d_1, f_2\}$, $\{c_1, d_2, f_1\}$, and $\{c_2, d_1, f_1\}$ has rank 2. Note that $r(\{c_1, d_1, f_2\}) > 2$, for otherwise $\mathrm{cl}_P(\{b_1, c_1, d_1\})$ is a plane that contains both $g_2$ and $f_2$, and so contains $a_2$ as well as $a_1$, contrary to (5.1). This leaves two options. To complete the proof, we show that both lead to contradictions.

First assume that $\{c_1, d_2, f_1\}$ has rank 2. Assume that in the plane $\pi$, the points $b$ and $f$ are collinear with $e$ on $\text{cl}_P(\{a, d\})$. As above, precisely one of $\{b_1, e_1, f_2\}$, $\{b_1, e_2, f_1\}$, and $\{b_2, e_1, f_1\}$ has rank 2. If $r(\{b_1, e_1, f_2\}) = 2$, then $f_2$ and $g_2$, and hence $a_2$, are in $\text{cl}_P(\{b_1, e_1, d_1\})$, as is $a_1$, contrary to (5.1). If $r(\{b_1, e_2, f_1\}) = 2$, then $e_2$ and $d_2$, and hence $a_2$ as well as $a_1$, are in the plane $\text{cl}_P(\{b_1, c_1, f_1\})$, contrary to (5.1). If $r(\{b_2, e_1, f_1\}) = 2$, then $g_1$ and $d_2$, and so $b_2$ as well as $b_1$, are in the plane $\text{cl}_P(\{a_1, c_1, f_1\})$, contrary to (5.1).

Finally, assume that $\{c_2, d_1, f_1\}$ has rank 2. Again assume that in the plane $\pi$, the points $b$ and $f$ are collinear with $e$ on $\text{cl}_P(\{a, d\})$. If $r(\{b_1, e_1, f_2\}) = 2$, then $f_2$ and $g_2$, and hence $a_2$ as well as $a_1$, are in $\text{cl}_P(\{b_1, e_1, d_1\})$, contrary to (5.1). If $r(\{b_2, e_1, f_1\}) = 2$, then $b_2$ and $c_2$, and hence $a_2$ as well as $a_1$, are in $\text{cl}_P(\{f_1, e_1, d_1\})$, contrary to (5.1). Therefore $r(\{b_1, e_2, f_1\}) = 2$. Assume that in the plane $\pi$, the points $c$ and $g$ are collinear with $h$ (which may be $e$) on $\text{cl}_P(\{a, d\})$. If $r(\{c_1, h_1, g_2\}) = 2$, then $r(\{c_2, h_2, g_1\}) = 2$, so $h_2$ is in $\text{cl}_P(\{c_2, f_1, g_1\})$, as is $a_1$ and $d_1$, and so $h_1$, contrary to (5.1). If $r(\{c_2, h_1, g_1\}) = 2$, then $h_1$ is in $\text{cl}_P(\{c_2, b_2, g_1\})$, as are $a_2$ and $d_2$, and hence $h_2$, contrary to (5.1). Lastly, $g_1$, $h_1$, and $c_2$ are in $\text{cl}_P(\{a_1, d_1, f_1\})$, so if $r(\{c_1, h_2, g_1\}) = r(\{c_2, h_1, g_2\}) = 2$, then $g_2$ is also in $\text{cl}_P(\{a_1, d_1, f_1\})$, contrary to (5.1). □

Using this lemma, we now find $ex_2(G; n)$ and $MAX(EX_2(G; n))$ where $G$ is a single-point deletion of a binary projective geometry.

**Theorem 6.** *Assume $n \geq m \geq 3$. Then*

$$ex_2(PG(m-1,2)\backslash x; n) = 2^n - 2^{n-m+2} + 1.$$

*If $m+1 \leq n \leq 2m-2$, then $MAX(EX_2(PG(m-1,2)\backslash x; n))$ contains precisely two geometries,*

$$PG(n-1,2)\backslash\big(PG(n-m+1,2) - y\big)$$

*and*

$$PG(n-1,2)\backslash\big(PG(n-m,2) \oplus PG(n-m,2)\big);$$

*otherwise $MAX(EX_2(PG(m-1,2)\backslash x; n))$ contains only one geometry,*

$$PG(n-1,2)\backslash(PG(n-m+1,2) - y).$$

*Proof.* To treat the size function, note that the examples discussed before the statement of Lemma 5 show $ex_2(PG(m-1,2)\backslash x; n) \geq 2^n - 2^{n-m+2} + 1$, so we need to prove $ex_2(PG(m-1,2)\backslash x; n) \leq 2^n - 2^{n-m+2} + 1$. Assume that $M$ is a subgeometry of $PG(n-1,2)$ in $EX_2(PG(m-1,2)\backslash x; n)$. If $M$ has no subgeometry isomorphic to $PG(m-2,2)$, then, by Theorem 1, $M$ has at most $2^n - 2^{n-m+2}$ points. Assume $M$ has a flat $T$ with the restriction $M|T$ isomorphic to $PG(m-2,2)$. At most $2^{n-m+1} - 1$ rank-$m$ flats of $M$ contain $T$, and each of these has at most $2^m - 3$ points. Each of these rank-$m$ flats has at most $(2^m - 3) - (2^{m-1} - 1)$, or $2^{m-1} - 2$, points not in $T$, so $M$ has at most $(2^{n-m+1} - 1)(2^{m-1} - 2) + (2^{m-1} - 1)$, or $2^n - 2^{n-m+2} + 1$, points. Therefore $ex_2(PG(m-1,2)\backslash x; n) = 2^n - 2^{n-m+2} + 1$.

Assume $M$ is in $MAX(EX_2(PG(m-1,2)\backslash x; n))$. From the last paragraph, we deduce that $M$ has a flat $T$ such that the restriction $M|T$ is isomorphic to $PG(m-2,2)$, and each such flat $T$ is contained in exactly $2^{n-m+1} - 1$ rank-$m$ flats of $M$, each of which contains exactly $2^m - 3$ points. The case of $n = m$ is obvious: $MAX(EX_2(PG(m-1,2)\backslash x; m)) = \{PG(m-1,2)\backslash(PG(1,2) - y)\}$.

Consider $n = m+1$. Fix a flat $T$ of $M$ with $M|T$ isomorphic to $PG(m-2,2)$. The flat $T$ is contained in three hyperplanes of $M$, each of which is a two-element deletion of $PG(m-1,2)$. Thus, $M = PG(m,2)\backslash S$ where $S$ is a set of six points. To show that $MAX(EX_2(PG(m-1,2)\backslash x; m+1))$ is as claimed in the theorem, we need to show that $PG(m,2)|S$ is isomorphic to either $M(K_4)$ or $U_{2,3} \oplus U_{2,3}$. Assume the former fails, so $r(S) > 3$.

We claim that no plane of $PG(m,2)$ contains a 3-point line $\ell$ of $M$ and three or four points of $S$, for this would force $M$ to have a $(PG(m-1,2)\backslash x)$-subgeometry. To see this, assume there were such a plane $\pi$ and line $\ell$. If $m = 3$, then $\ell$ is contained in three planes of $PG(3,2)$, so at least one of these planes contains at most one point from $S$, yielding the excluded subgeometry $PG(2,2)\backslash x$ in $M$. If $m > 3$, then $\ell$ is contained in at least seven planes of $PG(m,2)$, so at least one of these, say $\pi'$, is disjoint from $S$. If $m = 4$, then $\pi'$ is contained in three rank-4 flats of $PG(4,2)$, so at least one of these rank-4 flats contains at most one point from $S$, yielding the excluded subgeometry $PG(3,2)\backslash x$ in $M$. If $m > 4$, we continue in this manner, eventually obtaining the excluded subgeometry $PG(m-1,2)\backslash x$. Thus, the claim holds.

It follows that no plane of $PG(m,2)$ contains five of the six points of $S$. Indeed, if there were such a plane $\pi$, then there is a 2-point line, say $\{a,b\}$, of $PG(m,2)|(S\cap\pi)$. Then $a$ and $b$ together with the point $c$ of $S-\pi$ span a plane of the type ruled out in the last paragraph. It follows from this and the last paragraph that each plane of the restriction $PG(m,2)|S$ is isomorphic to $U_{2,3}\oplus U_{1,1}$. From this, we deduce that $PG(m,2)|S$ is isomorphic to $U_{2,3}\oplus U_{2,3}$, as needed.

Now consider $n > m+1$. Fix a flat $T$ of $M$ with $M|T$ isomorphic to $PG(m-2,2)$. Since $T$ is contained in $2^{n-m+1}-1$ rank-$m$ flats of $M$, the contraction $M/T$ is isomorphic to $PG(n-m,2)$. Note that $M/T$ has rank at least three. A key tool in the remainder of the proof is the labeling of the points of $M/T$ that we now describe. The points of $M/T$ are the rank-$m$ flats of $M$ containing $T$. For each rank-$m$ flat $X$ of $M$ containing $T$, there is a rank-$m$ flat $X^*$ of $PG(n-1,2)$ and a pair of points $x, x'$ of $X^*$ with $X = X^* - \{x,x'\}$. Note that $\mathrm{cl}_P(\{x,x'\})$ intersects $T$ in a point. Label the point $X$ of $M/T$ with the point $a$ of $T$ if $\mathrm{cl}_P(\{x,x'\})\cap T = \{a\}$.

