A BRIEF INTRODUCTION TO MATROID THEORY

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1. Introduction

Many different topics in mathematics, ranging from applied subjects such as optimization to pure areas such as arrangements of hyperplanes, naturally lead to matroids. The approach we will explore is matroid theory as an abstraction of affine and projective geometry. Therefore the first several sections will survey some elementary, although perhaps unfamiliar, aspects of geometry. We will discuss affine and projective geometry in general, although our main interest will be in *finite* geometries.

While affine geometry has been studied, in varying degrees of generality, for thousands of years, and projective geometry grew out of investigations into perspective during the Renaissance, chiefly by Girard Desargues (1591–1661), the study of finite geometries largely began in the 1930's and 1940's with the work of such mathematicians as Marshall Hall, Jr., Richard H. Bruck, and Herbert Ryser. We will focus on basic aspects of affine and projective geometries; this part of our survey is *not* meant as an introduction to current research in finite geometry, which continues to be a subject of intense research activity.

At its most fundamental level, geometry is concerned with such simple notions as points, lines, planes, and their higher-dimensional counterparts. We can consider such concepts even if we do not have a notion of distance (which would be made precise by a metric). This is exactly what we will consider: non-metric geometry. Thus, we will not have angles, curvature, or even the notion of "between". At first it may seem that such a minimalistic version of geometry would be too limited to be interesting, but this is far from the case.

2. Affine Geometries

We start with affine geometry, which abstracts the familiar properties of $\mathbb{R}^n$.

Definition 2.1. *An* affine geometry *is a set S of* points *and two collections of subsets of S, the set of* lines *and the set of* planes*, subject to these axioms:*

(A1) *each pair A, B of distinct points is contained in a unique line, which is denoted $\ell(A, B)$,*
(A2) *each triple of noncollinear points is contained in a unique plane,*
(A3) *if P is a point not in a line ℓ, then there is a unique line ℓ^* with P in ℓ^* and ℓ parallel to ℓ^* (parallel lines are coplanar and disjoint),*
(A4) *the relation "parallel or equal" is an equivalence relation, and*
(A5) *each line has at least two points.*

Axiom (A3) is the parallel postulate. Note that the reflexive and symmetric properties automatically hold for the relation in axiom (A4); thus, the only issue is the transitive property. Axiom (A5) excludes certain degenerate cases.

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The interpretation of these axioms in $\mathbb{R}^n$ is familiar: points are n-tuples in $\mathbb{R}^n$, lines are affine lines in $\mathbb{R}^n$ (i.e., lines that need not go through the origin), and planes are affine planes in $\mathbb{R}^n$ (i.e., planes that need not go through the origin). It is useful to think of these from a slightly more algebraic perspective: points are translations (or cosets) of the zero subspace, lines are the translations of the 1-dimensional (linear) subspaces, and planes are the translations of the 2-dimensional (linear) subspaces.

To get more examples of affine geometries, we could replace $\mathbb{R}$ by the elements of any division ring F (a structure that satisfies all the axioms of a field except perhaps the commutative law of multiplication). All basic results of linear algebra (in particular, all theorems about dimension and subspaces) are valid for vector spaces over arbitrary division rings. The case of most interest for us will be that in which F is a finite field, the Galois field $GF(q)$ for some prime power q. If q is prime, this field is $\mathbb{Z}_q$, the integers $0, 1, \ldots, q-1$ with arithmetic modulo q.

Thus, let F be a division ring. Let $AG(n, F)$ be the affine geometry with the following points, lines, and planes: the points are the n-tuples of F^n, i.e., the translations of the zero subspace of F^n, the lines are the translations of the 1-dimensional subspaces of F^n, and the planes are the translations of the 2-dimensional subspaces of F^n. Of course, F^n could be replaced by any n-dimensional vector space over F. Verifying axioms (A1)–(A5) is straightforward; for instance, the unique line that contains vectors A and B is $\{A + \alpha(A - B) \mid \alpha \in F\}$, the translation of the 1-dimensional subspace $\{\alpha(A - B) \mid \alpha \in F\}$ by A.

Consider $AG(2, 3)$, the affine plane over $GF(3)$, the field of three elements (i.e., $\{0, 1, 2\}$ under arithmetic modulo 3). (The notation $AG(n, GF(q))$ is shortened to $AG(n, q)$.) There are nine points, $(0, 0)$, $(1, 0)$, $(2, 0)$, $(0, 1)$, $(1, 1)$, $(2, 1)$, $(0, 2)$, $(1, 2)$, and $(2, 2)$. Note that $\{(0, 0), (0, 1), (0, 2)\}$ is a subspace; its translations are $\{(1, 0), (1, 1), (1, 2)\}$ and $\{(2, 0), (2, 1), (2, 2)\}$. Note that these three lines are parallel. In this manner, we get three other equivalence classes of parallel lines,

$$\{(0,0),(1,0),(2,0)\} \text{ and its cosets } \{(0,1),(1,1),(2,1)\} \text{ and } \{(0,2),(1,2),(2,2)\},$$

$$\{(0,0),(2,1),(1,2)\} \text{ and its cosets } \{(1,0),(0,1),(2,2)\} \text{ and } \{(2,0),(1,1),(0,2)\},$$

and

$$\{(0,0),(1,1),(2,2)\} \text{ and its cosets } \{(1,0),(2,1),(0,2)\} \text{ and } \{(2,0),(0,1),(1,2)\}.$$

These sets are shown in Figure 1. Together these give the points and lines of $AG(2, 3)$ as shown in Figure 2.

The affine geometry $AG(n, q)$ has q^n points since this is the number of n-tuples over the q-element field $GF(q)$. There are q points on each line of $AG(n, q)$ since the lines are the translations $\{v + \alpha u \mid \alpha \in GF(q)\}$ of the 1-dimensional subspaces $\{\alpha u \mid \alpha \in GF(q)\}$. Likewise there are q^2 points in each plane of $AG(n, q)$ since the planes are the translations $\{v + \alpha u + \beta w \mid \alpha, \beta \in GF(q)\}$ of the 2-dimensional subspaces $\{\alpha u + \beta w \mid \alpha, \beta \in GF(q)\}$.

