TUTTE POLYNOMIALS OF q-CONES

JOSEPH E. BONIN AND HONGXUN QIN

ABSTRACT. We derive a formula for the Tutte polynomial $t(G'; x, y)$ of a q-cone G' of a $GF(q)$-representable geometry G in terms of $t(G; x, y)$. We use this to construct collections of infinite sequences of $GF(q)$-representable geometries in which corresponding geometries are not isomorphic and yet have the same Tutte polynomial. We also use this to construct, for each positive integer k, sets of non-isomorphic $GF(q)$-representable geometries all of which have the same Tutte polynomial and vertical (or Whitney) connectivity at least k.

1. INTRODUCTION

To study tangential blocks, Whittle [9] introduced q-cones of geometries (simple matroids) representable over $GF(q)$. (In [9] and [5], q-cones are called q-lifts.)

Definition 1. *Let G be a rank-r geometry representable over $GF(q)$. A geometry G' is a q-*cone *of G with* base S *and* apex a *if*

(i) *the restriction $PG(r,q)|S$ of $PG(r,q)$ to the subset S of $PG(r,q)$ is isomorphic to G,*
(ii) *a is a point of $PG(r,q)$ not contained in the closure, $\mathrm{cl}_P(S)$, of S in $PG(r,q)$, and*
(iii) *G' is the restriction of $PG(r,q)$ to the set $\bigcup_{p\in S} \mathrm{cl}_P(\{a,p\})$.*

In other words, one represents G as a set S of points in $PG(r,q)$ and constructs G' by restricting $PG(r,q)$ to the set of points on the lines joining the points of S to a fixed point a outside the hyperplane of $PG(r,q)$ spanned by S.

It is easy to see that equivalent representations of a rank-r geometry G in $PG(r,q)$ give rise to isomorphic q-cones of G. In Section 8.6 of [5], Kung raised a natural question: Can one get non-isomorphic q-cones of G from inequivalent representations? Oxley and Whittle [8] answered this question in the affirmative with explicit examples.

While there can be many q-cones of a geometry G, Kung showed that all q-cones of G have the same characteristic polynomial and that this polynomial can be expressed in terms of the characteristic polynomial $\chi(G;\lambda)$ of G. Specifically, Kung proved the following result [5, Theorem 8.23]).

Theorem 2. *For every q-cone G' of a rank-r geometry G, we have*

$$\chi(G';\lambda) = (\lambda-1)q^r\chi(G;\lambda/q).$$

The main result of this paper is Theorem 6 which gives a formula for the Tutte polynomial of all q-cones of G in terms of the Tutte polynomial $t(G; x, y)$ of G; in particular, all q-cones of G have the same Tutte polynomial. In Section 3, we extend this in two ways. First, given any collection of m non-isomorphic $GF(q)$-representable geometries all of which have the same Tutte polynomial, one can construct at least m infinite sequences of $GF(q)$-representable geometries in which corresponding geometries are not isomorphic and yet have the same Tutte polynomial. Second, such sequences will contain geometries of arbitrarily large vertical (or Whitney) connectivity.

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We assume familiarity with basic matroid theory. Our notation and terminology follow [7] with this exception: we use the term geometry (short for combinatorial geometry) for a simple matroid. Typically we consider a geometry G that is a specific restriction of a projective geometry $PG(r,q)$; to distinguish the two closure operators, we use cl for the closure operator of G and cl_P for the closure operator of $PG(r,q)$. If there might otherwise be ambiguity, we denote the closure operator of G by cl_G. We use the following properties of q-cones from [9].

Lemma 3. *Let G' be a q-cone of G with base S and apex a, and let T be the ground set of G'.*

(i) *If H is a hyperplane of $PG(r,q)$ with $a \notin H$, then $H \cap T$ is a hyperplane of G'.*
(ii) *If H' is a hyperplane of G' with $a \notin H'$, then $G'|H'$ is isomorphic to G.*
(iii) *If H' is a hyperplane of G' with $a \in H'$, then $G'|H'$ is a q-cone of $G|F$ where F is the hyperplane $H' \cap S$ of G.*

For ease of reading, we describe situations such as (iii) above by saying a flat F' of G' is a q-cone of a flat F of G, where it is understood that this actually refers to the restrictions to these flats.