We consider how this labeling reflects the structure of the rank-$(m+1)$ flats of $M$ containing $T$. Let $X$ and $Y$ be rank-$m$ flats of $M$ containing $T$. Thus $\mathrm{cl}(X\cup Y)$ is a rank-$(m+1)$ flat of $M$ containing $T$. Therefore $\mathrm{cl}(X\cup Y)$ contains a third rank-$m$ flat $Z$ of $M$ containing $T$. Let $X = X^* - \{x,x'\}$, $Y = Y^* - \{y,y'\}$, and $Z = Z^* - \{z,z'\}$ for flats $X^*, Y^*, Z^*$ of $PG(n-1,2)$, and let $\{a\} = \mathrm{cl}_P(\{x,x'\})\cap T$, $\{b\} = \mathrm{cl}_P(\{y,y'\})\cap T$, and $\{c\} = \mathrm{cl}_P(\{z,z'\})\cap T$. By the work in the case $n = m+1$, we know that $PG(n-1,2)|\{x,x',y,y',z,z'\}$ is isomorphic to either $M(K_4)$ or $U_{2,3}\oplus U_{2,3}$. In the former case, $a = b = c$; in the latter case, $\{a,b,c\}$ is a line of $T$, namely, the intersection of the rank-4 flat $\mathrm{cl}_P(\{x,x',y,y',z,z'\})$ of $PG(n-1,2)$ with $T$. It follows that the set of points of $T$ used in the labeling of $M/T$ is a flat of $M|T$. The lines of $M/T$ correspond to rank-$(m+1)$ flats of $M$ containing $T$; the points on these lines for which the corresponding restriction $PG(n-1,2)|\{x,x',y,y',z,z'\}$ is isomorphic to $M(K_4)$ receive the same label; the points on these lines for which this restriction is isomorphic to $U_{2,3}\oplus U_{2,3}$ receive three different labels, with the labels being the points on a line of $T$.

Assume the labeling of $M/T$ has a line $L$ with all points of $L$ labeled $a$, and consider any other point $P$ of $M/T$. If $P$ were labeled $b\neq a$, then it follows that the three other points in the plane $\mathrm{cl}_{M/T}(L\cup P)$ are labeled with the third point, say $c$, on the line $\mathrm{cl}_M(\{a,b\})$ of $T$. However, the plane $\mathrm{cl}_{M/T}(L\cup P)$ then has three lines, each having two points labeled $c$ and one labeled $a$, contrary to the restrictions on labelings observed above. We deduce that if the points on any line of $M/T$ are labeled with the same label, then all points of $M/T$ have that label. It follows that either the restriction $PG(n-1,2)|(F\cap S)$ is isomorphic to $M(K_4)$ for every rank-$(m+1)$ flat $F$ of $M$ containing $T$ or every such restriction is isomorphic to $U_{2,3}\oplus U_{2,3}$.

Note that when $n\geq 2m-1$, there are $2^{m-1}-1$ points of $T$ to be used as possible labels for $M/T$ and at least $2^m-1$ points of $M/T$ to be labeled. It follows that at least two points of $M/T$ receive the same label, and so all points of $M/T$ receive the same label.

Let $M = PG(n-1,2)\backslash S$. Thus, $|S| = 2^{n-m+2}-2$. If no two points of $M/T$ receive the same label, then conditions (1) and (2) of Lemma 5 are satisfied, so $M$ is isomorphic to $PG(n-1,2)\backslash\big(PG(n-m,2)\oplus PG(n-m,2)\big)$. Thus, assume all points of $M/T$ receive the same label, say $a$. (Thus, this is the only option when $n\geq 2m-1$, while it is one of two options when $m+1\leq n\leq 2m-2$.) To prove that $M$ is isomorphic to $PG(n-1,2)\backslash\big(PG(n-m+1,2)-y\big)$, we need to show that $S\cup a$ is a flat of $PG(n-1,2)$ of rank $n-m+2$. Since $|S\cup a| = 2^{n-m+2}-1$, it suffices to show that $S\cup a$ is a flat of $PG(n-1,2)$, i.e., that each pair of points in $S\cup a$ is on a line with a third point of $S\cup a$. This follows since the pair together with $T$ span a flat of rank at most $m+1$, and for any rank-$(m+1)$ flat $F$ containing $T$, the restriction of $PG(n-1,2)$ to $(F\cap S)\cup a$ is isomorphic to $PG(2,2)$. □

For $q > 2$, we have the following, more limited, counterpart of Theorem 6. (Note that Theorem 15 gives a counterpart of Theorem 6 of a different flavor.)

**Theorem 7.** *Assume* $4 \leq m+1 \leq n \leq 2m-2$ *and* $q > 2$*. Then*

$$ex_q(PG(m-1,q)\backslash x; n) = \frac{q^n - 2q^{n-m+1} + 1}{q-1}$$

*and* $MAX(EX_q(PG(m-1,q)\backslash x; n))$ *contains a single geometry, namely*

$$PG(n-1,q)\backslash\big(PG(n-m,q) \oplus PG(n-m,q)\big).$$

*Proof.* That $PG(n-1,q)\backslash\big(PG(n-m,q)\oplus PG(n-m,q)\big)$ has no subgeometry isomorphic to $PG(m-1,q)\backslash x$ follows, as in the case $q = 2$, from the modular law in $PG(n-1,q)$. Thus if $4 \leq m+1 \leq n \leq 2m-2$, then

$$ex_q(PG(m-1,q)\backslash x; n) \geq \frac{q^n - 2q^{n-m+1} + 1}{q-1}.$$

Assume that $M$ is a subgeometry of $PG(n-1,q)$ in $EX_q(PG(m-1,q)\backslash x; n)$ where $n \geq m+1$. Essentially the same counting arguments as in the proof of Theorem 6 give the following. If $M$ has no $PG(m-2,q)$-subgeometry, then $M$ has at most $(q^n - q^{n-m+2})/(q-1)$ points, while if $M$ has a $PG(m-2,q)$-subgeometry, then $M$ has at most

$$\frac{q^{n-m+1}-1}{q-1}(q^{m-1}-2) + \frac{q^{m-1}-1}{q-1},$$

or $(q^n - 2q^{n-m+1} + 1)/(q-1)$, points. Since

$$\frac{q^n - 2q^{n-m+1} + 1}{q-1} > \frac{q^n - q^{n-m+2}}{q-1},$$

we get $ex_q(PG(m-1,q)\backslash x; n) = (q^n - 2q^{n-m+1} + 1)/(q-1)$.

Assume $M$ is in $MAX(EX_q(PG(m-1,q)\backslash x; n))$. Let $M = PG(n-1,q)\backslash S$. From the preceding paragraph, it follows that $M$ has a flat $T$ such that the restriction $M|T$ is isomorphic to $PG(m-2,q)$, and each such flat $T$ is contained in exactly $(q^{n-m+1}-1)/(q-1)$ rank-$m$ flats of $M$, each of which contains exactly $(q^m-1)/(q-1) - 2$ points. Fix such a flat $T$. The proof will be complete once we show that $S$ and $T$ satisfy conditions (1) and (2) of Lemma 5. Since each rank-$m$ flat of $M$ containing $T$ contains exactly $(q^m-1)/(q-1) - 2$ points, condition (1) holds.

Verifying condition (2) is equivalent to proving this condition in the case $n = m+1$, so assume $n = m+1$. Therefore $|S| = 2q+2$. We first need to show that, unlike the case of $q = 2$, the rank of $S$ must be at least four. Assume, to the contrary, that $r(S)$ is 3. We claim that some line $\ell$ of $PG(m,q)$ contained in $\mathrm{cl}_P(S)$ contains at most one point of $S$. To see this, note that it is not possible that all lines of $\mathrm{cl}_P(S)$ containing a fixed point $x$ of $S$ contain either exactly two or exactly $q+1$ points of $S$. Therefore either we have a line of $\mathrm{cl}_P(S)$ containing exactly one point of $S$, as desired, or there is a line $\ell^*$ of $\mathrm{cl}_P(S)$ containing exactly $i$ points of $S$ with $3 \leq i \leq q$. Assume the latter holds, and let $c$ be in $\ell^* - S$. Since three or more points of $S$ are on the same line, namely $\ell^*$, through $c$, it follows that at least one of the other $q$ lines of $\mathrm{cl}_P(S)$ through $c$ contains at most one point of $S$, as needed. Let $\ell$ be such a line and assume $\ell \cap S \subseteq \{s\}$. There is a rank-$m$ flat $F$ of $PG(m,q)$ with $\mathrm{cl}_P(S) \cap F = \ell$. This yields a contradiction since the flat $F - s$ of $M$ has a subgeometry isomorphic to $PG(m-1,q)\backslash x$. Thus, $r(S) > 3$.