The examples of affine geometries that we have seen so far suggest basic structural features of affine geometries. In particular, there is a bijection between the points on any two lines of an affine geometry, and between the points in any two planes. The proofs are good exercises.

If all lines in an affine geometry are finite and have exactly q points, we say that q is the *order* of the affine geometry. Thus, $AG(n, q)$ has order q.

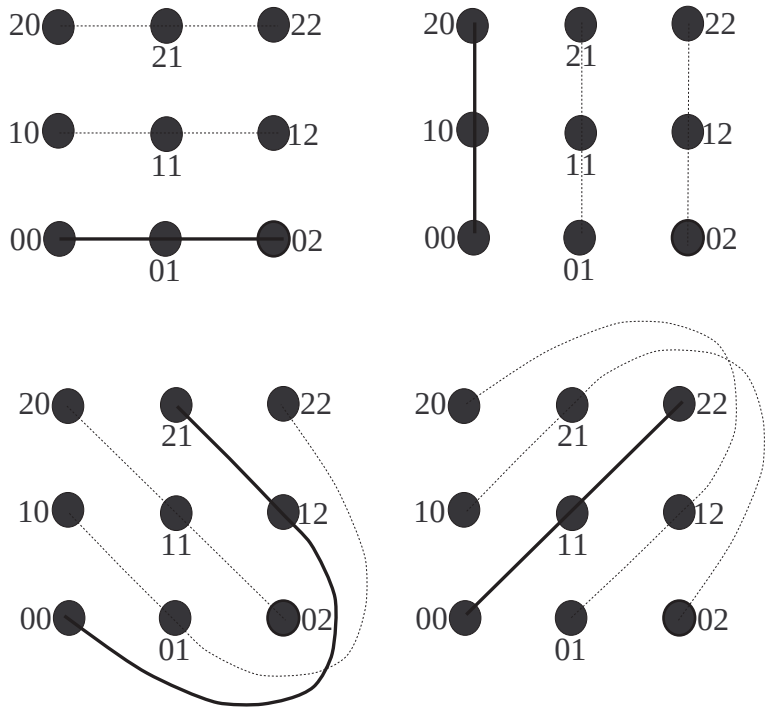

FIGURE 1. The four families of parallel lines in $AG(2,3)$. Subspaces are depicted with heavy lines; the cosets are dotted.

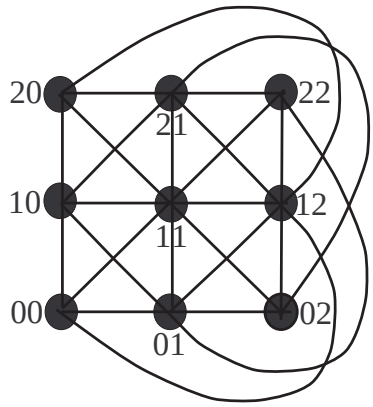

FIGURE 2. The affine plane AG(2,3).

We have mentioned points, lines, and planes. There may be affine subspaces of higher dimension. Must we list these separately or are they somehow determined by the points, lines, and planes? The notion of a *flat*, which is the geometric term for subspace, shows how to capture all subspaces using only points and lines.

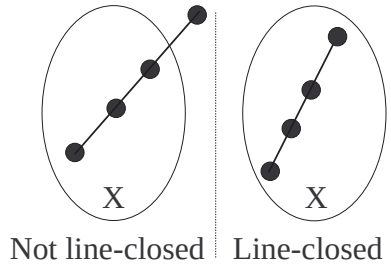

FIGURE 3

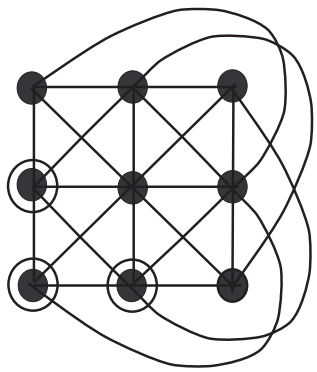

FIGURE 4

Definition 2.2. *In an affine geometry in which there are at least three points on each line, a* flat *is a subset X of the set of points that satisfies the* line-closure *condition: if $A, B \in X$, then $\ell(A, B) \subseteq X$.*

Figure 3 gives a generic picture of the condition of line-closure. Figure 4 shows a subset (the circled points) of $AG(2, 3)$ that is *not* a flat.

Note that Definition 2.2 uses only lines. Apart from affine geometries in which all lines have exactly two points, the only role planes have is in allowing us to talk about parallel lines. (Recall that parallel lines are coplanar lines that are disjoint, so this relies on knowing what planes are.) Planes have a more important role in affine geometries in which lines have exactly two points, but we will gloss over this.

The flats of an affine geometry on a set S of points have the following important properties that will be abstracted in the definition of a matroid. (Since the finite case is ultimately our main interest, we are implicitly assuming S is finite; the general case uses the same ideas and only slightly more cumbersome notation for the third property.)

(F1) The set S is a flat.
(F2) The intersection of any collection of flats is a flat.

(F3) If X is a flat and $X_1, X_2, \ldots, X_t$ are the flats that cover X (i.e., X is a proper subset of X_i, denoted $X \subset X_i$, and there is no flat Y with $X \subset Y \subset X_i$), then the differences $X_1 - X, X_2 - X, \ldots, X_t - X$ partition $S - X$.

Properties (F1) and (F2) are immediate from Definition 2.2. Property (F3) is more complicated and would require a distracting digression to prove in this setting. We will justify it indirectly later in two ways: in the first justification, we will state that our model $AG(n, F)$ covers almost all instances of affine geometries, and one can check (F3) directly for this model; in the second justification, we will prove the analogous property for projective geometries and cite a connection that allows us to translate between the two settings.

Rather than proving (F3) here, we focus on what it is saying. It is the natural generalization of the observation that given a line and a point not on the line, the point and the line determine a unique plane; in other words, the planes through a line partition the set of points that are not on that line. This special case of (F3) follows easily from axioms (A2) and (A3).

