A CONSTRUCTION OF INFINITE SETS OF INTERTWINES FOR PAIRS OF MATROIDS

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ABSTRACT. An intertwine of a pair of matroids is a matroid such that it, but none of its proper minors, has minors that are isomorphic to each matroid in the pair. For pairs for which neither matroid can be obtained, up to isomorphism, from the other by taking free extensions, free coextensions, and minors, we construct a family of rank-k intertwines for each sufficiently large integer k. We also treat some properties of these intertwines.

1. INTRODUCTION

If the classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of matroids are minor-closed, then so is $\mathcal{C}_1 \cup \mathcal{C}_2$. If M is an excluded minor for $\mathcal{C}_1 \cup \mathcal{C}_2$, then some minor of M is an excluded minor for $\mathcal{C}_1$ and another is an excluded minor for $\mathcal{C}_2$; furthermore, no proper minor of M has this property. These remarks motivate the following definition. A matroid M is an *intertwine* of matroids M_1 and M_2 if M but none of its proper minors has both an M_1-minor (i.e., a minor isomorphic to M_1) and an M_2-minor. Thus, each excluded minor for $\mathcal{C}_1 \cup \mathcal{C}_2$ is an intertwine of some excluded minor for $\mathcal{C}_1$ and some excluded minor for $\mathcal{C}_2$.

Many important results and problems in matroid theory involve the question of whether the set of excluded minors for a given minor-closed class of matroids is finite; this leads to the question of whether some pairs of matroids have infinitely many intertwines. This question was raised by Tom Brylawski [3]; see also [9, Problem 14.4.6], where it is also attributed to Neil Robertson and, in a different form, to Dominic Welsh. The question was settled affirmatively by Dirk Vertigan in the mid 1990's in unpublished work; we sketch his construction in Section 5. Jim Geelen gave another construction [5, Section 5]: for each pair of spikes, neither being a minor of the other and all elements of which are in dependent transversals, he constructed infinitely many intertwines that are also spikes. (That the class of spikes contains such infinite sets of intertwines follows from Vertigan's construction along with his embedding of the minor ordering on all matroids into that on the class of spikes (for this intriguing embedding, see [5, Section 3]); Geelen's construction is an attractive realization of this phenomenon.) In this paper, we take weaker hypotheses than the earlier constructions used; we assume only that neither M_1 nor M_2 can be obtained, up to isomorphism, from the other via free extensions, free coextensions, and minors; for such a pair (M_1, M_2), we show that particular amalgams of certain free coextensions of M_1 and M_2 are intertwines. This yields many intertwines of each sufficiently large rank; indeed, for some pairs, a variation on our basic construction produces a family of intertwines that grows roughly doubly-exponentially as a function of the rank.

Key background topics are collected in Section 2. The construction and a variation are given in Section 3. In Section 4, we treat properties of these intertwines; for instance, we show that for large ranks, the intertwines we construct have large connectivity and

1991 *Mathematics Subject Classification.* Primary: 05B35.

Key words and phrases. Matroid, intertwine, cyclic flat, free extension, free coextension.

uniform minors of large rank and corank; we show that if both matroids have no free elements, no cofree elements, no coloops, and no loops, then, for a fixed integer k, the intertwines we construct cover the full range of possible sizes for the ground sets of rank-k intertwines of the pair; we also show that the construction preserves certain properties, such as being transversal and being a gammoid. In Section 5, we explain the relation between our construction and Dirk Vertigan's.

We assume readers know basic matroid theory, an excellent account of which is in [9]. For further information on intertwines and their role in matroid theory, and for several tantalizing open problems involving intertwines, we highly recommend [5].

2. Background

The intertwines we construct are defined via cyclic flats and their ranks. A *cyclic set* in a matroid M is a (possibly empty) union of circuits. The cyclic flats of M, ordered by inclusion, form a lattice; indeed, $F \vee G = \mathrm{cl}_M(F \cup G)$ and $F \wedge G$ is the union of the circuits in $F \cap G$. We let $\mathcal{Z}(M)$ denote both the set and the lattice of cyclic flats of M. The following well-known results are easy to prove [9, Problem 2.1.13]:

(1) $\mathcal{Z}(M^*) = \{S - F : F \in \mathcal{Z}(M)\}$, where S is the ground set of M,
(2) $\mathcal{Z}(M_1 \oplus M_2) = \{F_1 \cup F_2 : F_1 \in \mathcal{Z}(M_1) \text{ and } F_2 \in \mathcal{Z}(M_2)\}$, and
(3) a matroid is determined by its cyclic flats and their ranks.

There are many ways to prove property (3); for instance, one can show how to get the circuits or the independent sets, or show that the rank of an arbitrary set Y in M is given by the formula

$$r(Y) = \min\{r(F) + |Y - F| : F \in \mathcal{Z}(M)\}. \tag{2.1}$$

The following result from [11, 2] carries property (3) further.

Proposition 2.1. *Let $\mathcal{Z}$ be a collection of subsets of a set S and let r be an integer-valued function on $\mathcal{Z}$. There is a matroid for which $\mathcal{Z}$ is the collection of cyclic flats and r is the rank function restricted to the sets in $\mathcal{Z}$ if and only if*

(Z0) *$\mathcal{Z}$ is a lattice under inclusion,*
(Z1) *$r(0_{\mathcal{Z}}) = 0$, where $0_{\mathcal{Z}}$ is the least element of $\mathcal{Z}$,*
(Z2) *$0 < r(Y) - r(X) < |Y - X|$ for all sets X, Y in $\mathcal{Z}$ with $X \subset Y$, and*
(Z3) *for all pairs of incomparable sets X, Y in $\mathcal{Z}$,*

$$r(X) + r(Y) \geq r(X \vee Y) + r(X \wedge Y) + |(X \cap Y) - (X \wedge Y)|.$$

Recall that the free extension $M + x$ of the matroid M on S by the element $x \notin S$ is the matroid on $S \cup x$ whose circuits are those of M along with the sets $B \cup x$ as B runs over the bases of M. We extend this notation to sets: $M + X$ is the result of applying free extension iteratively to add all elements of X to M. From the perspective of Proposition 2.1, $M + X$, for $X \neq \emptyset$, is the matroid on $S \cup X$ whose cyclic flats and ranks are (i) the proper cyclic flats F of M, with rank $r_M(F)$, and (ii) $S \cup X$, of rank $r(M)$. Dually, the cyclic flats and ranks of the free coextension $M \times X = (M^* + X)^*$ are (i) the sets $F \cup X$, of rank $r_M(F) + |X|$, for $F \in \mathcal{Z}(M)$ with $F \neq \emptyset$, and (ii) the empty set, of rank 0.