The same argument as in the fourth paragraph of the proof of Theorem 6 implies that no plane of $PG(m,q)$ contains a $(q+1)$-point line of $M$ and three or more points of $S$. It follows that each plane spanned by points of $S$ has at least $q+2$ points; note that such a plane contains exactly $q+2$ points if and only if $q+1$ of these points are on a line. (Both of these statements are easy to see directly; they are also special cases of Theorem 2 in [5].) Let $\pi$ be a plane of $PG(m,q)$ spanned by points of $S$ and let $t$ be a point of $S$ not in $\pi$. Since $|S| = 2q+2$, it follows that for some $s \in \pi \cap S$, we have $\mathrm{cl}_P(\{t,s\}) \cap S = \{t,s\}$; let $s$ be such a point. Let $s_1$ and $s_2$ be points of $S \cap \pi$ that are not collinear with $s$. Both $\mathrm{cl}_P(\{t,s,s_1\})$ and $\mathrm{cl}_P(\{t,s,s_2\})$ must contain at least

$q+2$ points of $S$. Since these planes intersect in a line that contains just two points of $S$, it follows that both $\mathrm{cl}_P(\{t,s,s_1\})$ and $\mathrm{cl}_P(\{t,s,s_2\})$ contain exactly $q+2$ points of $S$. By the observation above, in each of these planes $q+1$ points of $S$ are collinear. Thus each of $\mathrm{cl}_P(\{s,s_1\})$, $\mathrm{cl}_P(\{s,s_2\})$, $\mathrm{cl}_P(\{t,s_1\})$, and $\mathrm{cl}_P(\{t,s_2\})$ contain either two or $q+1$ points of $S$. Since $\pi$ also contains at least $q+2$ points of $S$ and since $|S|=2q+2$, it follows that at least one of $\mathrm{cl}_P(\{s,s_1\})$ or $\mathrm{cl}_P(\{s,s_2\})$ is a subset of $S$; we may assume $\mathrm{cl}_P(\{s,s_1\})\subset S$. If $\mathrm{cl}_P(\{t,s_2\})$ contained just two points of $S$, then each plane $\mathrm{cl}_P(\{t,s_2,s'\})$, for $s'\in \mathrm{cl}_P(\{s,s_1\})$, contains at least $q+2$ points of $S$, with only two on the common line $\mathrm{cl}_P(\{t,s_2\})$, so $|S|\geq 2+(q+1)q$, which contradicts $|S|=2q+2$. Thus both $\mathrm{cl}_P(\{s,s_1\})$ and $\mathrm{cl}_P(\{t,s_2\})$ are $(q+1)$-point lines in $S$, proving that condition (2) of Lemma 5 is satisfied, hence completing the proof of this theorem. □

From the first paragraph of the proof of Theorem 7, we can observe that

$$ex_q(PG(m-1,q)\backslash x;n)\leq \frac{q^n-2q^{n-m+1}+1}{q-1}$$

for all $n\geq m$; however, for $n>2m-2$ it is likely that $ex_q(PG(m-1,q)\backslash x;n)$ is much smaller than $(q^n-2q^{n-m+1}+1)/(q-1)$.

In the cycle matroid $M(K_4)$, there is a flat (a point) whose rank is two less than the rank of the matroid, and which is contained in three hyperplanes (lines), two of which are flats of the ambient projective geometry and the other of which is a single-point deletion of a flat of the projective geometry. By analyzing geometries of this type beyond the special case of $q=2$ and rank 3, we will get Theorem 11 which determines, for a large class of geometries $G$, both $ex_q(G;n)$ and $MAX(EX_q(G;n))$. Both $ex_q(G;n)$ and $MAX(EX_q(G;n))$ are the same as given by Theorem 1 when $G$ is $PG(m-2,q)$, but Theorem 11 applies to a larger class of geometries $G$. Among the geometries $G$ addressed are many that play an important role in matroid theory, including the Reid geometries $R_{cycle}[q]$ for primes $q$ exceeding 2. (See page 516 of [6] for the ternary Reid geometry and page 52 of [3] for Reid geometries in general.)

We start with a lemma that reduces the work to considering excluded subgeometries of the special form suggested by the description of $M(K_4)$ above. The proof is easy and so is omitted. Lemma 8, reformulated in the obvious way for minors, holds in the context of minor-closed classes, to which most of [3] is devoted. Again with the obvious modifications, this lemma could be formulated without the underlying assumption of representability.

**Lemma 8.** *Assume that $G_1$ and $G_2$ are geometries representable over $GF(q)$ and that $G_1$ is a subgeometry of $G_2$. Then $EX_q(G_1;n)\subseteq EX_q(G_2;n)$, so $ex_q(G_1;n)\leq ex_q(G_2;n)$.*

*Assume, furthermore, that $G_2$ is a subgeometry of a $GF(q)$-representable geometry $G_3$.*

(1) *If $EX_q(G_1;n)=EX_q(G_3;n)$, then $EX_q(G_2;n)=EX_q(G_3;n)$.*
(2) *If $ex_q(G_1;n)=ex_q(G_3;n)$, then $ex_q(G_2;n)=ex_q(G_3;n)$.*
(3) *If $MAX(EX_q(G_1;n))=MAX(EX_q(G_3;n))$, then $MAX(EX_q(G_2;n))=MAX(EX_q(G_3;n))$.*

From Theorem 1, $ex_q(PG(m-2,q);n)=(q^n-q^{n-m+2})/(q-1)$ and

$$MAX(EX_q(PG(m-2,q);n))=\{PG(n-1,q)\backslash PG(n-m+1,q)\}.$$

Thus, by Lemma 8, if $G'$ is a $GF(q)$-representable geometry with a subgeometry isomorphic to $PG(m-2,q)$ and if $ex_q(G';n)=(q^n-q^{n-m+2})/(q-1)$ and $MAX(EX_q(G';n))=\{PG(n-1,q)\backslash PG(n-m+1,q)\}$, then the same is true of every subgeometry $G$ of $G'$ that contains a $PG(m-2,q)$-subgeometry. The specific geometry $G'$ we are concerned with in Theorem 11 is constructed as follows. Let $L$ be a rank-$(m-2)$ flat in $PG(m-1,q)$ and let $H_1,H_2,H_3$ be three hyperplanes of $PG(m-1,q)$ that contain $L$. Assume $x$ is a point of $H_3-L$. The geometry $G'$ of interest is the restriction $PG(m-1,q)|\big(H_1\cup H_2\cup(H_3-x)\big)$. (The geometry obtained when $m=3$ and $q=2$ is $M(K_4)$; that obtained when $m=3$ and $q=3$ is the ternary Reid geometry.)

We first need to know that there is only one such restriction of $PG(m-1,q)$. To see this, we need Lemma 9, which is a matroid-theoretic reformulation of what is often called the fundamental theorem of projective geometry (see Section 2.1.2 of [2]). To make this paper self-contained, and since this perspective on the result may not be widely known, we include the brief matroid-theoretic proof.

**Lemma 9.** *Let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis of $PG(n-1,q)$ and let $b$ be a point in $PG(n-1,q)$ such that the fundamental circuit $C(b,B)$ of $b$ with respect to the basis $B$ is $B \cup b$. Let $B' = \{b'_1, b'_2, \ldots, b'_n\}$ be a basis of $PG(n-1,q)$ and let $b'$ be a point in $PG(n-1,q)$ such that $C(b',B') = B' \cup b'$. Then there is an automorphism $\phi$ of $PG(n-1,q)$ such that $\phi(b_i) = b'_i$ for $i = 1,2,\ldots,n$ and $\phi(b) = b'$.*

*Proof.* Let $A$ be a matrix representation of $PG(n-1,q)$. By the standard operations (row reduction, interchanging rows, scaling rows, and scaling columns), we may assume that the columns of $A$ corresponding to $b_1, b_2, \ldots, b_n$, in this order, form an identity matrix and the column corresponding to $b$ has all entries equal to 1. The standard operations also allow us to obtain from $A$ a matrix $A'$ in which the columns of $A'$ corresponding to $b'_1, b'_2, \ldots, b'_n$, in this order, form an identity matrix and the column corresponding to $b'$ has all entries equal to 1. Consider the map $\phi : PG(n-1,q) \to PG(n-1,q)$ for which $\phi(x) = y$ if the column in $A$ corresponding to $x$ is a scalar multiple of the column in $A'$ corresponding to $y$. Clearly $\phi$ is an automorphism of $PG(n-1,q)$ with $\phi(b_i) = b'_i$ for $i = 1,2,\ldots,n$ and $\phi(b) = b'$. □

With this lemma, we can show the uniqueness of the excluded subgeometry that is of interest in Theorem 11.

**Lemma 10.** *Let $L$ and $L'$ be rank-$(m-2)$ flats in the rank-$m$ projective geometry $PG(m-1,q)$. Let $H_1, H_2, H_3$ be three hyperplanes of $PG(m-1,q)$ that contain $L$ and let $H'_1, H'_2, H'_3$ be three hyperplanes of $PG(m-1,q)$ that contain $L'$. Assume $x$ is a point of $H_3 - L$ and $x'$ is a point of $H'_3 - L'$. Let $G$ be $PG(m-1,q)|\big(H_1 \cup H_2 \cup (H_3 - x)\big)$ and $G'$ be $PG(m-1,q)|\big(H'_1 \cup H'_2 \cup (H'_3 - x')\big)$. Then $G$ and $G'$ are isomorphic. Furthermore, $G$ is uniquely representable over $GF(q)$.*