One can easily show that the flats of $AG(n, F)$ are the translations of the linear subspaces of F^n. Note that property (F3) holds for an i-dimensional linear subspace X of F^n: this is simply saying that each vector u not in X is in a unique $(i+1)$-dimensional subspace that contains X, namely $\operatorname{span}(X \cup \{u\})$. To get the general case for $AG(n, F)$ from this, just translate: pick x in a flat X; the translation $\{y - x \mid y \in X\}$ is a linear subspace, so the flats covering it partition the set of points that are not in X; the translations of these flats by x are the covers of X and the partitioning property is preserved.

Property (F2) has an important consequence: Given a set T of points of an affine geometry, there is a unique smallest flat that contains T, namely the intersection of all flats that contain T. By property (F1), this is the intersection of a nonempty collection of sets.

Definition 2.3. *The* closure $\operatorname{cl}(T)$ *of a set T of points in an affine geometry is given by*

$$\operatorname{cl}(T) = \bigcap_{\substack{\text{flats } X \\ \text{with } T \subseteq X}} X.$$

One can think of the closure $\operatorname{cl}(T)$ of T as the flat spanned by T. In any affine geometry, $\operatorname{cl}(\emptyset) = \emptyset$; in particular, $\emptyset$ is a flat.

The closure of the set of circled points in Figure 4 is the entire plane, $AG(2, 3)$.

The notion of closure allows us to define the rank of a flat.

Definition 2.4. *The* rank $r(X)$ *of a flat X in an affine geometry is given by*

$$r(X) = \min\{|T| : T \subseteq X \text{ and } \operatorname{cl}(T) = X\}.$$

The rank of a flat captures how many points it takes to determine the flat. For instance, it takes two points to determine a line, so lines have rank two; likewise, it takes three points to determine a plane, so planes have rank three. The rank of an affine geometry is the rank of its ground set.

Rank is closely linked to dimension. One can show that in $AG(n, F)$, the affine geometry constructed from F^n, the flats of rank i are precisely the cosets of the $(i-1)$-dimensional subspaces of F^n. In particular, F^n, and hence $AG(n, F)$, has rank $n + 1$. Because of this we prefer to shift the notation; we will focus on $AG(n-1, F)$, the rank-n affine geometry that is constructed from F^{n-1}.

It is natural to ask: Are there affine geometries in addition to the examples $AG(n-1, F)$ we constructed from division rings? The following important theorem says that *all* affine geometries of rank 4 and greater are of the form $AG(n-1, F)$. (This is one of several different theorems that various authors cite as the fundamental theorem of affine geometry.)

Theorem 2.5 (The Fundamental Theorem of Affine Geometry)**.** *Each affine geometry of rank n, where $n \geq 4$, is isomorphic to $AG(n-1, F)$ for some division ring F.*

Thus, apart from affine geometries of low rank (specifically, affine lines and affine planes), affine geometry is precisely the study of the cosets of the subspaces of a vector space over a division ring.

Theorem 2.5 has evolved over time; it may not be possible to attribute it to any one person.

Rank two affine geometries, i.e., affine lines, obviously are not very interesting and have no meaningful correspondence with any algebraic structure. Rank three affine geometries, i.e., affine planes, are the subject of intense research, largely through the corresponding projective planes. We end this section with a very brief mention of some intriguing aspects of affine planes. For more about this fascinating area, see, e.g., [24].

There are many algebraic structures (e.g., near fields, Veblen-Wedderburn systems) that are much less constrained than are division rings that, nonetheless, have enough structure to give rise to affine planes.

Recall that the order of a finite affine plane is the number of points in each line. Thus, for a prime power q, the affine plane $AG(2, q)$ has order q. It is known that for every proper prime power $q = p^k$ ("proper" means $k > 1$) apart from $q = 4$ and $q = 8$, there are at least two nonisomorphic affine planes of order q. The following question has resisted all attacks for well over half a century.

Open Problem 2.6. *Must the order of a finite affine plane be a prime power?*

The Bruck Ryser theorem is a powerful tool for showing that a particular number is not the order of any affine plane; however this theorem addresses only numbers that are congruent to 1 or 2 modulo 4 and the implication is valid in only one direction.

Theorem 2.7 (Bruck and Ryser, 1949)**.** *If q is congruent to 1 or 2 modulo 4 and there is an affine plane of order q, then q is a sum of two squares.*

This theorem rules out affine planes of order 6, for instance, since 6 is congruent to 2 modulo 4 but is not a sum of two squares. (Tarry's proof on the non-existence of 6 by 6 orthogonal Latin squares, given around 1910, also shows that affine planes of order 6 do not exist.) Note that 10 is congruent to 2 modulo 4 and 10 is $3^2 + 1^2$. However, in 1988 it was shown that there is no affine plane of order 10. (The proof involved a massive computer search.) Thus, the converse of the Bruck Ryser theorem is false. The smallest positive integer for which we currently do not know whether there is an affine plane of that order is 12.

The most important things to remember from this section are properties (F1)–(F3) since abstracting these gives the definition of a matroid. The affine geometries $AG(n-1, q)$ will also play an important role in what follows, so we close this section by summarizing the basic properties of these geometries.

$AG(n-1,q)$
$AG(n-1,q)$ has rank n.
$AG(n-1,q)$ has q^{n-1} points.
Lines of $AG(n-1,q)$ have q points.
For $i>0$, rank-i flats of $AG(n-1,q)$ have q^{i-1} points.

3. Projective Geometries

According to Theorem 2.5, for ranks exceeding three affine geometry is the study of the cosets of the subspaces of a vector space over a division ring. Since the subspaces themselves, rather than their cosets, are the main focus of linear algebra, you might wonder what geometric structures arise from subspaces. Subspaces give rise to projective geometries.