We use only the simplest type of lift and truncation. The *i-fold lift* $L^i(M)$ of M is $(M \times X)\backslash X$ where $|X| = i$; dually, $(M + X)/X$ is the *i-fold truncation,* $T^i(M)$.

The *nullity* of a set Y is $\eta(Y) = |Y| - r(Y)$. Let $\mathcal{Z}'(M)$ be the set of nonempty proper cyclic flats of M and let $\eta(\mathcal{Z}'(M))$ be the sum of the nullities of these flats. The following lemma is easy to prove.

Lemma 2.2. *If $F \in \mathcal{Z}(M\backslash x)$, then $\mathrm{cl}_M(F) \in \mathcal{Z}(M)$. If $\mathrm{cl}_M(F) = F$, then $\eta_{M\backslash x}(F)$ is $\eta_M(F)$; if $\mathrm{cl}_M(F) = F \cup x$, then $\eta_{M\backslash x}(F) = \eta_M(F \cup x) - 1$.*

Dually, if $F \in \mathcal{Z}(M/y)$, then exactly one of F and $F \cup y$ is in $\mathcal{Z}(M)$. The nullities of F in M/y and the corresponding cyclic flat of M agree unless y is a loop of M, in which case $\eta_{M/y}(F) = \eta_M(F \cup y) - 1$.

Thus, if N is a minor of M, then $\eta(\mathcal{Z}'(N)) \leq \eta(\mathcal{Z}'(M))$.

While a cyclic flat of a matroid may give rise to cyclic flats in its restrictions, the next lemma identifies a situation in which this does not happen.

Lemma 2.3. *Let Z be a cyclic flat of M. If a subset U of Z with $|U| \geq \eta(Z)$ is contained in all nonempty cyclic flats that are contained in Z, then $Z - U$ is independent.*

Proof. Assume, to the contrary, that $Z - U$ contains some circuit C. The nonempty cyclic flat $\mathrm{cl}(C)$ is contained in Z, so $U \subseteq \mathrm{cl}(C)$. Thus $U \subseteq \mathrm{cl}(Z - U) = Z$. Now $\eta(Z - U) > 0$ and $U \subseteq \mathrm{cl}(Z - U)$ give $\eta(Z) > |U|$, contrary to the assumed inequality. □

For matroids M_1 on S_1 and M_2 on S_2, a matroid M on $S_1 \cup S_2$ with $M|S_1 = M_1$ and $M|S_2 = M_2$ is called an *amalgam* of M_1 and M_2.

An element x in a matroid M is *free in M* if $M = (M\backslash x) + x$. Dually, an element y is *cofree in M* if $M = (M/y) \times y$. Let $\mathrm{F}(M)$ be the set of all elements of M that are in no proper cyclic flat of M; thus, $\mathrm{F}(M)$ consists of the free elements and coloops of M. Note that $\mathrm{F}(M^*)$ is the intersection of all nonempty cyclic flats of M; it consists of the cofree elements and loops of M.

3. Intertwines

We now construct the matroids of interest. The notation established in this paragraph is used in the rest of the paper. Assume the matroids M_1 and M_2 have positive rank and are defined on disjoint ground sets, S_1 and S_2, respectively. Let r_1 and r_2 be their rank functions, and let η_1 and η_2 be their nullity functions. Fix subsets S_1' of S_1 and S_2' of S_2, an integer k with

$$k \geq r(M_1) + \eta_1(S_1') + r(M_2) + \eta_2(S_2'), \tag{3.1}$$

and sets T_1 and T_2 with

$$|T_1| = k - r(M_1) - |S_2'| \qquad \text{and} \qquad |T_2| = k - r(M_2) - |S_1'| \tag{3.2}$$

where T_1, T_2, and $S_1 \cup S_2$ are mutually disjoint. Let

$$\mathcal{Z} = \mathcal{Z}'\big(M_1 \times (T_1 \cup S_2')\big) \cup \mathcal{Z}'\big(M_2 \times (T_2 \cup S_1')\big) \cup \{\emptyset, S_1 \cup S_2 \cup T_1 \cup T_2\}.$$

(Note that both $M_1 \times (T_1 \cup S_2')$ and $M_2 \times (T_2 \cup S_1')$ have rank k by equations (3.2). Also, inequality (3.1) gives $|T_1| + |S_2'| \geq \eta_1(S_1') + r(M_2) + \eta_2(S_2')$, which is positive; therefore $M_1 \times (T_1 \cup S_2')$ is a proper coextension of M_1 and so has no loops. Likewise $M_2 \times (T_2 \cup S_1')$ has no loops, so $\emptyset$ is a cyclic flat of both matroids.) Define $r : \mathcal{Z} \to \mathbb{Z}$ by

(1) $r(F \cup T_1 \cup S_2') = r_1(F) + |T_1| + |S_2'|$ for $F \in \mathcal{Z}'(M_1)$,
(2) $r(F \cup T_2 \cup S_1') = r_2(F) + |T_2| + |S_1'|$ for $F \in \mathcal{Z}'(M_2)$,
(3) $r(\emptyset) = 0$, and
(4) $r(S_1 \cup S_2 \cup T_1 \cup T_2) = k$.