*Proof.* Let $B = \{b_1, b_2, \ldots, b_{m-2}, b_{m-1}, x\}$ be a basis of $PG(m-1,q)$ for which $\{b_1, b_2, \ldots, b_{m-2}\}$ is a basis of $L$ and $b_{m-1}$ is in $H_1 - L$. Likewise, let $B' = \{b'_1, b'_2, \ldots, b'_{m-2}, b'_{m-1}, x'\}$ be a basis of $PG(m-1,q)$ for which $\{b'_1, b'_2, \ldots, b'_{m-2}\}$ is a basis of $L'$ and $b'_{m-1} \in H'_1 - L'$. View the points of $PG(m-1,q)$ as column vectors in a matrix representation of $PG(m-1,q)$. Note that the points $b_1 + b_2 + \cdots + b_{m-1} + \alpha x$, as $\alpha$ ranges over the $q-1$ nonzero elements of $GF(q)$, are in different hyperplanes over $L$, and none of these points is in $H_1 \cup H_3$. Since $L$ is contained in $q-1$ hyperplanes besides $H_1$ and $H_3$, for some nonzero $\alpha$ in $GF(q)$, the point $b = b_1 + b_2 + \cdots + b_{m-1} + \alpha x$ is in $H_2$. Note that $C(b,B) = B \cup b$. Similarly, there is a $b' \in H'_2 - L$ with $C(b',B') = B' \cup b'$. By Lemma 9, there is an automorphism $\phi$ of $PG(m-1,q)$ with $\phi(b_i) = b'_i$ for $i = 1,2,\ldots,m-1$, $\phi(x) = x'$, and $\phi(b) = b'$. The image of $G$ under $\phi$ is $G'$, as needed. That $G$ is uniquely representable over $GF(q)$ is clear. □

Let $G_{m,q}$ denote the geometry $PG(m-1,q)|\big(H_1 \cup H_2 \cup (H_3 - x)\big)$ constructed above. Thus $G_{3,2}$ is $M(K_4)$ and $G_{3,3}$ is the ternary Reid geometry $R_{cycle}$[3], which is often denoted $R_9$. By Lemma 10, $G_{m,q}$ is well-defined. We now find, for $q > 2$, both $ex_q(G;n)$ and $MAX(EX_q(G;n))$ for all subgeometries of $G_{m,q}$ that contain a $PG(m-2,q)$-subgeometry. One geometry that plays a role in this result is $Q_3(GF(3)^*)$, the rank-3 ternary Dowling lattice. In general, the rank-3 Dowling lattice $Q_3(GF(q)^*)$ over $GF(q)^*$ is formed by restricting the projective plane $PG(2,q)$ to the set of points on three nonconcurrent lines. The rank-3 ternary Dowling lattice $Q_3(GF(3)^*)$ is also obtained by deleting the uniform matroid $U_{3,4}$ from $PG(2,3)$. (See page 27 of [3] and the references there for more information on Dowling lattices in their full generality.)

**Theorem 11.** *Assume $q > 2$ and $m \geq 3$. Let $G$ be a subgeometry of $G_{m,q}$ that contains a $PG(m-2,q)$-subgeometry. For all $n \geq m$, we have*

$$ex_q(G;n) = \frac{q^n - q^{n-m+2}}{q-1}.$$

*Furthermore $MAX(EX_q(G;n)) = \{PG(n-1,q) \backslash PG(n-m+1,q)\}$ with the single exception that $MAX(EX_3(R_9;3)) = \{AG(2,3), Q_3(GF(3)^*)\}$.*

*Proof.* By Lemma 8, apart from finding $MAX(EX_3(G;3))$ when $G$ is a subgeometry of $R_9$ containing a 4-point line, it suffices to prove the result for $G = G_{m,q}$. Since $G_{m,q}$ has a subgeometry isomorphic to $PG(m-2,q)$, it follows that $PG(n-1,q)\backslash PG(n-m+1,q)$ has no subgeometry isomorphic to $G_{m,q}$. Therefore $ex_q(G;n) \geq (q^n - q^{n-m+2})/(q-1)$. Thus we need only prove $ex_q(G;n) \leq (q^n - q^{n-m+2})/(q-1)$ and analyze geometries in $EX_q(G;n)$ with $(q^n - q^{n-m+2})/(q-1)$ points.

We first treat $n = m$. For $n = m$, we have

$$\frac{q^n - q^{n-m+2}}{q-1} = \frac{q^m - q^2}{q-1} = \frac{q^m - 1}{q-1} - (q+1).$$

Therefore we need to show that if $M$ is $PG(m-1,q)\backslash X$ where $|X| = q$, then $M$ has a $G_{m,q}$-subgeometry. By Theorem 1, $M$ has a restriction $M|H$ isomorphic to $PG(m-2,q)$. Fix two points $x, y \in X$ and let $\mathrm{cl}_P(\{x,y\}) \cap H = \{z\}$. Let $L$ be a rank-$(m-2)$ flat of $M|H$ that contains $z$. Since $L$ is contained in $q+1$ hyperplanes of $PG(m-1,q)$ and one of these, namely $\mathrm{cl}_P(L \cup \{x\})$, contains at least two points of $X$, it follows that two of these hyperplanes are disjoint from $X$ and a third contains at most one point of $X$. Thus $M$ has a $G_{m,q}$-subgeometry.

We now find $MAX(EX_q(G;m))$. First assume $m > 3$. We need to show that if $M$ is $PG(m-1,q)\backslash X$ where $X$ is a set of $q+1$ noncollinear points, then $M$ has a subgeometry isomorphic to $G_{m,q}$. By Theorem 1, $M$ has a restriction $M|H$ isomorphic to $PG(m-2,q)$. Fix three points $x, y, z \in X$ and let $\mathrm{cl}_P(\{x,y,z\}) \cap H = Z$. Thus $r(Z)$ is 1 or 2, and $Z$ is properly contained in $H$ since $m > 3$. Therefore there is a rank-$(m-2)$ flat $L$ of $M|H$ that contains $Z$. Since one of the $q+1$ hyperplanes of $PG(m-1,q)$ containing $L$ contains three points of $X$, it follows that two of these hyperplanes are disjoint from $X$ while a third contains at most one point of $X$. Thus $M$ has a subgeometry isomorphic to $G_{m,q}$.

Now assume $m$ is 3, so $n$ is also 3. Let $M$ be $PG(2,q)\backslash X$ where $X$ is a spanning set of $q+1$ points. First assume there are three collinear points $x, y, z \in X$. Note that $\mathrm{cl}_P(\{x,y,z\})$ is not $X$; let $u$ be in $\mathrm{cl}_P(\{x,y,z\}) - X$. At least two lines of $M$ through $u$ contain $q+1$ points and a third line of $M$ through $u$ contains at least $q$ points, so $M$ has a $G_{3,q}$-subgeometry. Thus we may assume that no three points of $X$ are collinear. Fix $x, y \in X$. Since no three points of $X$ are collinear, $\mathrm{cl}_P(\{x,y\}) \cap X = \{x,y\}$ and for any pair of points $u, w \in X - \{x,y\}$, the line $\mathrm{cl}_P(\{u,w\})$ intersects $\mathrm{cl}_P(\{x,y\})$ in a point $v$ of $M$. Counting the points on lines through $v$ shows there is a $G_{3,q}$-subgeometry of $M$ if $q > 3$. We also deduce that the only potential exception for $q = 3$ is $PG(2,3)\backslash U_{3,4}$, i.e., $Q_3(GF(3)^*)$. Indeed, $Q_3(GF(3)^*)$ has no $R_9$-subgeometry since $R_9$ is $PG(2,3)\backslash\big(U_{2,3} \oplus U_{1,1}\big)$. On the other hand, it also follows that both single-element deletions of $R_9$ having a 4-point line are subgeometries of $Q_3(GF(3)^*)$, so there are no other exceptions for $q = 3$.

Finally, assume $n > m$ and let $M$ be a subgeometry of $PG(n-1,q)$ with rank $n$ and no $G_{m,q}$-subgeometry. If $M$ has no subgeometry isomorphic to $PG(m-2,q)$, then $M$ has at most $(q^n - q^{n-m+2})/(q-1)$ points by Theorem 1, and among such geometries, only $PG(n-1,q)\backslash PG(n-m+1,q)$ has $(q^n - q^{n-m+2})/(q-1)$ points. To finish the proof, it suffices to show that if $M$ has a restriction $M|T$ isomorphic to $PG(m-2,q)$, then $M$ has fewer than $(q^n - q^{n-m+2})/(q-1)$ points. Such a rank-$(m-1)$ flat $T$ of $PG(n-1,q)$ is contained in at most $(q^{n-m+1}-1)/(q-1)$ flats of $M$ of rank $m$; each of these rank-$m$ flats has at most $(q^m - q^2)/(q-1)$ points by the case of rank $m$; therefore such flats have at most $(q^m - q^2)/(q-1) - (q^{m-1}-1)/(q-1)$, or $q^{m-1} - q - 1$, points not in $T$. Therefore the number of points in $M$ is at most

$$\frac{q^{n-m+1}-1}{q-1}(q^{m-1} - q - 1) + \frac{q^{m-1}-1}{q-1}.$$

Since this is strictly less than $(q^n - q^{n-m+2})/(q-1)$, the proof is complete. □

We single out the case of Reid geometries as being of special interest.

**Corollary 12.** *If $q$ is an odd prime, then $ex_q(R_{cycle}[q];n) = q^{n-1}$ for $n \geq 3$. Furthermore*

$$MAX(EX_3(R_9;3)) = \{AG(2,3), Q_3(GF(3)^*)\},$$

*while for $q > 3$ or $n > 3$, we have $MAX(EX_q(R_{cycle}[q]; n)) = \{AG(n-1, q)\}$.*

The cycle matroid $M(K_4)$ of the complete graph on four vertices is not affine over $GF(3)$, so we have $ex_3(M(K_4); n) \geq 3^{n-1}$. Indeed, an easy argument yields the following theorem.