Let's start with $(GF(2))^3$, the 3-dimensional vector space over the two-element field $GF(2)$. There are eight vectors in this vector space:

$$(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1).$$

There is a unique 0-dimensional subspace, $\{(0,0,0)\}$. There are seven 1-dimensional subspaces, corresponding to the seven nonzero vectors in $(GF(2))^3$:

$$\{(0,0,0),(0,0,1)\},\quad \{(0,0,0),(0,1,0)\},\quad \{(0,0,0),(0,1,1)\},\quad \{(0,0,0),(1,0,0)\},$$

$$\{(0,0,0),(1,0,1)\},\qquad \{(0,0,0),(1,1,0)\},\quad \text{and}\quad \{(0,0,0),(1,1,1)\}.$$

There are seven 2-dimensional subspaces:

$$\{(0,0,0),(0,0,1),(0,1,0),(0,1,1)\},$$
$$\{(0,0,0),(0,0,1),(1,0,0),(1,0,1)\},$$
$$\{(0,0,0),(0,0,1),(1,1,0),(1,1,1)\},$$
$$\{(0,0,0),(0,1,0),(1,0,0),(1,1,0)\},$$
$$\{(0,0,0),(0,1,0),(1,0,1),(1,1,1)\},$$
$$\{(0,0,0),(0,1,1),(1,0,0),(1,1,1)\},$$
$$\{(0,0,0),(1,0,1),(1,1,0),(0,1,1)\}.$$

Of course, the eight vectors of $(GF(2))^3$ form the unique 3-dimensional subspace of this vector space.

We can draw a diagram of these subspaces, just as we drew diagrams for affine geometries. Since $(0,0,0)$ is in all subspaces, we lose no information by suppressing it in the diagrams. The resulting diagram is given in Figure 5.

Notice that this geometry has some properties quite unlike those of affine geometries. In particular, every pair of lines has a point of intersection: there are no parallel lines. This property will hold in general for coplanar lines if we take as points the 1-dimensional subspaces of a vector space, as lines the 2-dimensional subspaces of the vector space, and as planes the 3-dimensional subspaces of the vector space. This follows from the familiar dimension theorem of linear algebra:

$$\dim(U_1)+\dim(U_2)=\dim(U_1+U_2)+\dim(U_1\cap U_2).$$

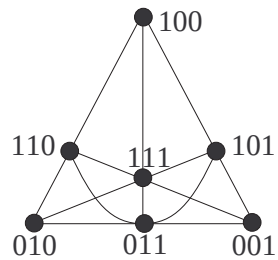

FIGURE 5. The Fano Plane, F_7 or $PG(2,2)$.

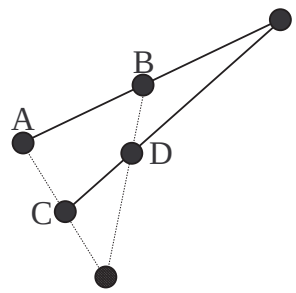

FIGURE 6. The configuration in the Pasch axiom.

Letting U_1 and U_2 be lines ℓ_1 and ℓ_2 (i.e., 2-dimensional subspaces) in a plane (i.e., a 3-dimensional subspace), we have

$$\underbrace{\dim(\ell_1)}_{2} + \underbrace{\dim(\ell_2)}_{2} = \underbrace{\dim(\ell_1 + \ell_2)}_{3} + \dim(\ell_1 \cap \ell_2),$$

so $\dim(\ell_1 \cap \ell_2)$ is 1, that is, $\ell_1 \cap \ell_2$ is a 1-dimensional subspace, or a point.

With this motivation, we can now define projective geometries.

Definition 3.1. *A* projective geometry *is a set S of* points *and a collection of subsets of S, the set of* lines*, subject to these axioms:*

(P1) *each pair A, B of distinct points is contained in a unique line, which is denoted $\ell(A, B)$,*
(P2) *if A, B, C, and D are distinct points for which $\ell(A, B) \cap \ell(C, D) \neq \emptyset$, then $\ell(A, C) \cap \ell(B, D) \neq \emptyset$, and*
(P3) *each line contains at least three points.*

Axiom (P2) is the Pasch axiom. It is a way of saying coplanar lines intersect without mentioning planes; intuitively (and in a sense we could easily make precise), since $\ell(A, B) \cap \ell(C, D) \neq \emptyset$, all four points A, B, C, and D lie in a plane, so the lines $\ell(A, C)$ and $\ell(B, D)$ are therefore coplanar and so should intersect nontrivially. This is illustrated in Figure 6.

Notice that the axiom system for projective geometry is considerably simpler than for affine geometry; there are three axioms, rather than five, and we mention only points and lines, not planes. This is typical: projective geometry is simpler

than affine geometry, even though, as we will see, it encompasses affine geometry. The reason is that, as we will make precise later, projective geometry is the natural completion of affine geometry.

Observe also that if the projective geometry is a plane, we have a beautiful symmetry in the axioms: *any* pair of points spans a unique line and *any* pair of lines intersects in a unique point. This has a striking resemblance to duality in linear algebra; this is no accident.

It should come as no surprise now that we get a projective geometry out of every vector space over a division ring; simply take the points to be the 1-dimensional subspaces and the lines to be the 2-dimensional subspaces, viewed as sets of the points. Alternatively, pick precisely one nonzero representative vector out of each 1-dimensional subspace, and let the points be the chosen vectors and the lines be the sets of these chosen vectors that are in 2-dimensional subspaces. The projective geometry constructed in this manner from $(GF(q))^n$ is denoted $PG(n-1,q)$. More generally, the projective geometry constructed in this manner from an n-dimensional vector space over a division ring F is denoted $PG(n-1,F)$.