Theorem 3.1. *The pair $(\mathcal{Z}, r)$ satisfies properties (Z0)-(Z3) of Proposition 2.1. The rank-k matroid M on $S_1 \cup S_2 \cup T_1 \cup T_2$ thus defined has the following properties:*

(i) *M is an amalgam of $M_1 \times (T_1 \cup S_2')$ and $M_2 \times (T_2 \cup S_1')$, and*

(ii) $\eta_1(F) = \eta_M(F \cup T_1 \cup S_2')$ *for* $F \in \mathcal{Z}'(M_1)$ *and* $\eta_2(F) = \eta_M(F \cup T_2 \cup S_1')$ *for* $F \in \mathcal{Z}'(M_2)$.

Proof. We first note a useful inequality: for any $i \in \{1,2\}$ and $F_i \in \mathcal{Z}(M_i)$,

$$|S_i' \cap F_i| - r_i(F_i) \leq \eta_i(S_i'), \tag{3.3}$$

which follows by comparing each side to $\eta_i(S_i' \cap F_i)$.

Property (Z0) holds since any pair of sets in $\mathcal{Z}$ that does not have a join in one of $\mathcal{Z}'\big(M_1 \times (T_1 \cup S_2')\big)$ and $\mathcal{Z}'\big(M_2 \times (T_2 \cup S_1')\big)$ has $S_1 \cup S_2 \cup T_1 \cup T_2$ as the join. Property (Z1) is item (3) above. Property (Z2) follows from this property in $\mathcal{Z}\big(M_1 \times (T_1 \cup S_2')\big)$ and $\mathcal{Z}\big(M_2 \times (T_2 \cup S_1')\big)$, as do all instances of property (Z3) except in the case $X = F_1 \cup T_1 \cup S_2'$ with $F_1 \in \mathcal{Z}(M_1)$ and $Y = F_2 \cup T_2 \cup S_1'$ with $F_2 \in \mathcal{Z}(M_2)$. In this case, the required inequality is

$$r_1(F_1) + |T_1| + |S_2'| + r_2(F_2) + |T_2| + |S_1'| \geq k + |F_1 \cap S_1'| + |F_2 \cap S_2'|,$$

which follows from inequalities (3.1) and (3.3), and equations (3.2).

By symmetry, assertion (i) follows if we show that for any $F \in \mathcal{Z}'(M_2)$, the difference $(F \cup T_2 \cup S_1') - (T_2 \cup (S_2 - S_2'))$ is independent in $M|S_1 \cup T_1 \cup S_2'$. All such differences are contained in $S_1' \cup S_2'$, so it suffices to show that $S_1' \cup S_2'$ is independent in M. To show this, by equation (2.1) it suffices to prove

$$r_M(Z) + |(S_1' \cup S_2') - Z| \geq |S_1'| + |S_2'|$$

for all $Z \in \mathcal{Z}(M)$; again by symmetry, it suffices to show this for $Z = F_1 \cup T_1 \cup S_2'$ with $F_1 \in \mathcal{Z}'(M_1)$. For such Z, the required inequality is

$$r_1(F_1) + |T_1| + |S_2'| + |S_1' - F_1| \geq |S_1'| + |S_2'|,$$

or, using equations (3.2) and manipulating,

$$k \geq r(M_1) + |S_1' \cap F_1| - r_1(F_1) + |S_2'|.$$

This inequality follows from inequalities (3.1) and (3.3) since $|S_2'| \leq r(M_2) + \eta_2(S_2')$.

Assertion (ii) is evident. □

The matroid so constructed depends on M_1, M_2, k, S_1', S_2', T_1, and T_2. If (as in the next result) listing all parameters aids clarity, we use $M_k(M_1, S_1', T_1; M_2, S_2', T_2)$ to denote this matroid; otherwise we simply write M.

The next result, which follows by comparing the cyclic flats and their ranks, shows that combining the construction with duality yields other instances of the same construction.

Theorem 3.2. *With* $j = |S_1| + |S_2| + |T_1| + |T_2| - k$, *we have*

$$\big(M_k(M_1, S_1', T_1; M_2, S_2', T_2)\big)^* = M_j(M_1^*, S_1 - S_1', T_2; M_2^*, S_2 - S_2', T_1).$$

Also, $j \geq r(M_1^*) + \eta_{M_1^*}(S_1 - S_1') + r(M_2^*) + \eta_{M_2^*}(S_2 - S_2')$ *if and only if* k *satisfies inequality (3.1).*

We now treat the main result. A similar but somewhat longer argument would modestly increase the range for k; we opt for the shorter proof since the main interest is in having infinitely many intertwines. Recall that $\mathrm{F}(M)$ is the set of free elements and coloops of M, so $\mathrm{F}(M^*)$ is the set of cofree elements and loops of M.

Theorem 3.3. *Assume that the ground sets* S_1 *and* S_2 *of* M_1 *and* M_2 *are disjoint and that no matroid isomorphic to* M_1 *(resp.,* M_2*) can be obtained from* M_2 *(resp.,* M_1*) by any combination of minors, free extensions, and free coextensions. For each* $i \in \{1,2\}$, *fix a* *set S_i' with* $\mathrm{F}(M_i) \subseteq S_i' \subseteq S_i - \mathrm{F}(M_i^*)$. *If $k \geq 4\max\{|S_1|, |S_2|\}$, then the matroid M defined above is an intertwine of M_1 and M_2.*