**Theorem 13.** *If $G$ is a subgeometry of $R_9$ that has an $M(K_4)$-subgeometry, then $ex_3(G; n) = 3^{n-1}$ for $n \geq 3$. Furthermore if $G$ has fewer than nine points, then $MAX(EX_3(G; n)) = \{AG(n-1, 3)\}$.*

Theorem 14 is a binary counterpart of Theorem 11 in that it shows that, with few exceptions, the type of result given by Theorem 1 holds for all binary geometries that have a hyperplane that is a projective geometry. Indeed, between Theorems 1, 6, and 14, we know $ex_2(G; n)$ and $MAX(EX_2(G; n))$ for all binary geometries $G$ having a hyperplane that is a projective geometry.

**Theorem 14.** *Assume $m \geq 3$. Let $G$ be a subgeometry of $PG(m-1, 2)$ with at most $2^m - 3$ points that contains a subgeometry isomorphic to $PG(m-2, 2)$. For all $n \geq m$, we have $ex_2(G; n) = 2^n - 2^{n-m+2}$. Furthermore,*

$$MAX(EX_2(PG(m-1, 2)\backslash\{x, y\}; m))$$

*consists of the two geometries $PG(m-1, 2)\backslash U_{2,3}$ and $PG(m-1, 2)\backslash U_{3,3}$, while in all other cases (i.e., if $n > m$ or if $n = m$ and $G$ has fewer than $2^m - 3$ points), we have*

$$MAX(EX_2(G; n)) = \{PG(n-1, 2)\backslash PG(n-m+1, 2)\}.$$

*Proof.* From Lemma 8, the key geometry to consider is $PG(m-1, 2)\backslash\{x, y\}$, with more work needed for $n = m$. That $PG(n-1, 2)\backslash PG(n-m+1, 2)$ has no $G$-subgeometry follows since $G$ has $PG(m-2, 2)$ as a subgeometry. Thus $ex_2(G; n) \geq 2^n - 2^{n-m+2}$. That $ex_2(G; n)$ is $2^n - 2^{n-m+2}$ for $n = m$ is trivial. That $MAX(EX_2(PG(m-1, 2)\backslash\{x, y\}; m))$ contains just $PG(m-1, 2)\backslash U_{2,3}$ and $PG(m-1, 2)\backslash U_{3,3}$ follows since there are no other binary geometries of rank $m$ with $2^m - 4$ points. That $MAX(EX_2(G; m))$ is $\{PG(m-1, 2)\backslash U_{2,3}\}$ for all other rank-$m$ binary geometries $G$ satisfying the hypothesis follows from Lemma 8 since it holds for the only such $G$ with $2^m - 4$ points, namely $G = PG(m-1, 2)\backslash U_{3,3}$. (Note that $PG(m-1, 2)\backslash U_{2,3}$ does not satisfy the hypothesis since it does not contain a $PG(m-2, 2)$-subgeometry.) Now assume $n > m$ and let $G$ be $PG(m-1, 2)\backslash\{x, y\}$. Let $M$ be a rank-$n$ subgeometry of $PG(n-1, 2)$ with no $G$-subgeometry. If $M$ has no $PG(m-2, 2)$-subgeometry, then $M$ has at most $2^n - 2^{n-m+2}$ points by Theorem 1, and among such geometries, only $PG(n-1, 2)\backslash PG(n-m+1, 2)$ has $2^n - 2^{n-m+2}$ points. By the same argument as in the last paragraph of the proof of Theorem 11, if $M$ has a $PG(m-2, 2)$-subgeometry, then $M$ has fewer than $2^n - 2^{n-m+2}$ points. This completes the proof. ☐

The geometry $PG(m-1, 2)\backslash x$ considered in Theorem 6 is the restriction of $PG(m-1, 2)$ to the set of all points that do not depend on all elements in some fixed basis of $PG(m-1, 2)$. From this perspective, Theorem 15 gives a counterpart to Theorem 6 for $q > 2$.

Assume $m \geq 3$. Let $B = \{b_1, b_2, \ldots, b_m\}$ be a basis of $PG(m-1, q)$ and let $H_i = \mathrm{cl}_P(B - \{b_i\})$. Let $Q_{m,q}$ be $PG(m-1, q)|(H_1 \cup H_2 \cup \cdots \cup H_m)$. Alternatively, $Q_{m,q}$ is the restriction of $PG(m-1, q)$ to the set of points $x$ such that either $x$ is in $B$ or $C(x, B) - x$ is a proper subset of $B$. Note that $Q_{m,2}$ is a single-element deletion of $PG(m-1, 2)$. Also, $Q_{3,q}$ is the rank-3 Dowling lattice $Q_3(GF(q)^*)$ over the group of units of $GF(q)$ while for $m > 3$, the geometry $Q_{m,q}$ contains many more points than the rank-$m$ Dowling lattice over $GF(q)^*$. (In the terminology and notation of Section 6 of [4], $Q_{m,q}$ is the weight-$(m-1)$ Dowling geometry $B_{m,m-1}(q)$.) Theorem 15 gives $ex_q(Q_{m,q}; n)$ and $MAX(EX_q(Q_{m,q}; n))$ for all prime powers $q > 2$.

Theorem 15 says that $PG(n-1, q)\backslash\big(PG(n-m+1, q) - y\big)$ is the only geometry in $MAX(EX_q(Q_{m,q}; n))$. Note that by Theorem 1, if $n \geq m$, then all subgeometries of $PG(n-1, q)\backslash\big(PG(n-m+1, q) - y\big)$ isomorphic to $PG(m-2, q)$ contain $y$ and $y$ is the only point with this property. For $n \geq m$, this distinguished point $y$ of $PG(n-1, q)\backslash\big(PG(n-m+1, q) - y\big)$ will be called the *apex* of $PG(n-1, q)\backslash\big(PG(n-m+1, q) - y\big)$. (This terminology comes from $q$-lifts, which have more recently been called $q$-cones. See [7] and Section 8.6 of [4]. Note that for $n \geq m$, the geometry $PG(n-1, q)\backslash\big(PG(n-m+1, q) - y\big)$ is a $q$-cone of the geometry $PG(n-2, q)\backslash PG(n-m, q)$.)

**Theorem 15.** *Assume $m \geq 3$ and $q > 2$. For $n \geq m-1$, we have*

$$ex_q(Q_{m,q}; n) = \frac{q^n - q^{n-m+2}}{q-1} + 1$$

*and $MAX(EX_q(Q_{m,q}; n)) = \{PG(n-1,q)\backslash\big(PG(n-m+1,q) - y\big)\}$.*

*Proof.* All subgeometries of $PG(n-1,q)\backslash\big(PG(n-m+1,q)-y\big)$ isomorphic to $PG(m-2,q)$ contain the apex $y$. Since this is not true of any point in $Q_{m,q}$, it follows that $PG(n-1,q)\backslash\big(PG(n-m+1,q)-y\big)$ has no $Q_{m,q}$-subgeometry. Therefore $ex_q(Q_{m,q}; n) \geq (q^n - q^{n-m+2})/(q-1)+1$.

The assertions about $ex_q(Q_{m,q}; m-1)$ and $MAX(EX_q(Q_{m,q}; m-1))$ are clear. The claimed value of $ex_q(Q_{m,q}; m)$ is $(q^m-1)/(q-1)-q$. The case $n = m$ will be addressed by proving the following assertion.

**(15.1)** *If $N$ is the deletion $PG(m-1,q)\backslash X$ where $X$ is any set of $q$ noncollinear points of $PG(m-1,q)$, then $N$ contains a subgeometry isomorphic to $Q_{m,q}$.*

Indeed, any subgeometry of $PG(m-1,q)$ with more than $(q^m-1)/(q-1)-q$ points contains such a geometry $N$ (simply delete more points to get $X$), so from (15.1) we deduce $ex_q(Q_{m,q}; m) \leq (q^m-1)/(q-1)-q$ and hence

$$ex_q(Q_{m,q}; m) = \frac{q^m-1}{q-1} - q.$$

Also, (15.1) gives $MAX(EX_q(Q_{m,q}; m)) = \{PG(m-1,q)\backslash\big(PG(1,q)-y\big)\}$.