2. The Tutte Polynomial of a q-Cone

Our main result, Theorem 6, gives a formula for the Tutte polynomial $t(G';x,y)$ of a q-cone G' of a geometry G in terms of $t(G;x,y)$. Recall that the Tutte polynomial $t(G;x,y)$ of a matroid G on the set S is given by

$$t(G;x,y) = \sum_{A \subseteq S} (x-1)^{r(S)-r(A)}(y-1)^{|A|-r(A)}.$$

The characteristic polynomial $\chi(G;\lambda)$ of G is, up to sign, a special evaluation of the Tutte polynomial of G:

$$\chi(G;\lambda) = (-1)^{r(G)} t(G;1-\lambda,0).$$

Brylawski [1] defined the weighted characteristic polynomial $\overline{\chi}(G;x,y)$ of a matroid G to be

$$\overline{\chi}(G;x,y) = \sum_{F \in \mathcal{L}(G)} x^{|F|}\chi(G/F;y)$$

where $\mathcal{L}(G)$ is the lattice of flats of G. (This sum could be taken over all subsets F of the ground set of G since if F is not a flat, G/F has loops, and so $\chi(G/F;y)$ is 0. The weighted characteristic polynomial is, upon switching the variables, the coboundary polynomial of [4]. See also Section 6.3.F of [3], where the notation $\overline{\chi}(G;x,y)$ is used for the coboundary polynomial.) The following formulas are well-known and easy to prove.

$$\begin{aligned} t(G;x,y) &= \frac{\overline{\chi}(G;y,(x-1)(y-1))}{(y-1)^{r(G)}} \\ \overline{\chi}(G;x,y) &= (x-1)^{r(G)} t(G;\frac{y}{x-1}+1,x) \end{aligned} \tag{1}$$

Thus, one can obtain $t(G;x,y)$ from $\overline{\chi}(G;x,y)$ and conversely.

To derive the formula for $t(G';x,y)$ in Theorem 6, we use the weighted characteristic polynomial along with Theorem 2, and then use (1) to recast the result in terms of the Tutte polynomial. To carry this out, we need to identify the flats F of G' and the contractions G'/F of G' by these flats. These steps are treated in the next two lemmas. Lemma 4 extends Lemma 3 by giving a description and enumeration of all proper flats of a q-cone of a geometry.

Lemma 4. *Let G be a rank-r geometry and let G' be a q-cone of G with base S and apex a. For each j with $0 \leq j \leq r$, let $F_{j1}, F_{j2}, \ldots, F_{jk_j}$ be the rank-j flats of G. Then for each j with $1 \leq j \leq r$, there are $k_{j-1} + q^j k_j$ flats of G' having rank j, with these corresponding to the flats of G as follows:*

(i) *one flat isomorphic to a q-cone of $F_{(j-1)i}$ for each i with $1 \leq i \leq k_{j-1}$, and*
(ii) *q^j flats A with $G'|A$ isomorphic to $G|F_{ji}$ for each i with $1 \leq i \leq k_j$.*

Proof. Let A be a rank-j flat of G' containing the apex a. Let $A' = S \cap A$. Note that A is a union of $(q+1)$-point lines of $PG(r,q)$, each containing a. Therefore each line of A through a contains one point in the hyperplane $\mathrm{cl}_P(S)$, and hence in S; it follows that A' has rank $j-1$. Also, A' is a flat of G since A' is $S \cap \mathrm{cl}_P(A)$, and A is a q-cone of A'. Conversely, for a rank-$(j-1)$ flat A' of G, the rank-j flat $\mathrm{cl}(A' \cup a)$ of G' is a q-cone of A'.

Let A be a rank-j flat of G' not containing the apex a. Let A^* be $\mathrm{cl}(A \cup \{a\})$. From the last paragraph, A^* is a q-cone of a rank-j flat A' of G with apex a. Since A is a hyperplane of $G'|A^*$ not containing a, it follows from Lemma 3 that $G'|A$ and $G|A'$ are isomorphic. Furthermore, since there are q^j hyperplanes of the rank-$(j+1)$ geometry $PG(r,q)|\mathrm{cl}_P(A^*)$ that do not contain a, it follows from Lemma 3 that each rank-j flat A' of G gives rise to precisely q^j flats A of G' with $G'|A$ isomorphic to $G|A'$, completing the proof of the lemma. □

Lemma 5 examines the contractions G'/F of G' by flats of G'. To aid in phrasing the second part of this lemma precisely, we assume that G is a particular subgeometry of $PG(r,q)$; this results in no loss of generality. Essentially the same lemma has appeared elsewhere (Lemma 2.5 of [9] and Lemma 8.19 of [5]), but since this lemma plays a key role in the proof of Theorem 6 and our formulation of the lemma is slightly more precise, we provide a proof. Here, and later, we use the notation $\widetilde{M}$ as in [7] for the simplification of a matroid M.