Proof. Theorem 3.1 part (i) shows that M_1 and M_2 are minors of M. By symmetry, to prove that M is an intertwine, it suffices to show that for $a \in S_1 \cup T_1$, neither $M\backslash a$ nor M/a has both an M_1-minor and an M_2-minor; furthermore, by Theorem 3.2 and the observation that the hypotheses are invariant under duality, it suffices to treat $M\backslash a$. Now for each $i \in \{1, 2\}$, we have $|T_1| \geq r(M_1^*) + r(M_2) + |S_i| + 1$ since we assumed $k \geq 4\max\{|S_1|, |S_2|\}$. If $M\backslash a\backslash X/Y \simeq M_i$ with $i \in \{1, 2\}$, then $M\backslash a\backslash X/Y$ has $|S_i|$ elements; at least $|T_1| - |X \cap T_1| - |Y \cap T_1| - 1$ of these elements are in T_1, so we have $|X \cap T_1| + |Y \cap T_1| \geq |T_1| - |S_i| - 1$, and therefore $|X \cap T_1| + |Y \cap T_1| \geq r(M_1^*) + r(M_2)$. Thus, either (i) $|X \cap T_1| \geq r(M_1^*)$ or (i*) $|Y \cap T_1| \geq r(M_2)$.

We claim that the three conclusions below follow when inequality (i) holds:

(1) $M\backslash(X \cap T_1) = \big(M_2 \times (T_2 \cup S_1')\big) + \big((S_1 - S_1') \cup (T_1 - X)\big)$,
(2) $M\backslash(X \cap T_1)$ has no M_1-minor, and
(3) $\eta_2(F) = \eta_{M\backslash(X\cap T_1)}(F \cup T_2 \cup S_1')$ for $F \in \mathcal{Z}'(M_2)$.

Item (1) holds since, using Lemma 2.3, we get that the proper cyclic flats and their ranks in the two matroids agree. Item (1) and the hypotheses give item (2). Item (3) is immediate.

Inequalities (i) and (i*) are related by duality, so Theorem 3.2 and the results in the last paragraph give the following conclusions if inequality (i*) holds:

(1*) $M/(Y \cap T_1) = \big(M_1 \times \big((T_1 - Y) \cup S_2'\big)\big) + \big((S_2 - S_2') \cup T_2\big)$,
(2*) $M/(Y \cap T_1)$ has no M_2-minor, and
(3*) $\eta_1(F) = \eta_{M/(Y\cap T_1)}\big(F \cup (T_1 - Y) \cup S_2'\big)$ for $F \in \mathcal{Z}'(M_1)$.

For $a \in (S_1 - S_1') \cup T_1$, assume $M\backslash a$ has an M_1-minor, say $M\backslash a\backslash X/Y$. By item (2), inequality (i*) holds. Since a is in at least one set in $\mathcal{Z}'(M/(Y \cap T_1))$, item (3*) gives $\eta\big(\mathcal{Z}'(M/(Y \cap T_1)\backslash a)\big) < \eta(\mathcal{Z}'(M_1))$; the contradiction $M\backslash a\backslash X/Y \not\simeq M_1$ now follows from Lemma 2.2.

Lastly, for $a \in S_1'$, assume $M\backslash a$ has an M_2-minor, say $M\backslash a\backslash X/Y$. Inequality (i) holds by item (2*). Now $a \in S_1'$ gives $\eta\big(\mathcal{Z}'(M\backslash(X \cap T_1)\backslash a)\big) < \eta(\mathcal{Z}'(M_2))$, which, with Lemma 2.2, gives the contradiction $M\backslash a\backslash X/Y \not\simeq M_2$. $\square$

Assume $\mathrm{F}(M_i) = \emptyset = \mathrm{F}(M_i^*)$ for both $i \in \{1, 2\}$. Reflecting on the proof above shows that $a \in S_1$ if and only if neither $M\backslash a$ nor M/a has an M_1-minor, and likewise for S_2 and M_2. These conclusions and the structure of the cyclic flats of M show that the counterparts of the sets S_1, S_2, T_1, T_2, S_1', and S_2' can be determined from any matroid that is isomorphic to M. This gives the following result.

Corollary 3.4. *Assume* $\mathrm{F}(M_i) = \emptyset = \mathrm{F}(M_i^*)$ *for both* $i \in \{1, 2\}$. *The construction gives at least* $(|S_1| + 1)(|S_2| + 1)$ *nonisomorphic rank-k intertwines of M_1 and M_2 for each integer $k \geq 4\max\{|S_1|, |S_2|\}$. If, in addition, both M_1 and M_2 have trivial automorphism groups, then the construction yields $2^{|S_1|+|S_2|}$ nonisomorphic rank-k intertwines.*

Knowing more about M_1 and M_2 may suggest variations on the construction that yield more intertwines, as we now illustrate. Assume that in addition to satisfying the conditions in Theorem 3.3, neither M_1 nor M_2 has circuit-hyperplanes. Let M be the intertwine constructed above. From the bound on k in Theorem 3.3 we get $|T_1 \cup T_2| > k$. Let $\mathcal{H}$ be a collection of k-subsets of $T_1 \cup T_2$ with $|H \cap H'| \leq k - 2$ whenever H and H' are distinct sets in $\mathcal{H}$. In the construction, replace $\mathcal{Z}$ by $\mathcal{Z} \cup \mathcal{H}$ and extend r to $\mathcal{Z} \cup \mathcal{H}$ by setting $r(H) = k - 1$ for all $H \in \mathcal{H}$. Properties (Z0)–(Z3) of Proposition 2.1 are easily verified. Let M' be the matroid thus constructed. The sets in $\mathcal{H}$ are the circuit-hyperplanes of M'. By comparing the cyclic flats and their ranks, it follows that if $M'\backslash X/Y$ has no circuit-hyperplanes, then $M'\backslash X/Y = M\backslash X/Y$. Since neither M_1 nor M_2 has circuit-hyperplanes, it follows that if some single-element deletion or contraction of M' had both an M_1-minor and an M_2-minor, then the same would be true of the corresponding single-element deletion or contraction of M, contrary to Theorem 3.3. Thus, M' is an intertwine of M_1 and M_2. Thus, we have the following result.