Note that this is precisely how the real projective plane $PG(2,\mathbb{R})$ is formed from $\mathbb{R}^3$ in topology and geometry. The standard constructions of the real projective plane take one of the following three equivalent approaches, each of which illustrates what is stated in the last paragraph. Take as the points of $PG(2,\mathbb{R})$ the lines of $\mathbb{R}^3$ through the origin, with the origin deleted; each plane of $\mathbb{R}^3$ through the origin gives rise to the line of $PG(2,\mathbb{R})$ that consists of all points of $PG(2,\mathbb{R})$ that lie in this plane, and all lines of $PG(2,\mathbb{R})$ are of this form. Alternatively, choose as the points of $PG(2,\mathbb{R})$ precisely one out of each pair of antipodal points on the unit sphere in $\mathbb{R}^3$ (for example, the points on the upper half sphere, with half of the edge included), and let the lines of $PG(2,\mathbb{R})$ be the sets of these points of $PG(2,\mathbb{R})$ that are intersections of the set of points of $PG(2,\mathbb{R})$ with planes of $\mathbb{R}^3$ through the origin. Alternatively, take as the points of $PG(2,\mathbb{R})$ the pairs of antipodal points on the unit sphere in $\mathbb{R}^3$, and let the lines of $PG(2,\mathbb{R})$ be the sets of these points of $PG(2,\mathbb{R})$ that are intersections of the set of points of $PG(2,\mathbb{R})$ with planes of $\mathbb{R}^3$ through the origin; finally, identify antipodal points.

As with affine geometries, we will focus on *finite* projective geometries. We illustrate the construction above with a second finite example (Figure 5 being the first example, although too small to illustrate the choice of representative vectors): Figure 7 shows the projective plane $PG(2,3)$ formed from the 3-dimensional vector space $(GF(3))^3$ over the field $GF(3)$. In this diagram, the point $(1,1,2)$ represents the 1-dimensional subspace $\{(0,0,0),(1,1,2),(2,2,1)\}$; the line

$$\{(0,1,2),(1,2,1),(1,1,2),(1,0,0)\}$$

represents the 2-dimensional subspace

$$\{(0,0,0),(0,1,2),(0,2,1),(1,2,1),(2,1,2),(1,1,2),(2,2,1),(1,0,0),(2,0,0)\}.$$

Notice that while $AG(2,3)$ has three point on each line, $PG(2,3)$ has four points on each line.

The geometry $PG(n-1,q)$ has $(q^n-1)/(q-1)$ points since the points correspond to the 1-dimensional subspaces of $(GF(q))^n$, and there are q^n-1 nonzero vectors in $(GF(q))^n$ with $q-1$ nonzero vectors in each 1-dimensional subspace. Alternatively, by choosing as a representative vector of a subspace $P=\langle v\rangle$ the unique multiple of v that has a 1 in the first nonzero position (as shown in Figure 7), we see that

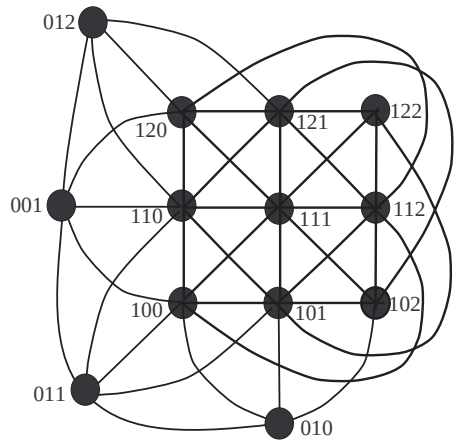

FIGURE 7. The projective plane $PG(2,3)$.

there are $(q^n-1)/(q-1)$, or $q^{n-1}+q^{n-2}+\cdots+q+1$, points since the first nonzero position could be the first (leaving q^{n-1} ways to fill in the other $n-1$ entries) or the second (leaving q^{n-2} ways to fill in the other $n-2$ entries), and so on.

The notion of a flat is essentially the same as in affine geometries.

Definition 3.2. *A* flat *in a projective geometry is a set X of points in the geometry that satisfies the* line-closure *condition: if $A, B \in X$, then $\ell(A,B) \subseteq X$.*

In keeping with our observation that projective geometry is simpler than affine geometry, there is no exceptional case like the case of two-point lines in affine geometries. Observe that Definition 3.2 is simply the geometric formulation of the familiar algebraic definition of a linear subspace. Thus, the flats in the projective geometry arising from a division ring are precisely the subspaces (or the sets of the representative vectors in the subspaces).

The flats of a projective geometry on a set S of points have the following important properties that we also saw for the flats of an affine geometry and that we will see again in the definition of a matroid. (As was the case for affine geometries, we are implicitly assuming S is finite since this is ultimately the case of greatest interest.)

(F1) The set S is a flat.
(F2) The intersection of any collection of flats is a flat.
(F3) If X is a flat and $X_1, X_2, \ldots, X_t$ are the flats that cover X, then the differences $X_1 - X, X_2 - X, \ldots, X_t - X$ partition $S - X$.

Properties (F1) and (F2) are immediate from Definition 3.2.

Given that the flats in the projective geometry arising from a vector space are (sets of representatives from) the subspaces, property (F3) has this familiar interpretation: for an i-dimensional linear subspace X of a vector space, each vector u not in X is in a unique $(i+1)$-dimensional subspace that contains X, namely $\mathrm{span}(X \cup \{u\})$.

It is easy to justify property (F3) in general (again, we focus on the case in which S is finite). Note that property (F3) trivially holds if X is the empty set (which

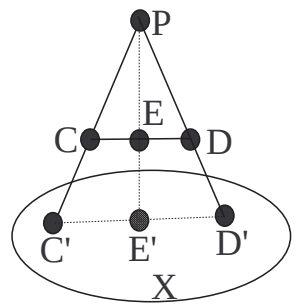

FIGURE 8

is a flat by Definition 3.2), for then the covering flats $X_1, X_2, \ldots, X_t$ are just the singleton sets of points (which are also flats). So assume that X is a nonempty flat. The key to proving property (F3) is to identify the flat that covers X and contains a point P that is not in X. We claim that the flat that covers X and contains a point P that is not in X is