Lemma 5. *Assume that the rank-r geometry G is the restriction $PG(r,q)|S$. Let G' be a q-cone of G with base S and apex a and let F be a flat of G'.*

(i) *If $a \in F$ and F is a q-cone of the flat X of G, then $\widetilde{G'/F}$ is isomorphic to $\widetilde{G/X}$.*
(ii) *If $a \notin F$, then $\widetilde{G'/F}$ is isomorphic to a q-cone of $\widetilde{G/X}$ where X is the flat of G corresponding to F, i.e., $\cup_{x \in F}\big(S \cap \mathrm{cl}(\{x,a\})\big)$.*

Proof. Assume first that a is in F. Thus, each flat of G' that contains F is a q-cone of a flat of G that contains X. Furthermore, from the proof of Lemma 4 there is an inclusion-preserving bijection between the flats of G' that contain F and the flats of G/X, so (i) follows.

Now assume that a is not in F and that F has rank k. Being a q-cone, G' is a restriction, say $PG(r,q)|T$, of $PG(r,q)$. Let H be a hyperplane of $PG(r,q)$ that contains F but not a. Thus, $G'|(T \cap H)$ is isomorphic to G, so it suffices to show that the lattice of flats of G'/F is isomorphic to the lattice of flats of a q-cone of the simplification of $G'|(T \cap H)/F$. To see this, note that G'/F is a restriction of $PG(r,q)/F$, which has $PG(r-k,q)$ as its simplification. Furthermore the ground set $(T \cap H) - F$ of $G'|(T \cap H)/F$ is contained in the hyperplane $H - F$ of $PG(r,q)/F$; also, $\mathrm{cl}_{G'}(F \cup \{a\})$ is a point of G'/F not contained in the hyperplane $H - F$. Thus it suffices to show that for each rank-$(k+1)$ flat X of G' containing F other than $\mathrm{cl}_{G'}(F \cup \{a\})$, the rank-$(k+2)$ flat $\mathrm{cl}_{G'}(F \cup \{a\} \cup X)$ contains exactly $q+1$ rank-$(k+1)$ flats of G' that contain F, with precisely one of these $q+1$ flats contained in $T \cap H$. This follows easily since G' is a q-cone of G. □

We are now ready to prove our main result.

Theorem 6. *If G is a rank-r geometry representable over $GF(q)$ and G' is a q-cone of G, then*

$$t(G';x,y) = \frac{y(y^q-1)^r}{(y-1)^{r+1}} t(G; \frac{(x-1)(y-1)}{y^q-1}+1, y^q) + \frac{q^r(xy-x-y)}{y-1} t(G; \frac{x-1}{q}+1, y).$$

Proof. The weighted characteristic polynomial of G' can be written as

$$\begin{aligned}\overline{\chi}(G';x,y) &= \sum_{F\in\mathcal{L}(G')} x^{|F|}\chi(G'/F;y) \\ &= \sum_{F\in\mathcal{L}(G'):a\in F} x^{|F|}\chi(G'/F;y) \;+\; \sum_{F\in\mathcal{L}(G'):a\notin F} x^{|F|}\chi(G'/F;y)\end{aligned}$$

where a is the apex of G'. We use Lemmas 4 and 5 along with Theorem 2 to compute the last two sums separately. The flats F of G' containing the apex a are q-cones of the flats of G, and for such flats, $\widetilde{G'/F}$ is isomorphic to $\widetilde{G/X}$ for the corresponding flat X of G, so the first sum is

$$\begin{aligned}\sum_{F\in\mathcal{L}(G'):a\in F} x^{|F|}\chi(G'/F;y) &= \sum_{X\in\mathcal{L}(G)} x^{q|X|+1}\chi(G/X;y) \\ &= x\sum_{X\in\mathcal{L}(G)} (x^q)^{|X|}\chi(G/X;y) \\ &= x\,\overline{\chi}(G;x^q,y).\end{aligned}$$

For flats F of G' not containing the apex, $\widetilde{G'/F}$ is isomorphic to a q-cone of $\widetilde{G/X}$ where X is the corresponding flat of G; furthermore, by Theorem 2 all q-cones of $\widetilde{G/X}$ have the same characteristic polynomial as any particular such q-cone $(G/X)'$. For each flat X of G, there are $q^{r(X)}$ corresponding flats of G', so the second sum is

$$\sum_{F\in\mathcal{L}(G'):a\notin F} x^{|F|}\chi(G'/F;y) = \sum_{X\in\mathcal{L}(G)} q^{r(X)}(x^{|X|}\chi((G/X)';y)).$$