Theorem 3.5. *Assume M_1 and M_2 satisfy the hypotheses of Theorem 3.3 and neither has circuit-hyperplanes. For each integer n, there is an integer k_0 so that if $k \geq k_0$, then M_1 and M_2 have at least n intertwines of rank k.*

To take these ideas a step further, we show that the number of nonisomorphic rank-k intertwines obtained from the variation on the basic construction grows roughly doubly-exponentially as a function of k. Set $a = r(M_1) + |S_2'|$ and $b = r(M_2) + |S_1'|$; also set $c = a+b$. None of a, b, and c depends on k. From equations (3.2) we have $|T_1| = k-a$ and $|T_2| = k - b$, so $|T_1 \cup T_2| = 2k - c$. Fix a set $C \subset T_1 \cup T_2$ with $|C| = c$. To get a neater lower bound on the number of intertwines, we focus only on intertwines of a relatively restricted type, namely, those for which the set C is contained in each circuit-hyperplane. Thus, the circuit-hyperplanes consist of C and $k - c$ other elements from $(T_1 \cup T_2) - C$. Now $|(T_1 \cup T_2) - C| = 2k - 2c$. To simplify the discussion slightly, we assume that $k - c$ is even. Let $h = (k - c)/2$. Pair off the elements of $(T_1 \cup T_2) - C$, thus giving $2h$ pairs. Note that a union of C and h of these pairs of elements has the right size to be a circuit-hyperplane; furthermore, since the elements not in C come in pairs, any two distinct sets of this form intersect in at most $k - 2$ elements. Thus, there are $2^{\binom{2h}{h}}$ intertwines of this special form. The remark before Corollary 3.4 applies as well to the variation on the basic construction: the sets S_1, S_2, T_1, T_2, S_1', and S_2' are distinguished by the isomorphism type of the intertwine. Therefore, the number of nonisomorphic intertwines of this type is at least

$$\frac{2^{\binom{2h}{h}}}{(k-a)!(k-b)!}.$$

To get a more revealing lower bound, apply the obvious inequality $\binom{2h}{h} > 2^{2h}/(2h+1)$ to the numerator (the middle binomial coefficient $\binom{2h}{h}$ exceeds the average of all binomial coefficients $\binom{2h}{i}$); use the inequality $n! \leq \sqrt{n+1}\; n^n\, e^{-n+1}$ for each term in the denominator (see, for example., [4, Exercise 3.13.13]); what results is a lower bound of the form 2^α where α is $4^h/(2h+1)$ minus several terms, each smaller than $k \log_2(k/e)$. Thus, we have roughly doubly-exponential growth for the number of nonisomorphic intertwines.

This discussion and these results suggest several problems. Let $i(k; M_1, M_2)$ denote the number of rank-k intertwines of M_1 and M_2 up to isomorphism. What can be said about $i(k; M_1, M_2)$? If M_1 and M_2 satisfy the hypotheses of Theorem 3.3, is $i(k; M_1, M_2)$ increasing as a function of k, at least for sufficiently large k? If so, under what conditions on M_1 and M_2 is the difference $i(k+1; M_1, M_2) - i(k; M_1, M_2)$ bounded above by a constant or by a polynomial? Under what conditions does $i(k; M_1, M_2)$ grow exponentially or super-exponentially?

A matroid M is a *labelled intertwine* of M_1 and M_2 if M but none of its proper minors has minors equal to M_1 and M_2. As we show next, weaker hypotheses than those in Theorem 3.3 suffice for our construction to yield labelled intertwines.

Theorem 3.6. *Assume S_1 and S_2 are disjoint. If inequality (3.1) holds, neither M_1 nor M_2 is uniform, $\mathcal{Z}'(M_1) \neq \{S_1'\}$, and $\mathcal{Z}'(M_2) \neq \{S_2'\}$, then the matroid M constructed above is a labelled intertwine of M_1 and M_2.*

Proof. By symmetry, to prove that no proper minor of M has both M_1 and M_2 as minors, it suffices to show that if $M\backslash X/Y = M_1$, then $X = (S_2 - S_2') \cup T_2$ and $Y = T_1 \cup S_2'$. Thus, assume $M\backslash X/Y = M_1$. Fix $F \in \mathcal{Z}'(M_1) - \{S_1'\}$. By Lemma 2.2, the cyclic flat F of $M\backslash X/Y$ must arise from the cyclic flat $F \cup T_1 \cup S_2'$ of M; from Theorem 3.1 part (ii), it follows that for $M\backslash X/Y$ to yield the same nullity on F as in M_1, each element of $T_1 \cup S_2'$ must be contracted; dually, each element of $(S_2 - S_2') \cup T_2$ must be deleted. Thus, $X = (S_2 - S_2') \cup T_2$ and $Y = T_1 \cup S_2'$. □

To close this section, we mention several open problems. Note that Theorem 3.3 does not apply if either M_1 or M_2 is a uniform matroid. Geelen [5] asked: can a matroid and a uniform matroid have infinitely many intertwines? More generally, in [6] he asked whether the hypotheses in Theorem 3.3 are optimal.

4. Further Results

Among the pairs of matroids that Theorem 3.3 applies to are any two spikes of rank at least 4, neither of which is a minor of the other, provided that the one of smaller rank (if the ranks differ) is not a free spike. (We use the definition of spikes in [5], which some sources call tip-less spikes. Free spikes are the only spikes that can be obtained from a spike by minors along with at least one lift or truncation.) Thus, the assumption in the construction in [5] that each element is in a dependent transversal is not needed here. However, unlike the construction in [5], the intertwine we get when M_1 and M_2 are spikes is not a spike.