We prove (15.1) by induction on $m$. Let $N = PG(m-1,q)\backslash X$ be as in (15.1). In the base case $m = 3$, the geometry $N$ is a plane formed from $PG(2,q)$ by deleting $q$ points that span $PG(2,q)$ and $Q_{3,q}$ consists of three nonconcurrent $(q+1)$-point lines. Let $x, y, z$ be noncollinear points of $X$. Let $t$ be a point in $\mathrm{cl}_P(\{x,y\})$ that is in $N$. Since $q$ points of $PG(2,q)$ have been deleted to form $N$ and at least two of these are on the line $\mathrm{cl}_P(\{x,y\})$ of $PG(2,q)$ which contains $t$, it follows that $t$ is on at least two $(q+1)$-point lines of $N$. Let $\ell_1$ and $\ell_2$ be two such lines. Now $\ell_1$ intersects $\mathrm{cl}_P(\{x,z\})$ in a point $s$ of $N$ different from $t$. As above, $s$ is on at least two $(q+1)$-point lines of $N$; let $\ell_3$ be one of these distinct from $\ell_1$. The points on $\ell_1 \cup \ell_2 \cup \ell_3$ form the required $Q_{3,q}$-subgeometry, completing the proof of (15.1) for $m = 3$. Now assume $m > 3$ and (15.1) holds for $m-1$. Again let $x, y, z$ be noncollinear points of $X$ and let $t$ be a point in $\mathrm{cl}_P(\{x,y\})$ that is in $N$. Consider the restriction $N|T$ of $N$ to the points that are on $(q+1)$-point lines of $N$ through $t$, and let $N'$ be the minor $(N|T)/t$ of $N$. It follows that $N'$ is isomorphic to $PG(m-2,q)\backslash X'$ where $|X'| < q$. Therefore by the case $m-1$ of (15.1), it follows that $N'$ has a $Q_{m-1,q}$-subgeometry. From this, it follows that there are $m-1$ lines $\ell_1, \ell_2, \ldots, \ell_{m-1}$ of $N$ though $t$ such that for all choices of points $b_1, b_2, \ldots, b_{m-1}$ with $b_i \in \ell_i - t$, we have that $t, b_1, b_2, \ldots, b_{m-1}$ is a basis of $N$ and $\mathrm{cl}_P(\{t, b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{m-1}\})$ is a flat of $N$. To complete the proof of (15.1), it suffices to show there is a hyperplane $H$ of $N$ isomorphic to $PG(m-2,q)$ and not containing $t$, for then if we let $H \cap \ell_i = \{b_i\}$ for $i = 1, 2, \ldots, m-1$, we have that each $\mathrm{cl}_P(\{t, b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{m-1}\})$ as well as $\mathrm{cl}_P(\{b_1, b_2, \ldots, b_{m-1}\})$ is a flat of $N$, thereby exhibiting the required $Q_{m,q}$-subgeometry. To see that there is such a hyperplane $H$, let $\mathrm{cl}_P(\{x,z\})$ intersect the hyperplane $\mathrm{cl}_P(\ell_1 \cup \ell_2 \cup \ldots \cup \ell_{m-2})$ in a point $s$ and let $Z$ be a rank-$(m-2)$ flat of $PG(m-1,q)$ contained in $\mathrm{cl}_P(\ell_1 \cup \ell_2 \cup \ldots \cup \ell_{m-2})$ that contains $s$ and not $t$. Since $Z$ is in $q+1$ hyperplanes of $PG(m-1,q)$ and $\mathrm{cl}_P(Z \cup \{x,z\})$ contains at least two of the $q$ points of $X$, it follows that at least two of these hyperplanes are flats of $N$. Since exactly one such hyperplane, namely $\mathrm{cl}_P(\ell_1 \cup \ell_2 \cup \ldots \cup \ell_{m-2})$, contains $t$, the required hyperplane not containing $t$ exists, completing the proof of (15.1).

To prove $ex_q(Q_{m,q}; n) \leq (q^n - q^{n-m+2})/(q-1)+1$ for $n > m$, let $M$ be a subgeometry of $PG(n-1,q)$ with no $Q_{m,q}$-subgeometry. If $M$ has no $PG(m-2,q)$-subgeometry, then, by Theorem 1, $M$ has at most $(q^n - q^{n-m+2})/(q-1)$ points. Thus we may assume there is a rank-$(m-1)$ flat $T$ of $M$ with $M|T$ isomorphic to $PG(m-2,q)$. At most $(q^{n-m+1}-1)/(q-1)$ flats of $M$ having rank $m$ contain $T$. By the rank-$m$ case, each rank-$m$ flat has at most $(q^m-1)/(q-1)-q$ points. Therefore each rank-$m$ flat containing $T$ has at most

$q^{m-1} - q$ points in addition to the points of $T$, and so $M$ has at most

$$\frac{q^{n-m+1}-1}{q-1}(q^{m-1}-q) + \frac{q^{m-1}-1}{q-1}$$

points, that is, at most $(q^n - q^{n-m+2})/(q-1) + 1$ points, as needed.

Assume $n > m$ and $M \in MAX(EX_q(Q_{m,q}; n))$. From (15.1) and the preceding paragraph, we draw the following conclusion.

**(15.2)** *The geometry $M$ has a flat $T$ of rank $m-1$ with $M|T$ isomorphic to $PG(m-2,q)$. Every such rank-$(m-1)$ flat is in exactly $(q^{n-m+1}-1)/(q-1)$ flats of $M$ having rank $m$ and all such rank-$m$ flats are isomorphic to $PG(m-1,q)\backslash\big(PG(1,q)-y\big)$.*

Let $M$ be $PG(n-1,q)\backslash X$. To complete the proof, we need to show that there is a point $a$ of $M$ such that $X \cup a$ is a flat of $PG(n-1,q)$. The key to this is the following assertion, which is proven below.

**(15.3)** *Let $T$ be a rank-$(m-1)$ flat of $M$ with $M|T$ isomorphic to $PG(m-2,q)$ and let $T'$ be a rank-$(m+1)$ flat of $PG(n-1,q)$ containing $T$ with $X' = T' \cap X$. Then $\mathrm{cl}_P(X')$ is a plane that contains $q^2+q$ points of $X$ and one point of $T$.*

Before proving (15.3), we note that the rest of the proof of the theorem follows easily from it. To see this, we first show that there is one point $a$ of $T$ such that for every rank-$m$ flat $T^*$ of $PG(n-1,q)$ containing $T$, the $q$ points of $X \cap T^*$ are collinear with $a$. Indeed, let $T_0$ be a rank-$m$ flat of $PG(n-1,q)$ containing $T$ and let $\mathrm{cl}_P(X \cap T_0) \cap T = \{a\}$. For any other such rank-$m$ flat $T^*$, applying (15.3) to the rank-$(m+1)$ flat $\mathrm{cl}_P(T_0 \cup T^*)$ shows that $\mathrm{cl}_P(X \cap T^*) \cap T$ must also be $\{a\}$. Finally, we show that $X \cup a$ is a flat of $PG(n-1,q)$. For this, we need to show that for $x, y \in X \cup a$, we have $\mathrm{cl}_P(\{x,y\}) \subseteq X \cup a$. This is obvious if $a \in \mathrm{cl}_P(\{x,y\})$, while if $a \notin \mathrm{cl}_P(\{x,y\})$, applying (15.3) to $T' = \mathrm{cl}_P(T \cup \{x,y\})$ gives the result.

It suffices to prove (15.3) in the case $n = m+1$. The proof for $m > 3$ rests on the case $m = 3$, so we treat this case first. This case has the following simpler formulation.

**(15.4)** *Let $M$ be $PG(3,q)\backslash X$. Then $r(X)$ is three.*

From (15.2), we know that $M$ has a line with exactly $q+1$ points, and each such line $\ell$ is contained in exactly $q+1$ planes of $M$, each of which is isomorphic to $PG(2,q)\backslash\big(PG(1,q)-y\big)$; note that the apex of each such plane is on $\ell$. We first show that (15.4) follows if there is a $(q+1)$-point line $\ell$ of $M$ for which all planes containing $\ell$ have the same apex, say $a$, on $\ell$. Assume there is such a line $\ell$ and point $a$ on $\ell$. Note that each line of $PG(3,q)$ through $a$ either contains $q$ points of $X$ or is disjoint from $X$. Pick $x, y$ in $X$ with $a \notin \mathrm{cl}_P(\{x,y\})$. If $r(X) > 3$, then the plane $\mathrm{cl}_P(\{x,y,a\})$ of $PG(3,q)$ shows that (15.2) is violated since $\mathrm{cl}_P(\{x,y,a\})$ contains a $(q+1)$-point line of $M$ through $a$ as well as at least $2q$ points of $X$, namely the points of $\mathrm{cl}_P(\{a,x\}) - a$ and $\mathrm{cl}_P(\{a,y\}) - a$. This contradiction proves $r(X) = 3$, as needed.

Thus we want to show that for some $(q+1)$-point line $\ell$ of $M$, all planes through $\ell$ have the same apex. Assume this is not the case. That is, assume that for each $(q+1)$-point line $\ell$ of $M$, there are at least two planes $\pi$ and $\pi'$ containing $\ell$ with the apex of $\pi$ differing from the apex of $\pi'$. We claim that as a consequence each point of such a line $\ell$ is the apex of some plane through $\ell$. If this were not the case, then there is a point $a$ on $\ell$ that is the apex of two planes, say $\pi_1$ and $\pi_2$, containing $\ell$, together with a different point $b$ on $\ell$ that is the apex of another plane, say $\pi_3$, containing $\ell$. Let $\ell^*$ be a $(q+1)$-point line of $M$ contained in $\pi_1$ other than $\ell$. Thus, $a$ is in $\ell^*$. Let $\ell_2$ and $\ell_3$ be the lines $\mathrm{cl}_P(X \cap \pi_2)$ and $\mathrm{cl}_P(X \cap \pi_3)$, respectively. Thus $a \in \ell_2$ and $b \in \ell_3$. It follows that $\mathrm{cl}_P(\ell^* \cup \ell_2)$ intersects $\ell_3$ in a point $x$ of $X$. This shows that the planes through $\ell^*$ violate the structure given in (15.2). Therefore we make the following assumption for the rest of the proof of (15.4).

**(15.5)** *For every $(q+1)$-point line $\ell$ of $M$, each point of $\ell$ is the apex of some plane of $M$ through $\ell$.*

This gives us much information about the lines of $M$. Indeed, we have the following.

**(15.6)** *Each point $a$ of $M$ is contained in exactly $q$ lines of $M$ that each have exactly $q+1$ points. All other lines of $M$ containing $a$ have exactly $q$ points; there are $q^2$ such lines.*

To see this, let $a$ be a point of $M$. Since $q^2+q+1$ lines of $PG(3,q)$ contain $a$ and $|X|=q^2+q$, it follows that $a$ is on at least one $(q+1)$-point line, say $\ell$, of $M$. Now (15.6) follows from (15.5) since $a$ is the apex of precisely one of the planes containing $\ell$.