By Theorem 2,

$$\begin{aligned}\chi((G/X)';y) &= (y-1)q^{r(G/X)}\chi(G/X;y/q) \\ &= (y-1)q^{r(G)-r(X)}\chi(G/X;y/q).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{F\in\mathcal{L}(G'):a\notin F} x^{|F|}\chi(G'/F;y) &= \sum_{X\in\mathcal{L}(G)} q^{r(X)}x^{|X|}(y-1)q^{r(G)-r(X)}\chi(G/X;y/q) \\ &= q^r(y-1)\sum_{X\in\mathcal{L}(G)} x^{|X|}\chi(G/X;y/q) \\ &= q^r(y-1)\,\overline{\chi}(G;x,y/q).\end{aligned}$$

Therefore,

$$\overline{\chi}(G';x,y) = x\,\overline{\chi}(G;x^q,y) + q^r(y-1)\overline{\chi}(G;x,y/q).$$

Using (1) to recast this in terms of the Tutte polynomial yields the formula stated in the theorem. □

Not surprisingly, setting x to $1-\lambda$ and y to 0 in Theorem 6 gives the formula for the characteristic polynomial stated in Theorem 2. In contrast, the formula for the characteristic polynomial of the dual, $(G')^*$, of G' that results from setting x to 0 and y to $1-\lambda$ involves polynomials other than the characteristic polynomial of G and G^*.

3. Vertical Connectivity and Chains of q-Cones

Given non-isomorphic matroids M and N with the same Tutte polynomial, several standard operations can be applied to produce additional examples of non-isomorphic matroids that have the same Tutte polynomial. One can obtain such examples, for instance, by adding loops or isthmuses, or by other applications of the operation of direct sum. This construction, of course, produces disconnected matroids. To obtain additional connected matroids that have the same Tutte polynomial, one can apply the operations of free extension and free coextension (see Proposition 4.2 in [2]); however, even if M and N are representable over $GF(q)$, the free extensions, $M + e$ and $N + e$, and free coextensions, $M \times e$ and $N \times e$, need not be representable over $GF(q)$. In this section, we use q-cones to provide collections of infinite sequences of connected geometries that are representable over $GF(q)$ where corresponding geometries in the sequences are not isomorphic and yet have the same Tutte polynomial. To produce such sequences, simply iterate the construction of q-cones starting with the geometries in any collection of non-isomorphic $GF(q)$-representable geometries that have the same Tutte polynomial. (For such examples, see [2, 3].) In addition to producing such sequences that are representable over a given finite field, the vertical connectivity of the geometries in such sequences increases by 1 for successive q-cones. These ideas are made precise in this section.

Definition 7. *A sequence $G_0, G_1, G_2, \dots$ of geometries is called a* chain of q-cones *if G_k is a q-cone of G_{k-1} for each $k \geq 1$.*

The examples in [8] show that there may be many chains of q-cones with the same first term. Note that for a chain $G_0, G_1, G_2, \dots$ of q-cones, one can compute $t(G_k; x, y)$ for $k \geq 1$ from $t(G_0; x, y)$ by iterating the formula in Theorem 6. (Since the projective geometries $PG(0, q)$, $PG(1, q)$, $PG(2, q), \dots$ form a chain of q-cones, one could compute the Tutte polynomial of $PG(n-1, q)$ recursively this way. However, the approach of Mphako [6] via the coboundary polynomial [4] yields a more elegant formula for the Tutte polynomial of $PG(n-1, q)$.)

Theorem 8 compares the vertical connectivity of G with that of a q-cone G' of G. We first recall the relevant definitions. Let k be a positive integer. A vertical k-separation of a matroid G with ground set S is a partition (X, Y) of S such that

$$\min\{r(X), r(Y)\} \geq k$$

and

$$r(X) + r(Y) - r(G) \leq k - 1.$$

The vertical connectivity $\kappa(G)$ of G is the least positive integer k such that G has a vertical k-separation; if there is no such k, then $\kappa(G)$ is the rank, $r(G)$, of G. Thus $\kappa(G) \leq r(G)$. (Vertical connectivity is also known as Whitney connectivity. See Section 8.2 of Oxley [7] for more information on vertical connectivity.)