The construction here and that in [5] give intertwines with contrasting properties and so show that some properties that hold for one construction need not hold for intertwines in general. For instance, the intertwines constructed here have neither small circuits nor small cocircuits, but those constructed in [5] have each element in a many 4-circuits and in many 4-cocircuits. Also, in our construction the number of cyclic flats does not depend on the rank, but in the construction in [5] the number of cyclic flats grows with the rank (as is true for the variation we discussed before Theorem 3.5).

4.1. Sizes of intertwines.

We show that the intertwines constructed above can exhibit the full range of possible sizes for each rank.

Theorem 4.1. *If S is the ground set of a rank-k intertwine of M_1 and M_2, then*

$$2k - r(M_1) - r(M_2) \le |S| \le 2k + r(M_1^*) + r(M_2^*).$$

If $\mathrm{F}(M_i) = \emptyset = \mathrm{F}(M_i^*)$ for both $i \in \{1,2\}$, then the construction in Section 3 gives intertwines of each cardinality in this range.

Proof. Let M, on the set S, be a rank-k intertwine of M_1 and M_2, so $M\backslash X/Y \simeq M_1$ and $M\backslash X'/Y' \simeq M_2$ for some subsets X, Y, X', Y' of S. Standard arguments about minors show that we may assume that Y and Y' are independent sets with $|Y| = k - r(M_1)$ and $|Y'| = k - r(M_2)$. No proper contraction of M has both M_1- and M_2-minors, so $Y \cap Y' = \emptyset$; thus, $|Y| + |Y'| \le |S|$, so $k - r(M_1) + k - r(M_2) \le |S|$. This lower bound on $|S|$ is achieved in the construction when $S_1' = S_1$ and $S_2' = S_2$. No proper deletion of M has both M_1- and M_2-minors, so $X \cap X' = \emptyset$; thus, $|X| \le |S_2| + |Y'|$. This inequality, the equation $|S| = |S_1| + |X| + |Y|$, and values for $|Y|$ and $|Y'|$ give the upper bound. This bound is attained when $S_1' = \emptyset = S_2'$. By varying $|S_1'|$ and $|S_2'|$, all cardinalities between these bounds can be realized. □

4.2. **Representable matroids.** All spikes are contained in $\mathcal{E}(U_{2,6}, U_{4,6})$, the class of matroids that have neither $U_{2,6}$- nor $U_{4,6}$-minors. The results in this subsection and the next are akin to a corollary that Vertigan got from his work on intertwines and spikes: some pairs of matroids in $\mathcal{E}(U_{2,6}, U_{4,6})$ have infinitely many intertwines in $\mathcal{E}(U_{2,6}, U_{4,6})$.

The result below uses the following equivalent formulations of two special cases of our construction. (The first assertion follows by comparing the cyclic flats and their ranks; the second is the dual of the first. Recall that T^k and L^j denote truncations and lifts.) If $k \geq r(M_1) + r(M_2)$, then

$$M_k(M_1, \emptyset, T_1; M_2, \emptyset, T_2) = T^k\big((M_1 \times T_1) \oplus (M_2 \times T_2)\big).$$

If $k \geq r(M_1) + |S_1| + r(M_2) + |S_2|$ and $j = k - r(M_1) - r(M_2)$, then

$$M_k(M_1, S_1, T_1; M_2, S_2, T_2) = L^j\big((M_1 + T_2) \oplus (M_2 + T_1)\big).$$

Corollary 4.2. *Assume a class $\mathcal{C}$ of matroids is closed under direct sum, free extension, free coextension, truncation, and lift. If $M_1, M_2 \in \mathcal{C}$ satisfy the hypotheses of Theorem 3.3 and if either $\mathrm{F}(M_1) = \emptyset = \mathrm{F}(M_2)$ or $\mathrm{F}(M_1^*) = \emptyset = \mathrm{F}(M_2^*)$, then $\mathcal{C}$ contains infinitely many intertwines of M_1 and M_2.*

Such classes $\mathcal{C}$ include the class of matroids that are representable over a given infinite field and the class of matroids that are representable over a given characteristic.

4.3. **Transversal matroids and gammoids.** We next show that the intertwine M that we constructed is transversal if and only if M_1 and M_2 are; we also treat the corresponding statements for several related types of matroids. We will use the characterization of transversal matroids in Lemma 4.3, which is a refinement, observed by Ingleton [7], of a result of Mason [8]. (For a proof, see [1].) For a collection $\mathcal{F}$ of sets, let $\cap\mathcal{F}$ be $\bigcap_{X\in\mathcal{F}} X$ and $\cup\mathcal{F}$ be $\bigcup_{X\in\mathcal{F}} X$.

Lemma 4.3. *A matroid M is transversal if and only if for all $\mathcal{A} \subseteq \mathcal{Z}(M)$ with $\mathcal{A} \neq \emptyset$,*

$$\sum_{\mathcal{F}\subseteq\mathcal{A}} (-1)^{|\mathcal{F}|+1} r(\cup\mathcal{F}) \geq r(\cap\mathcal{A}). \tag{4.1}$$

In this result, it suffices to consider only antichains $\mathcal{A}$ of cyclic flats since if $X, Y \in \mathcal{A}$ with $X \subset Y$, then using $\mathcal{A} - \{Y\}$ in place of $\mathcal{A}$ does not change either side of inequality (4.1); with $\mathcal{A}$, the terms on the left side that include Y cancel via the involution that adjoins X to, or omits X from, $\mathcal{F}$. Also, it suffices to focus on inequality (4.1) for $|\mathcal{A}| > 2$ since equality holds when $|\mathcal{A}| = 1$ and the case of $|\mathcal{A}| = 2$ is the semimodular inequality.