$$X_P = \bigcup_{A \in X} \ell(A, P), \tag{1}$$

that is, the set of all points that are collinear with P and a point of X. Clearly any flat that contains X and P contains all of X_P, so all we need to show is that X_P satisfies the line-closure condition that defines flats. Toward this end assume C and D are in X_P. By the definition of X_P, there are points C' and D' in X with $C \in \ell(C', P)$ and $D \in \ell(D', P)$. If $C' = D'$, then $\ell(C, D) = \ell(P, C')$, hence $\ell(C, D) \subseteq X_P$ as desired, so assume $C' \neq D'$. (See Figure 8.) Let E be in $\ell(C, D)$; we need to show that E is in X_P, i.e., that there is a point E' in X with $E \in \ell(E', P)$. Now (skipping the applications of the Pasch axiom that fully justify this) note that since $\ell(C', D')$ and $\ell(E, P)$ are coplanar lines, they intersect at some point E', which is necessarily in X since X is a flat that contains both C' and D'. Thus E is indeed in X_P, so X_P is a flat. Now to verify property (F3), note that we have shown that each point P not in X is in a smallest flat X_P that contains X and P and equation (1) gives an explicit expression for this flat. Assume that for two points P and Q, the sets $X_P - X$ and $X_Q - X$ are not disjoint, say both contain the point R. Since X_P is a flat that contains X and R, we have $X_R \subseteq X_P$; on the other hand, since R is in X_P, we know that R and P are collinear with a point of X, so P is in X_R, and so $X_P \subseteq X_R$; therefore $X_P = X_R$. Similarly, $X_Q = X_R$. Therefore $X_P = X_Q$. Thus we have shown that the only flats that cover X and intersect in a proper superset of X are identical, which is the required partitioning property.

Several more ideas we saw for affine geometries carry over to projective geometries. In particular, as a consequence of property (F2), for each set T of points in a projective geometry there is a unique smallest flat containing T, namely the intersection of all flats that contain T. By Property (F1), this is the intersection of a nonempty collection of sets.

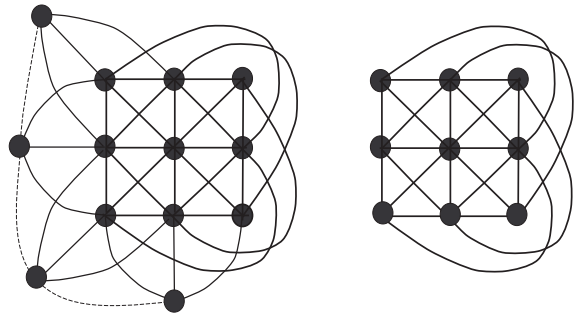

FIGURE 9. Remove the points of a hyperplane (dotted) of $PG(2,3)$ to get $AG(2,3)$.

Definition 3.3. *The* closure $\mathrm{cl}(T)$ *of a set* T *of points in a projective geometry is given by*

$$\mathrm{cl}(T) = \bigcap_{\substack{\text{flats } X \\ \text{with } T \subseteq X}} X.$$

Again, closure gives rise to the notion of rank.

Definition 3.4. *The* rank $r(X)$ *of a flat* X *in a projective geometry is given by*

$$r(X) = \min\{|T| : T \subseteq X \text{ and } \mathrm{cl}(T) = X\}.$$

In the projective geometry arising from a vector space, the flats of rank i are (the sets of representatives of) the i-dimensional subspaces. Thus, rank is the projective geometry counterpart of dimension in linear algebra. Note that $PG(n-1,q)$ has rank n, and the rank-i flats of $PG(n-1,q)$ have $(q^i-1)/(q-1)$ points. In particular, lines of $PG(n-1,q)$ have $(q^2-1)/(q-1)$, or $q+1$, points.

In linear algebra, the subspaces of dimension $n-1$ in an n-dimensional vector spaces are called the hyperplanes. Analogously, the flats of rank $n-1$ in a projective geometry of rank n are called *hyperplanes*.

We have alluded to the fact that projective geometry is the natural completion of affine geometry. This is exemplified by the connection between $AG(2,3)$ and $PG(2,3)$ suggested in Figure 9. The precise formulation of this is the following theorem, which is easy to prove.

Theorem 3.5. *Let* H *be a hyperplane of a projective geometry on the set* S *of points and let* $\mathcal{L}$ *and* $\mathcal{P}$ *be the set of lines and the set of planes (i.e., flats of rank* 3*) of the geometry. We get an affine geometry with* $S' = S - H$ *as the set of points by taking as the set of lines*

$$\mathcal{L}' = \{\ell \cap S' \mid \ell \in \mathcal{L} \text{ with } \ell \not\subseteq H\}$$

and as the set of planes

$$\mathcal{P}' = \{\pi \cap S' \mid \pi \in \mathcal{P} \text{ with } \pi \not\subseteq H\}.$$

Conversely, assume that $\mathcal{L}'$ *is the set of lines of an affine geometry on a set* S'*, and that* $\mathcal{P}'$ *is the set of planes of this geometry. Let* $\{\mathcal{L}'_i \mid i \in \mathcal{I}\}$ *be the set of equivalence classes of parallel lines. With each equivalence class* $\mathcal{L}'_i$*, let* A_i *be a point not in* S'*. Let* S *be the set* $S' \cup \{A_i \mid i \in \mathcal{I}\}$*. With each line* ℓ' *of* $\mathcal{L}'$*, let* ℓ *be* $\ell' \cup \{A_i\}$

where ℓ' is in $\mathcal{L}'_i$. With each plane π of $\mathcal{P}'$, let ℓ_π be $\{A_i \mid \pi$ contains a line in $\mathcal{L}'_i\}$. Then S together with the set

$$\mathcal{L} = \{\ell \mid \ell' \in \mathcal{L}'\} \cup \{\ell_\pi \mid \pi \in \mathcal{P}'\}$$

of lines is a projective geometry.

Thus, affine and projective geometry are intimately linked. To get an affine geometry from a projective geometry, remove all points in a hyperplane and consider the induced sets of lines and planes. (The operation of deletion that this illustrates is a basic matroid operation.) To get a projective geometry from an affine geometry, add one point to all lines in each equivalence class of parallel lines and let the lines of new points correspond to the planes of the affine geometry.

As there are affine planes that do not arise from our construction based on vector spaces over a division ring, there are projective planes that do not arise from our construction based on vector spaces over a division ring. Such planes may have some rather unexpected properties. For instance, with a projective plane that does not arise from a division ring, it may be possible to remove different lines and get *nonisomorphic* affine planes. Thus, while each affine plane has a unique completion to a projective plane given by the construction in Theorem 3.5, many nonisomorphic affine planes can have the same completion to a projective plane.

