ON BASIS-EXCHANGE PROPERTIES FOR MATROIDS

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ABSTRACT. We give a counterexample to a conjecture by Wild about binary matroids. We connect two equivalent lines of research in matroid theory: a simple type of basis-exchange property and restrictions on the cardinalities of intersections of circuits and cocircuits. Finally, we characterize direct sums of series-parallel networks by a simple basis-exchange property.

In [9], Wild proposed a characterization of binary matroids. In order to state his conjecture compactly, we recall his notation. (We also use standard matroid notation and terminology as found, for example, in [6].) For a basis B of a matroid M and an element x in the matroid, $R(x \to B)$ denotes the set of all elements $y \in B$ so that $(B - y) \cup x$ is a basis of M. Informally, $R(x \to B)$ is the set of elements of B that can be replaced by x. Thus if x is in B, the set $R(x \to B)$ is the singleton $\{x\}$; if x is not in B, the set $R(x \to B)$ consists of all elements in the fundamental circuit, $C(x, B)$, of x with respect to B, except for x itself. Wild observed that every binary matroid M satisfies the following property:

> For all bases B and elements x, y of M, if $R(x \to B) = R(y \to B)$, then x and y are either equal or parallel.

He conjectured that this property characterizes binary matroids. We present a counterexample.

Our counterexample, M, is simpler to describe via its dual M^*. Let M^* be the matroid on $\{a, a', b, b', c, c', d, d'\}$ whose underlying simple matroid is the 4-point line (i.e., $U_{2,4}$) and for which $\{a, a'\}$, $\{b, b'\}$, $\{c, c'\}$, and $\{d, d'\}$ are parallel classes. (Thus M is simple and representable over every field other than $GF(2)$.) The automorphism group of M is transitive on the bases, so it suffices to show that for a particular basis B of M and the two elements x, y of M not in B, $R(x \to B)$ and $R(y \to B)$ are unequal (thus proving the condition above for this non-binary matroid). Let B be the basis $\{a', b', c, c', d, d'\}$ of M. To find $R(a \to B)$, we find the cocircuit of M^* contained in $\{a, a', b', c, c', d, d'\}$, which is $\{a, a', c, c', d, d'\}$, and then delete a; thus $R(a \to B) = \{a', c, c', d, d'\}$. Similarly $R(b \to B) = \{b', c, c', d, d'\}$, proving that $R(a \to B)$ and $R(b \to B)$ are unequal, as needed.

It is well-known that some basis-exchange properties characterize certain classes of matroids. To describe this efficiently, we adopt another notational convention from [9]. For bases B and B' of a matroid M and an element x of B, let $\mathrm{Sym}(x, B, B')$ be the set of elements y of B' such that *both* $(B - x) \cup y$ and $(B' - y) \cup x$ are bases of M. Some basis-exchange properties discussed in [9] impose restrictions on the cardinality $|\mathrm{Sym}(x, B, B')|$ of $\mathrm{Sym}(x, B, B')$. The following proposition, conjectured by Rota and proven by Greene, was the first result of this type. (See [2, Section XI, Theorem 1]. For the motivation for studying this type of basis-exchange property, see [3] and [8].)

Proposition 1. *A matroid M is binary if and only if for each pair of bases B, B' of M and each $x \in B$, $|\mathrm{Sym}(x, B, B')|$ is odd.*

Greene showed the equivalence of several statements, all characterizing binary matroids, including the basis-exchange property in Proposition 1 and the following circuit-cocircuit intersection property from [4].

Proposition 2. *A matroid M is binary if and only if for every circuit C and cocircuit C^* of M, $|C \cap C^*|$ is even.*

Proposition 3 gives a general connection between results of these types.

Proposition 3. *For a matroid M and an integer $k \geq 2$, there are bases B and B' of M and an element x of B with $|\mathrm{Sym}(x, B, B')| = k$ if and only if there is a circuit C and a cocircuit C^* of M such that $|C \cap C^*| = k + 1$.*

Proof. Assume there are bases B and B' of M and an element x of B with $|\mathrm{Sym}(x, B, B')| = k$. Since $|\mathrm{Sym}(x, B, B')| > 1$, it follows that $x \notin B'$. Now $\mathrm{Sym}(x, B, B')$ consists of the elements y other than x in the fundamental circuit $C := C(x, B')$ (so that $(B' - y) \cup x$ is a basis) that are also in the fundamental cocircuit $C^* := C^*(x, B)$ (so that $(B - x) \cup y$ is a basis). Thus $\mathrm{Sym}(x, B, B') \cup x = C \cap C^*$, so $|C \cap C^*| = k + 1$.