Corollary 4.4. *A matroid M is transversal if and only if $M \times x$ is.*

Cotransversal matroids are duals of transversal matroids. Bitransversal matroids are both transversal and cotransversal. Gammoids are minors of transversal matroids. Restrictions of transversal matroids are transversal, so any gammoid is a contraction of some transversal matroid; it follows that any gammoid is a nullity-preserving contraction of some transversal matroid. The class of gammoids is closed under duality, so any gammoid has a rank-preserving extension to a cotransversal matroid.

Theorem 4.5. *Assume inequality (3.1) holds. The matroids M_1 and M_2 are transversal if and only if $M = M_k(M_1, S_1', T_1; M_2, S_2', T_2)$ is. The corresponding statements hold for cotransversal matroids, bitransversal matroids, and gammoids.*

Proof. Since $M_1 \times (T_1 \cup S_2')$ and $M_2 \times (T_2 \cup S_1')$ are restrictions of M, from Corollary 4.4 it follows that if M is transversal, then so are M_1 and M_2. Now assume M_1 and M_2 are transversal. Let $\mathcal{A}$ be an antichain in $\mathcal{Z}(M)$ with $|\mathcal{A}| > 2$. Set

$$\mathcal{A}_1 = \mathcal{A} \cap \mathcal{Z}'(M_1 \times (T_1 \cup S_2')) \quad \text{and} \quad \mathcal{A}_2 = \mathcal{A} \cap \mathcal{Z}'(M_2 \times (T_2 \cup S_1')).$$

Thus, $\mathcal{A}$ is the disjoint union of $\mathcal{A}_1$ and $\mathcal{A}_2$. By Corollary 4.4 and Lemma 4.3, inequality (4.1) holds for $\mathcal{A}_1$ if it is nonempty, and likewise for $\mathcal{A}_2$; thus, this inequality holds for $\mathcal{A}$ if one of $\mathcal{A}_1$ and $\mathcal{A}_2$ is empty. Assume neither is empty. For $F_1 \in \mathcal{A}_1$ and $F_2 \in \mathcal{A}_2$, we have $r_M(F_1 \cup F_2) = r(M)$, so

$$\begin{aligned}\sum_{\mathcal{F} \subseteq \mathcal{A}} (-1)^{|\mathcal{F}|+1} r(\cup \mathcal{F}) &= \sum_{\mathcal{F}_1 \subseteq \mathcal{A}_1} (-1)^{|\mathcal{F}_1|+1} r(\cup \mathcal{F}_1) + \sum_{\mathcal{F}_2 \subseteq \mathcal{A}_2} (-1)^{|\mathcal{F}_2|+1} r(\cup \mathcal{F}_2) \\ &\qquad + \sum_{\substack{\mathcal{F}_1 \subseteq \mathcal{A}_1,\, \mathcal{F}_1 \neq \emptyset \\ \mathcal{F}_2 \subseteq \mathcal{A}_2,\, \mathcal{F}_2 \neq \emptyset}} (-1)^{|\mathcal{F}_1|+|\mathcal{F}_2|+1}\, r(M) \\ &\geq r(\cap \mathcal{A}_1) + r(\cap \mathcal{A}_2) - r(M) \\ &\geq r(\cap \mathcal{A}),\end{aligned}$$

where the last line follows from semimodularity along with the inclusions $T_1 \cup S_2' \subseteq \cap \mathcal{A}_1$ and $T_2 \cup S_1' \subseteq \cap \mathcal{A}_2$, and the fact that $T_1 \cup T_2 \cup S_1' \cup S_2'$ spans M (a consequence of equation (2.1) and inequalities (3.1) and (3.3)). Thus, inequality (4.1) holds, so M is transversal.

The assertions about cotransversal and bitransversal matroids follow by Theorem 3.2.

If M is a gammoid, then so are its minors M_1 and M_2. Now assume M_1 and M_2 are gammoids. Let M_1' and M_2' be rank-preserving cotransversal extensions of M_1 and M_2. Thus, $M_k(M_1', S_1', T_1; M_2', S_2', T_2)$ is cotransversal since inequality (3.1) holds with M_1' and M_2' in place of M_1 and M_2. Comparing the cyclic flats and their ranks shows that M is a restriction of $M_k(M_1', S_1', T_1; M_2', S_2', T_2)$, so M is a gammoid. □

Corollary 4.6. *If M_1 and M_2 satisfy the hypotheses of Theorem 3.3 and are transversal, then infinitely many intertwines of M_1 and M_2 are transversal. The corresponding statements hold for cotransversal matroids, bitransversal matroids, and gammoids.*

These results raise questions along similar lines. Does the construction preserve the class of matroids that are algebraic over a given field? Also, does it preserve the class of base-orderable matroids or that of strongly base-orderable matroids?

4.4. **Uniform minors.** We claim that $M|B_1 \cup B_2 \cup T_1 \cup T_2$, where B_i is a basis of M_i, is the uniform matroid $U_{k,2k-|S_1'|-|S_2'|}$. To see this, note that if C were a circuit in this restriction with $r(C) < k$, then $\mathrm{cl}_M(C) \in \mathcal{Z}'(M)$; however, it follows from the construction that flats in $\mathcal{Z}'(M)$ intersect $B_1 \cup B_2 \cup T_1 \cup T_2$ in independent sets.

Corollary 4.7. *If M_1 and M_2 satisfy the hypotheses of Theorem 3.3, then for any integer n, some intertwine of M_1 and M_2 has a $U_{n,2n}$-minor.*

4.5. **Connectivity.** Recall that for any non-uniform matroid, $\lambda(M) \leq \kappa(M)$ where $\lambda(M)$ is the (Tutte) connectivity of M and $\kappa(M)$ is its vertical connectivity. Thus, showing that the connectivity of intertwines can be arbitrarily large gives the counterpart for vertical connectivity.