Given Theorems 2.5 and 3.5, the next theorem should come as no surprise.

Theorem 3.6 (The Fundamental Theorem of Projective Geometry)**.** *Every projective geometry of rank n, where $n \geq 4$, is isomorphic to $PG(n-1, F)$ for some division ring F.*

Thus, apart from planes, projective geometry is the study of the subspaces of a vector space over a division ring.

Given that projective geometry is simpler than affine geometry, although the two subjects are equivalent in the sense made precise in Theorem 3.5, the problems mentioned at the end of Section 2 are typically studied for projective planes rather than affine planes.

As in the last section, the most important things to remember from this section are properties (F1)–(F3). The projective geometries $PG(n-1, q)$ will also play an important role in what follows, so we close this section by summarizing the basic properties of these geometries.

$PG(n-1,q)$
$PG(n-1,q)$ has rank n.
$PG(n-1,q)$ has $(q^n-1)/(q-1)$, or $q^{n-1}+q^{n-2}+\cdots+q+1$, points.
Lines of $PG(n-1,q)$ have $q+1$ points.
Rank-i flats of $PG(n-1,q)$ have $(q^i-1)/(q-1)$ points.

4. Coordinates

To motivate some of the topics we will see in matroid theory, it is useful to sketch some of the elements that go into the proofs of Theorems 2.5 and 3.6. These ideas go back well over a hundred years; they are used in Hilbert's book [23], which first appeared in 1899, although they were known well before then. For a complete, elementary presentation from a modern perspective, see [2].

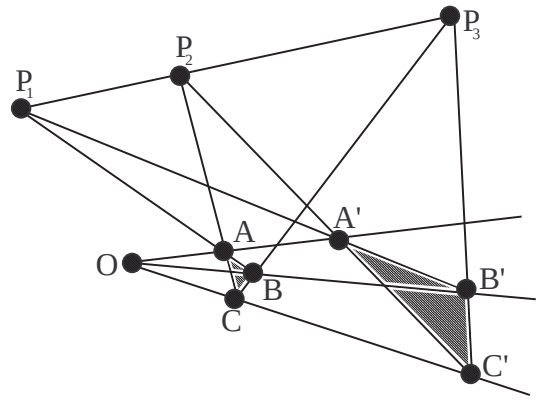

FIGURE 10. Desargues' configuration.

The problem of realizing the flats of a projective geometry as the subspaces of a vector space over a division ring is the problem of coordinatizing the geometry. This is equivalent to coordinatizing the corresponding affine geometry. We will shift freely between affine and projective geometry.

We mentioned that not all projective planes arise from division rings. It is natural to ask: Is it possible to characterize the projective planes that arise from division rings? This is precisely what Desargues' theorem does.

Theorem 4.1. *A projective plane is isomorphic to a projective plane arising from a division ring if and only if it satisfies the following condition:*

> DESARGUES' THEOREM. *Given any triples A, B, C and A', B', C' of noncollinear points, if the lines $\ell(A, A')$, $\ell(B, B')$, and $\ell(C, C')$ are concurrent (i.e., meet in a point), then the three points $\ell(A, B) \cap \ell(A', B')$, $\ell(A, C) \cap \ell(A', C')$, and $\ell(B, C) \cap \ell(B', C')$ are collinear.*

Thus, Desargues' theorem (which, in some form, dates back to the 1600's) is, for us, not a theorem; it is a condition, or axiom, that characterizes the projective planes that arise from division rings. (With the earlier, more limited view of geometry, this was indeed a theorem since the only geometries considered satisfied this condition.)

Briefly, Desargues' theorem says that two triangles that are perspective from a point are perspective from a line. What this means is the following. The two triangles A, B, C and A', B', C' being perspective from the point O (see Figure 10) has this physical interpretation in Euclidean spaces: if one places one's eye at point O, the two triangles A, B, C and A', B', C' line up exactly. The triangles being perspective from a line is the dual notion: instead of saying "the lines through corresponding points are concurrent," interchange points and lines to get "the points at the intersection of corresponding lines are collinear."

Although Theorem 4.1 deals with planes, Desargues' Theorem can be interpreted as describing a configuration in a plane or in rank 4. Note that if this is a configuration in rank 4, then P_1, P_2, and P_3 are in the plane $\mathrm{cl}(\{A, B, C\})$ and in the plane $\mathrm{cl}(\{A', B', C'\})$; since the intersection of flats is a flat by (F2), the points P_1, P_2,

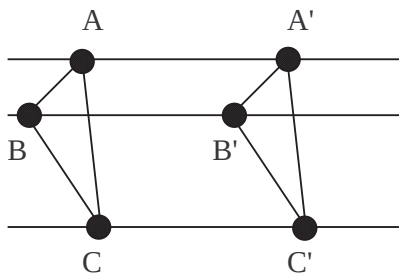

FIGURE 11. One affine version of Desargues' configuration.

and P_3, being in $\mathrm{cl}(\{A, B, C\}) \cap \mathrm{cl}(\{A', B', C'\})$, must indeed be on a line. This begins to suggest why Theorems 2.5 and 3.6 apply when the rank is at least four.

It is an easy exercise to show that Desargues' Theorem is equivalent to its converse.

Theorem 4.2. *A projective plane satisfies Desargues' Theorem if and only if it satisfies the following condition:*

> *Given any triples A, B, C and A', B', C' of noncollinear points, if the points $\ell(A, B) \cap \ell(A', B')$, $\ell(A, C) \cap \ell(A', C')$, and $\ell(B, C) \cap \ell(B', C')$ are collinear, then the lines $\ell(A, A')$, $\ell(B, B')$, and $\ell(C, C')$ are concurrent.*

We have mentioned several times that projective geometry is simpler than affine geometry. This is in part because one projective configuration such as Desargues' configuration can give rise to many affine configuration. For instance, if none of the points in Desargues' configuration are in the hyperplane of a projective geometry that is deleted to form an affine geometry, then we get the identical configuration in the affine geometry. However, if O is collinear with P_1, P_2, and P_3 in the projective geometry, and these are all in the hyperplane that is deleted, the corresponding affine configuration is given in Figure 11. The affine interpretation of this configuration is the following:

> Given triangles A, B, C and A', B', C' with the lines $\ell(A, A')$, $\ell(B, B')$, and $\ell(C, C')$ all parallel, if $\ell(A, B) \parallel \ell(A', B')$ and $\ell(A, C) \parallel \ell(A', C')$, then $\ell(B, C) \parallel \ell(B', C')$.