Conversely, assume there is a circuit C and a cocircuit C^* of M such that $|C \cap C^*| = k + 1$. Let $x \in C \cap C^*$. Extend the independent set $C - x$ to a basis B' of M. Let B be a basis of M containing x so that $B - x$ spans the hyperplane complementary to C^*. It is clear that $\mathrm{Sym}(x, B, B') = (C \cap C^*) - x$, so $|\mathrm{Sym}(x, B, B')| = k$. □

In [7], Seymour proved that a matroid M is binary if and only if for every circuit C and cocircuit C^* of M, $|C \cap C^*|$ is not 3. In the language of basis-exchange properties, this is the following proposition.

Proposition 4. *A matroid M is binary if and only if for each pair of bases B, B' of M and each $x \in B$, $|\mathrm{Sym}(x, B, B')|$ is not 2.*

One can prove Proposition 4 directly: from a $U_{2,4}$-minor, one can construct a pair of bases violating the property in Proposition 4, and the converse follows from Proposition 1.

From Theorem 3.1 in [5], it follows that M is a direct sum of series-parallel networks if and only if for every circuit C and cocircuit C^* of M, $|C \cap C^*|$ is either 0 or 2. From this and Proposition 3, we deduce that a matroid M is a direct sum of series-parallel networks if and only if for each pair of bases B, B' of M and each $x \in B$, $|\mathrm{Sym}(x, B, B')| = 1$. Rephrased, this is the following attractive basis-exchange characterization of these matroids.

Proposition 5. *A matroid M is a direct sum of series-parallel networks if and only if for each pair of bases B, B' of M and each x in B, there is a unique element y in B' such that both $(B - x) \cup y$ and $(B' - y) \cup x$ are bases of M.*

We note that this theorem is easy to prove directly from the perspective of basis-exchange properties. One checks that the stated basis-exchange property is inherited by minors, and that the excluded minors for the class of direct sums of series-parallel networks (i.e., $U_{2,4}$ and $M(K_4)$) do not have this property. Conversely, one checks that series extensions and parallel extensions of matroids with the stated basis-exchange property also have this property.

An immediate corollary of Proposition 5 is that direct sums of series-parallel networks are base-orderable (see [1]).

We close by noting that few of the well-studied classes of matroids can be characterized by basis-exchange properties of the simple type given in Propositions 1, 4, and 5. To make this precise, for any subset A of the set $\mathbf{N}$ of positive integers let $\mathcal{C}(A)$ denote the class of matroids that satisfy the basis-exchange property:

For each pair of bases B, B' of M and each $x \in B$, $|\mathrm{Sym}(x, B, B')|$ is in A.

Thus the class of binary matroids is $\mathcal{C}(\mathbf{N} - 2\mathbf{N})$, by Proposition 1, and $\mathcal{C}(\mathbf{N} - \{2\})$, by Proposition 4. The class of direct sums of series-parallel networks is $\mathcal{C}(\{1\})$ by Proposition 5.

It is easy to check that $\mathcal{C}(A)$ is closed under minors, duals, and direct sums; these statements can be shown either from the perspective of the given basis-exchange property or from the equivalent formulation in terms of intersections of circuits and cocircuits. It follows from Proposition 2.2 in [5] that the excluded minors of $\mathcal{C}(A)$ have twice as many points as their rank. Thus, while the class of binary matroids and the class of direct sums of series-parallel networks are among the classes $\mathcal{C}(A)$, such simple basis-exchange properties cannot be used to characterize regular matroids, graphic matroids, cographic matroids, or F-representable matroids for any field F other than $GF(2)$.

We note that it also follows from Proposition 2.2 in [5] that if $\mathbf{N} - A$ is finite, then $\mathcal{C}(A)$ has a finite number of excluded minors.

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