Qin [10] proved $\lambda((M+p) \times q) - \lambda(M) \in \{1, 2\}$ for any matroid M. For the matroid M constructed above, fix a subset T_2' of T_2 with $|T_2'| = \eta(M_2)$. Comparing the cyclic flats and their ranks, with the help of Lemma 2.3, gives

$$M\backslash T_2' = \big(M_1 \times (T_1 \cup S_2')\big) + \big((T_2 - T_2') \cup (S_2 - S_2')\big).$$

After some number of free coextensions or free extensions of M_1 according to the difference between $|T_1 \cup S_2'|$ and $|(T_2 - T_2') \cup (S_2 - S_2')|$ (which does not change as k increases), the deletion $M\backslash T_2'$ can be seen as resulting from free extension/free coextension pairs, so the connectivity of such deletions $M\backslash T_2'$ grows with k. Since extending as needed by the elements in T_2' to obtain M preserves the rank and introduces no circuits of size $|T_2| + |S_1'|$ or smaller, $\lambda(M)$ also grows with k.

Corollary 4.8. *If M_1 and M_2 satisfy the hypotheses of Theorem 3.3, then for any integer n, some intertwine of M_1 and M_2 is n-connected.*

With the truncation that cuts the rank of the direct sum in half, it follows that the intertwine $T^k\big((M_1 \times T_1) \oplus (M_2 \times T_2)\big)$ (arising from $S_1' = \emptyset = S_2'$) is rounded, that is, the ground set is not the union of two proper flats, or, equivalently, each cocircuit spans. (This notion, also called non-splitting, is equivalent to having $\kappa(M) = r(M)$.)

Note that in a rank-n spike M with $n \geq 4$, if H is a hyperplane spanned by $n-2$ legs (using the terminology of [5]), then $(H, E(M) - H)$ is a vertical 3-separation of M. Thus, the construction in this paper and that in [5] yield intertwines with contrasting connectivity and vertical connectivity properties.

5. The Relation to Vertigan's Construction

As mentioned in the introduction, the first construction of infinite sets of intertwines for pairs of matroids was given by Dirk Vertigan. In this section we briefly outline his construction and show that, although the approaches differ, some instances of the two constructions coincide; furthermore, both approaches can be extended to yield the same collections of intertwines. Vertigan's theorem is as follows.

Theorem 5.1. *Assume neither M_1 nor M_2 can be obtained, up to isomorphism, from the other by any combination of minors, free extensions, and free coextensions. If* $\mathrm{F}(M_i) = \emptyset = \mathrm{F}(M_i^*)$ *for both $i \in \{1, 2\}$, then M_1 and M_2 have infinitely many intertwines.*

The intertwines he constructed to prove this result are defined as follows. Let S_1 and S_2 be the ground sets of M_1 and M_2, which, in contrast to Theorem 3.3, need not be disjoint. Let X and Y be disjoint k-element sets, where $k \geq 10\max\{|S_1|, |S_2|\}$, such that (i) $S_1 \cup S_2 \subseteq X \cup Y$, (ii) $X \cap S_1$ has $r(M_1)$ elements and is dependent in M_1, and (iii) $Y \cap S_2$ has $r(M_2)$ elements and is dependent in M_2. Set

$$M_1' = \big(M_1 + (Y - S_1)\big) \times (X - S_1) \quad \text{and} \quad M_2' = \big(M_2 + (X - S_2)\big) \times (Y - S_2).$$

Thus, $r(M_1') = k = r(M_2')$. He argues that the intersection of the collections of bases of M_1' and M_2' is the collection of bases of a matroid on $X \cup Y$, and that this matroid is an intertwine of M_1 and M_2. Thus, this intertwine has rank k and has $2k$ elements. Vertigan observed that, as in Corollary 4.7, these intertwines have uniform minors of large rank and corank.

To relate this construction to ours, we first show that the bases of the intertwines we constructed can be described in a similar manner. Using the notation in Section 3, set

$$M_1'' = \big(M_1 \times (T_1 \cup S_2')\big) + \big(T_2 \cup (S_2 - S_2')\big)$$

and

$$M_2'' = \big(M_2 \times (T_2 \cup S_1')\big) + \big(T_1 \cup (S_1 - S_1')\big).$$

Both M_1'' and M_2'' have rank k. Observe that $\mathcal{Z}(M) = \mathcal{Z}(M_1'') \cup \mathcal{Z}(M_2'')$. Using equation (2.1), it follows that a subset of $S_1 \cup S_2 \cup T_1 \cup T_2$ is a basis of M if and only if it is a basis of both M_1'' and M_2''. In particular, the constructions coincide when applied under the same set up, and the basis approach can be extended to cover the results in this paper. In the other direction, it is easy to check that if we replace inequality (3.1) with a slightly stronger inequality, then Theorem 3.1 applies even when S_1 and S_2 are not disjoint; of course, then we need $S_1' \subseteq S_1 - S_2$ and $S_2' \subseteq S_2 - S_1$. Likewise, Theorem 3.2 can be adapted (for instance, instead of $S_1 - S_1'$ on the right, we need $S_1 - (S_2 \cup S_1')$). Consistent with the hypotheses in Theorem 5.1, Theorem 3.3 also applies provided that $S_1 \cap S_2$ is disjoint from $\mathrm{F}(M_i)$ and $\mathrm{F}(M_i^*)$ for both $i \in \{1, 2\}$. Thus, an advantage of dealing with disjoint ground sets is that it eliminates the need for assumptions about $\mathrm{F}(M_i)$ and $\mathrm{F}(M_i^*)$.

ACKNOWLEDGEMENTS

I am grateful to Jim Geelen for comments that led me to turn my attention to the topic of intertwines and for valuable feedback on this work. I thank Dirk Vertigan for providing copies of the slides of his talks on intertwines and spikes, and for permission to sketch his construction. I thank Anna de Mier for comments that improved the exposition and for discussions that led to improvements over the original lower bound in the material that immediately follows Theorem 3.5. I am grateful to the referee who provided several comments that led to improved exposition.

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