Let $\ell_X$ be a line of $PG(3,q)$ containing exactly $q$ points of $X$ and let $a$ be the only point of $M$ in $\ell_X$. Let $x$ be a point of $X$ not in $\ell_X$. To prove (15.4), we need to show that the plane $\mathrm{cl}_P(\ell_X \cup \{x\})$ contains $X$. Assume this is not the case. Therefore $\mathrm{cl}_P(\ell_X \cup \{x\})$ contains points of $M$ in addition to $a$; let $b$ be such a point. Note that $\mathrm{cl}_P(\{a,b\})$ cannot be a line of $M$, for then the plane $\mathrm{cl}_P(\ell_X \cup \{x\})$ would show that (15.2) fails. Thus $\mathrm{cl}_P(\{a,b\})$ contains exactly $q$ points of $M$.

Assume first that the only points of $M$ in $\mathrm{cl}_P(\ell_X \cup \{x\})$ are in $\mathrm{cl}_P(\{a,b\})$. Let the points in $\mathrm{cl}_P(\{a,b\})$ that are not in $X$ be $a_1, a_2, \dots, a_q$. Let $s$ be a point of $M$ not in $\mathrm{cl}_P(\ell_X \cup \{x\})$. Since $s$ is on exactly $q$ lines of $M$ that have exactly $q+1$ points, it follows that for $i=1,\dots,q$, the line $\mathrm{cl}_P(\{s,a_i\})$ is a line of $M$. Choose $u$ in $\mathrm{cl}_P(\{s,a_3\}) - \{s,a_3\}$ and let $\{v\} = \mathrm{cl}_P(\{u,a_1\}) \cap \mathrm{cl}_P(\{s,a_2\})$. It follows that $\mathrm{cl}_P(\{s,u\})$, $\mathrm{cl}_P(\{s,v\})$, and $\mathrm{cl}_P(\{u,v\})$ are lines of $M$, thereby exhibiting a $Q_{3,q}$-subgeometry of $M$. This contradiction shows that not all points of $M$ in $\mathrm{cl}_P(\ell_X \cup \{x\})$ are in $\mathrm{cl}_P(\{a,b\})$.

Let $c$ be a point of $M$ in $\mathrm{cl}_P(\ell_X \cup \{x\}) - \mathrm{cl}_P(\{a,b\})$. Now $\mathrm{cl}_P(\{c,b\}) - X$ is a $q$-point line of $M$, say $\{c_1, c_2, \dots, c_q\}$. Also, each $\mathrm{cl}_P(\{a,c_i\}) - X$ is a $q$-point line of $M$. It follows that the points of $M$ in $\mathrm{cl}_P(\ell_X \cup \{x\})$ are on $q$ lines through $a$, each of which contains $q$ points of $M$. Let these lines of $M$ be $\ell_1, \ell_2, \dots, \ell_q$ with $\{x_i\} = \mathrm{cl}_P(\ell_i) - \ell_i$. Let $\mathrm{cl}_P(\{x_1,x_2\}) \cap \ell_3 = \{u\}$ and choose $v \in \ell_3 - \{u,a\}$. Note that $\mathrm{cl}_P(\{v,x_2\})$ contains at least two points of $X$, namely $x_2$ and one point from $\ell_X - a$, as well as at least two points of $M$, namely $v$ and the point of intersection $\mathrm{cl}_P(\{v,x_2\}) \cap \ell_1$. Thus $\mathrm{cl}_P(\{v,x_2\}) - X$ is a line of $M$ with fewer than $q$ points. This contradiction to (15.6) completes the proof that $r(X)$ is three, and so establishes (15.4).

We now turn to proving (15.3) for $m>3$. It suffices to consider $n=m+1$. The following observation will be useful.

> **(15.7)** *If $n=m+1$ and $m>3$, then each $(q+1)$-point line of $M$ is contained in a rank-$(m-1)$ flat of $PG(m,q)$ that is disjoint from $X$.*

Indeed, assume $\ell$ is a $(q+1)$-point line of $M$. Since $n=m+1$, we have $|X|=q^2+q$. Since $\ell$ is contained in $(q^{m-1}-1)/(q-1)$ planes of $PG(m,q)$ and $(q^{m-1}-1)/(q-1) > q^2+q$, it follows that $\ell$ is contained in a plane $\pi$ of $PG(m,q)$ that is disjoint from $X$, which proves the claim if $m=4$. If $m>4$, then argue in the same manner to get $\pi$ contained in a rank-4 flat of $PG(m,q)$ that is disjoint from $X$, and so on, until the required rank-$(m-1)$ flat is obtained.

Now assume $m>3$ and $n=m+1$. Let $T$ be a rank-$(m-1)$ flat of $M$ with $M|T$ isomorphic to $PG(m-2,q)$. Let $H_1, H_2, \dots, H_{q+1}$ be the hyperplanes of $PG(m,q)$ that contain $T$. Let $X_i = H_i \cap X$ and let $\mathrm{cl}_P(X_i) \cap T = \{a_i\}$ for $i=1,\dots,q+1$. We first argue that $a_1 = a_2 = \cdots = a_{q+1}$.

Assume this is not the case; in particular, assume $a_1 \neq a_2$. We show that this yields the contradiction that $M$ has a $Q_{m,q}$-subgeometry. Note that $X_1$ and $X_2$ are not coplanar, for otherwise they would be in the same rank-$m$ flat of $PG(m,q)$ containing $T$, contradicting (15.2). Let $X_1 = \{x_1, x_2, \dots, x_q\}$. Consider the $q$ planes $\mathrm{cl}_P(X_2 \cup \{x_i\})$. Note that no line $\ell$ of $PG(m,q)$ through $a_2$ and contained in $\mathrm{cl}_P(X_2 \cup \{x_i\})$ is disjoint from $X$, otherwise the rank-$(m-1)$ flat containing $\ell$ given by (15.7) would be contained in a rank-$m$ flat of $PG(m,q)$ containing more than $q$ points of $X$, contrary to (15.2). It follows that each of the $q$ lines in $\mathrm{cl}_P(X_2 \cup \{x_i\})$ through $a_2$ besides $\mathrm{cl}_P(X_2)$ contains at least one point of $X$, so $\mathrm{cl}_P(X_2 \cup \{x_i\})$ contains at least $2q$ points of $X$. Since any two of these planes intersect precisely in $\mathrm{cl}_P(X_2)$, it follows that $\mathrm{cl}_P(X_1 \cup X_2)$ contains at least $q + q \cdot q$ points of $X$. Thus $X \subset \mathrm{cl}_P(X_1 \cup X_2)$, so $r(X)=4$. Since $|X| = q^2+q$ and $r(X)=4$, statement (15.4) applies: $\mathrm{cl}_P(X)$ contains a subgeometry of $M$ isomorphic to $Q_{3,q}$, so there are noncollinear points $p_1, p_2, p_3$ of $\mathrm{cl}_P(X)$ such that $\mathrm{cl}_P(\{p_1,p_2\})$, $\mathrm{cl}_P(\{p_1,p_3\})$, and $\mathrm{cl}_P(\{p_2,p_3\})$ are lines of $M$. Let $Z$ be a rank-$(m-3)$ flat of $PG(m,q)$ such that $X \cup Z$ spans $PG(m,q)$. From modularity, it follows that $\mathrm{cl}_P(Z \cup \{p_3\}) \cap \mathrm{cl}_P(X) = \{p_3\}$. Let $B = \{b_1, b_2, \dots, b_{m-2}\}$ be a basis of $\mathrm{cl}_P(Z \cup \{p_3\})$ for which $C(p_3,B)$ is $B \cup p_3$. We claim that $B \cup \{p_1,p_2\}$ generates a subgeometry of $M$ isomorphic to $Q_{m,q}$, producing the desired contradiction. We need to argue that the $m$ flats $\mathrm{cl}_P(\{p_2\}\cup B)$, $\mathrm{cl}_P(\{p_1\}\cup B)$, and $\mathrm{cl}_P(\{p_1,p_2\}\cup(B-\{b_i\}))$ of $PG(m,q)$ are contained in $M$. Thus we need only argue that $\mathrm{cl}_P(\{p_2\}\cup B)\cap\mathrm{cl}_P(X)$, $\mathrm{cl}_P(\{p_1\}\cup B)\cap\mathrm{cl}_P(X)$, and $\mathrm{cl}_P(\{p_1,p_2\}\cup(B-\{b_i\}))\cap\mathrm{cl}_P(X)$ are flats of $M$. By modularity, $\mathrm{cl}_P(\{p_2\}\cup B)$ intersects $\mathrm{cl}_P(X)$ in a line. Since $p_3\in\mathrm{cl}_P(B)$, this line is $\mathrm{cl}_P(\{p_2,p_3\})$, which is contained in $M$. The argument in the second case is similar. Again by modularity, $\mathrm{cl}_P(\{p_1,p_2\}\cup(B-\{b_i\}))$ intersects $\mathrm{cl}_P(X)$ in a line, so we have $\mathrm{cl}_P(\{p_1,p_2\}\cup(B-\{b_i\}))\cap\mathrm{cl}_P(X)=\mathrm{cl}_P(\{p_1,p_2\})$, which is contained in $M$. Thus $B\cup\{p_1,p_2\}$ generates a $Q_{m,q}$-subgeometry of $M$. This contradiction arose from assuming $a_1\neq a_2$, so we deduce that there is a single point $a$ of $T$ such that $\mathrm{cl}_P(X_i)\cap T=\{a\}$ for $i=1,2,\ldots,q+1$.