Theorem 8. *If G' is a q-cone of G, then $\kappa(G') = \kappa(G) + 1$.*

Proof. Assume that the q-cone G' of G has apex a, that G has rank r, and that S and T are the ground sets of G and G', respectively. To get the desired conclusion, we prove the two inequalities $\kappa(G') \leq \kappa(G) + 1$ and $\kappa(G') \geq \kappa(G) + 1$.

The inequality $\kappa(G') \leq \kappa(G) + 1$ is obvious if $\kappa(G) = r$, so assume $\kappa(G) = k < r$. Let (X, Y) be a vertical k-separation of G. Let

$$X' = \bigcup_{x \in X} \mathrm{cl}(\{a, x\})$$

and

$$Y' = \bigcup_{y \in Y} \big(\mathrm{cl}(\{a, y\}) - \{a\}\big).$$

It follows easily that (X', Y') is a vertical $(k+1)$-separation of G', so $\kappa(G') \leq k+1$, as needed.

The inequality $\kappa(G') \geq \kappa(G) + 1$, or $\kappa(G') - 1 \geq \kappa(G)$, is obvious if $\kappa(G') = r+1$, so assume $\kappa(G') = k \leq r$. We need to show that G has a vertical j-separation for some j with $1 \leq j \leq k-1$. Let (X', Y') be a vertical k-separation of G'. Therefore

$$\min\{r(X'), r(Y')\} \geq k$$

and

$$r(X') + r(Y') - (r+1) \leq k - 1.$$

Let $X = X' \cap S$ and $Y = Y' \cap S$. Thus $X \cup Y = S$. Let $x = r(X') - r(X)$ and $y = r(Y') - r(Y)$. Thus $x, y \geq 0$. If $x = 0$, then $X' = X \subseteq S$, so $r(Y') = r+1$. However, this yields the contradiction

$$k = k + (r+1) - (r+1) \leq r(X') + r(Y') - (r+1) \leq k - 1.$$

Thus x, and by symmetry y, is positive. If $r(X) = 0$, then $r(Y) = r$ and either $r(Y') = r+1$ or $r(X') = r+1$. From this, we get the same contradiction as above, so $r(X) \geq 1$ and, by symmetry, $r(Y) \geq 1$. Thus (X, Y) is a partition of S. Without loss of generality, assume $x \leq y$. Then

$$\min\{r(X), r(Y)\} \geq k - y$$

and

$$r(X) + r(Y) - r = r(X') - x + r(Y') - y - r \leq k - x - y \leq (k-y) - 1.$$

By semimodularity and the last inequality, it follows that $k-y$ is positive, so (X, Y) is a vertical $(k-y)$-separation. Since $y \geq 1$, we have

$$\kappa(G) \leq k - y \leq k - 1 = \kappa(G') - 1,$$

as needed to complete the proof □

To apply our results to the types of examples in [2, 3], we need the following lemma.

Lemma 9. *Assume that G and M are rank-r geometries that are representable over $GF(q)$. Assume that G' and M' are q-cones of G and M, respectively. If G and M are not isomorphic, then G' and M' are not isomorphic.*

Proof. From Lemma 3, it follows that G' has q^r hyperplanes isomorphic to G and M' has q^r hyperplanes isomorphic to M. Thus if G' and M' were isomorphic, each would have at least $2q^r$ hyperplanes. This is a contradiction since G' and M' are rank-$(r+1)$ geometries representable over $GF(q)$ and so have at most $(q^{r+1} - 1)/(q-1)$ hyperplanes. □

From this, it follows that two chains of q-cones starting with non-isomorphic geometries of the same rank will contain no isomorphic geometries. Combining this with Theorems 6 and 8 gives the following result.

Theorem 10. *Assume that M and N are non-isomorphic geometries that are representable over $GF(q)$ and for which $t(M; x, y) = t(N; x, y)$. Assume that $M = M_0, M_1, M_2, \ldots$ and $N = N_0, N_1, N_2, \ldots$ are chains of q-cones. For all $i \geq 0$, the geometries M_i and N_i are not isomorphic yet $t(M_i; x, y) = t(N_i; x, y)$. Furthermore $\kappa(M_i) = \kappa(M_0) + i$ and $\kappa(N_i) = \kappa(N_0) + i$ for all $i \geq 0$.*

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DEPARTMENT OF MATHEMATICS, THE GEORGE WASHINGTON UNIVERSITY, WASHINGTON, DC 20052
E-mail address: jbonin@gwu.edu, hongxun@gwu.edu