To complete the proof of (15.3), and hence the theorem, we need to show that $X_1,X_2,\ldots,X_{q+1}$ are coplanar. If this were not the case, then the plane $\mathrm{cl}_P(X_1\cup X_2)$ would contain a $(q+1)$-point line $\ell$ of $M$ through $a$. By (15.7), $\ell$ would be contained in a rank-$(m-1)$ flat $T$ of $PG(m,q)$ that is also a flat of $M$. However, $\mathrm{cl}_P(T\cup X_1)$ is a rank-$m$ flat of $PG(m,q)$ that contains $T$ and at least $2q$ points of $X$, namely the points of $X_1\cup X_2$, contrary to (15.2). Thus, $X_1,X_2,\ldots,X_{q+1}$ are coplanar, as needed. □

We single out the case of rank-3 Dowling lattices as being of special interest. The geometry $AG^+(n-1,q)$ in Corollary 16 and Theorem 17 is

$$PG(n-1,q)\backslash\big(PG(n-2,q)-y\big),$$

the unique single-element extension of $AG(n-1,q)$ in $PG(n-1,q)$.

**Corollary 16.** *For $n\geq 2$ and $q>2$, we have $ex_q(Q_3(GF(q)^*);n)=q^{n-1}+1$ and*

$$MAX(EX_q(Q_3(GF(q)^*);n))=\{AG^+(n-1,q)\}.$$

Recall that an alternative way to construct $Q_3(GF(q)^*)$ is to delete a $U_{3,4}$-subgeometry from $PG(2,3)$. Theorem 17 finds $ex_3(PG(2,3)\backslash U_{3,3};n)$ and the geometries that have this number of points. The proof is omitted since all ideas needed appear in the proof of Theorem 15 and the proof is much more straightforward than that of Theorem 15.

**Theorem 17.** *Let $G=PG(2,3)\backslash U_{3,3}$. For $n\geq 2$ we have $ex_3(G;n)=3^{n-1}+1$ and*

$$MAX(EX_3(G;n))=\{AG^+(n-1,3)\}.$$

## 3. Excluding binary affine geometries.

In contrast to the results in Section 2, finding the exact value of $ex_q(G;n)$ for most geometries $G$ appears to be quite difficult. A more tractable problem in some cases is finding the limit of the ratio of $ex_q(G;n)$ to the number of points in $PG(n-1,q)$,

$$\lim_{n\to\infty}\frac{ex_q(G;n)}{(q^n-1)/(q-1)}.$$

Upon dividing by the number of points in $PG(n-1,q)$, Lemma 2 gives

$$\frac{q^n-q^{n-c+1}}{q^n-1}\leq\frac{ex_q(G;n)}{(q^n-1)/(q-1)}\leq\frac{q^n-q^{n-m+1}}{q^n-1} \tag{2}$$

where $c$ is the critical exponent of $G$ over $GF(q)$ and $m$ is the rank of $G$. Thus, the greater the gap between the rank of $G$ and the critical exponent of $G$, the greater the gap between the upper and lower bounds in this inequality.

It is mentioned in Section 3.1 of [3] that little is known about $ex_q(G;n)$ when the critical exponent of $G$ is 1. This is the case in which $G$ is affine over $GF(q)$ or, equivalently, $G$ is a subgeometry of $AG(m-1,q)$. In this case, the lower bound in (2) is zero. Theorem 20 shows that when $q$ is 2 and $G$ is affine, this lower bound is actually the limit of the ratio $ex_2(G;n)/(2^n-1)$. Before turning to this, we improve the lower bound in (2) when $G$ is the affine geometry $AG(m-1,q)$. Theorem 18 shows that $ex_q(AG(m-1,q);n)$ is bounded below by an exponential function.

**Theorem 18.** *For $n, m \geq 3$, we have*

$$\frac{6^{1/3}q^{n/3}}{q-1} \leq ex_q(AG(m-1,q);n).$$

Theorem 18 follows from Lemmas 8 and 19 since the uniform matroid $U_{3,4}$ is a subgeometry of every affine geometry of rank three or more. The key idea in the proof of Lemma 19 is essentially the same as in the proof of the Gilbert-Varshamov bound in coding theory.

**Lemma 19.** *The size function $ex_q(U_{3,4};n)$ is at least $k+1$ if*

$$k(q-1) + \binom{k}{3}(q-1)^3 < q^n - 1.$$

*In particular, for $n \geq 3$ we have*

$$\frac{6^{1/3}q^{n/3}}{q-1} \leq ex_q(U_{3,4};n).$$

*Proof.* To show $k+1 \leq ex_q(U_{3,4};n)$, we construct an $n \times (k+1)$ matrix $A$ over $GF(q)$ having no column of zeros, having no column that is a scalar multiple of another column, and having no set of four columns that is a circuit. Construct the matrix $A$ one column at a time, starting with an arbitrary nonzero column. Having $i$ columns that satisfy these conditions, the next column can be any $n$-tuple over $GF(q)$ except the $i(q-1)$ multiples of columns already in $A$ and the $\binom{i}{3}(q-1)^3$ or fewer columns that form a 4-circuit with three columns already in $A$. There is such a column if

$$i(q-1) + \binom{i}{3}(q-1)^3 < q^n - 1,$$

which proves the first assertion. The second follows from this by an elementary computation. □

Theorem 20 is the main result of this section.

**Theorem 20.** *Let $G$ be a binary affine geometry. Then*

$$\lim_{n\to\infty} \frac{ex_2(G;n)}{2^n - 1} = 0.$$

By Lemma 8, to prove Theorem 20, we may assume $G$ is $AG(m-1,2)$. In particular, we need an upper bound for $ex_2(AG(m-1,2);n)$ that is considerably smaller than $2^n - 1$. This is provided by Lemma 21, which improves the upper bound in (2) when $q$ is 2 and $G$ is affine.

**Lemma 21.** *For $n \geq m \geq 3$, we have $ex_2(AG(m-1,2);n) < 2^{n\,t_m+1}$ where $t_m = 1 - 1/2^{m-2}$.*

*Proof.* The proof is by induction on $m$. Throughout we assume $M$ is a rank-$n$ binary geometry with $s$ points. We consider $M$ to be a particular restriction of $PG(n-1,2)$, say $PG(n-1,2)|S$. We say that points $x$ and $y$ of $M$ *determine* the point $z$ of $PG(n-1,2)$ if $\{x,y,z\}$ is a line of $PG(n-1,2)$. Thus, $z$ may or may not be in $S$.

Note that for $M$ to have no $AG(2,2)$-subgeometry, no two pairs of points of $M$ can determine the same point of $PG(n-1,2)$. Thus $\binom{s}{2} \leq 2^n - 1$. Since $m \geq 3$, we have $s \geq 3$, so $s^2/3 \leq \binom{s}{2}$. Thus $s^2/3 < 2^n$, so $s < 2^{(n+\log_2 3)/2}$. This establishes the case $m = 3$ since $(\log_2 3)/2 < 1$.

Now assume $m \geq 3$ and the lemma holds for $m$. A subgeometry $M'$ of $M$ isomorphic to $AG(m,2)$ would consist of $2^{m-1}$ pairs of points in $M$ (corresponding to the lines in a pencil of parallel lines), each determining the same point $z$ in the complement $PG(n-1,2) - M'$ of $M'$, and for which $AG(m-1,2)$ is obtained by restricting $PG(n-1,2)$ to the ground set of $M'$ together with $z$ and then contracting $z$. Thus, if $M$ has no $AG(m,2)$-subgeometry, each point in $PG(n-1,2)$ must be determined by fewer than $2^{(n-1)t_m+1}$ pairs in

$M$, or fewer than $2^{(n-1+\log_2 3)/2}$ pairs in the case of $m = 3$. Therefore, if $M$ has no $AG(m, 2)$-subgeometry, we have

$$\binom{s}{2} < (2^n - 1)2^{(n-1)t_m+1},$$

while for $m = 3$, we get the stronger inequality

$$\binom{s}{2} < (2^n - 1)2^{(n-1+\log_2 3)/2}.$$

Replace $\binom{s}{2}$ by $s^2/3$ and manipulate as above. The resulting inequalities are

$$s < 2^{n\, t_{m+1} + (\frac{1}{2})^{m-1} + \frac{1}{2}\log_2 3}$$

and

$$s < 2^{\frac{3}{4}n - \frac{1}{4} + \frac{3}{4}\log_2 3},$$

respectively. These yield the inequality of the lemma since $(\frac{1}{2})^{m-1} + \frac{1}{2}\log_2 3 < 1$ for $m \geq 4$ and $-\frac{1}{4} + \frac{3}{4}\log_2 3 < 1$. □

While it seems likely that the analog of Theorem 20 is true for all prime powers $q$, this appears to be considerably more difficult to prove.

The final corollary follows from Lemmas 19 and 21. The upper and lower bounds are far apart; it would be of interest to narrow this gap.

**Corollary 22.** *For $n \geq 3$, we have* $6^{\frac{1}{3}} 2^{\frac{n}{3}} \leq ex_2(AG(2, 2); n) \leq 2^{\frac{n}{2}+1}$.

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