In the Euclidean plane, this statement is obvious: it follows immediately from basic results on parallelograms. There are two other affine cases of Desargues' configuration (it is a good exercise to identify them), but the two we have mentioned are the most relevant for us.

Theorem 4.1 suggests that we should be able to define the operations of addition and multiplication of some division ring in any affine plane in which Desargues' theorem holds. Let's briefly sketch this so we can see the role Desargues' theorem plays. (See Figure 12.) To add points A and B on a line on which a zero has been designated, construct a parallel auxiliary line and choose a point D on it. Construct $\ell(0, D)$ and let the parallel line through A intersect the auxiliary line at D'. Construct $\ell(B, D)$ and the parallel line through D'; where this parallel line intersects the original line is $A + B$.

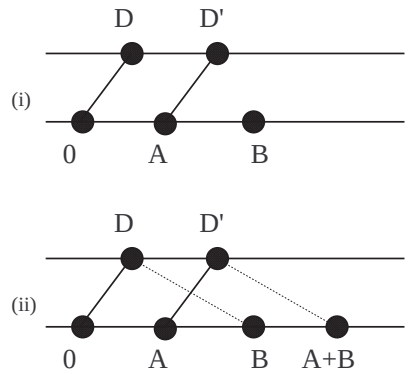

FIGURE 12. Addition in Desarguesian planes.

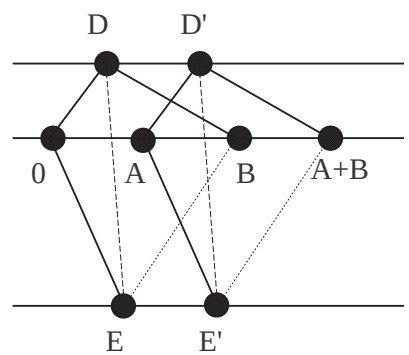

FIGURE 13

This construction has the obvious interpretation in the Euclidean plane of using parallel lines to translate the distance from 0 to A up to the second line and then back down to the original line, but starting at B rather than at 0. However the construction makes sense in any affine plane since it uses only parallel lines. The issue is: When is this operation well-defined? Specifically, when is this construction independent of the choice of the auxiliary line and the auxiliary point? This is the role that Desargues' theorem plays.

To see that this operation is well-defined, consider $A + B$ as constructed from an auxiliary point D, and consider an auxiliary point E on a different auxiliary line. (See Figure 13.) By applying Desargues' theorem to triangles $D, 0, E$ and D', A, E', we conclude that $\ell(E, D)$ is parallel to $\ell(E', D')$. This allows us to apply Desargues' theorem to triangles D, B, E and $D', A+B, E'$, so we can conclude that $\ell(E, B)$ is parallel to $\ell(E', A + B)$, which says that computing $A + B$ using E gives the same result as using D. The remaining case is that of using two auxiliary points on the same line parallel to the original line, but this follows from what we have

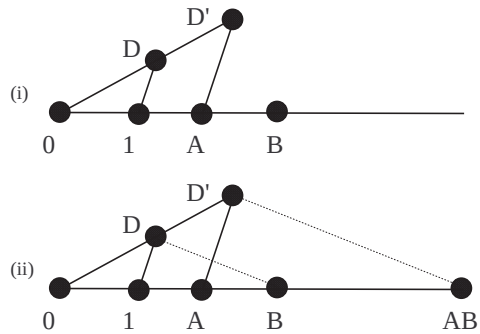

FIGURE 14. Multiplication in Desarguesian planes.

shown by comparing these with the result obtained by using an auxiliary point on a second auxiliary line.

Having that this operation of addition is well-defined, it is an interesting series of elementary exercises to show that the points on a line form an abelian group under this operation and that the groups defined by a different choice of 0 or a different line are isomorphic.

The operation of multiplication is defined similarly, except that the auxiliary line is one that goes through the additive identity. (See Figure 14.) Starting with the additive identity 0, the multiplicative identity 1, and two points A and B on the line $\ell(0,1)$, to multiply A and B choose an auxiliary line through 0 and a point D on this line. Construct $\ell(1,D)$ and the parallel line through A, meeting the auxiliary line at D'. Next construct $\ell(B,D)$ and the parallel line through D', meeting the original line at AB.

In the Euclidean plane, an elementary argument using similar triangles shows that this gives the correct product. However, the definition uses only parallel lines and so makes sense in any affine plane. The issue is whether this product is well-defined. For this, one of the other affine realizations of Desargues' theorem comes into play. (See Figure 15.) Construct AB using D and consider and auxiliary point E on a different auxiliary line. By applying one version of Desargues' theorem to triangles $D, 1, E$ and D', A, E', we conclude that $\ell(E,D)$ is parallel to $\ell(E',D')$. This allows us to apply Desargues' theorem to triangles D, B, E and D', AB, E', so we can conclude that $\ell(E,B)$ is parallel to $\ell(E',AB)$, which says that computing AB using E gives the same result as using D. As we saw for addition, the general case follows from this.

Having that this operation of multiplication is well-defined, it is another sequence of elementary exercises to show the remaining properties for a division ring, namely that multiplication is associative, that 1 is a multiplicative identity, that each nonzero element has a multiplicative inverse, and that multiplication distributes over addition. Elementary arguments also show that the resulting division ring is, up to isomorphism, independent of the line chosen and the points 0 and 1 chosen on any line. One can then use this division ring to coordinatize the plane.

Theorem 4.1 states that Desargues' theorem characterizes the projective planes that arise from 3-dimensional vector spaces over division rings. What characterizes

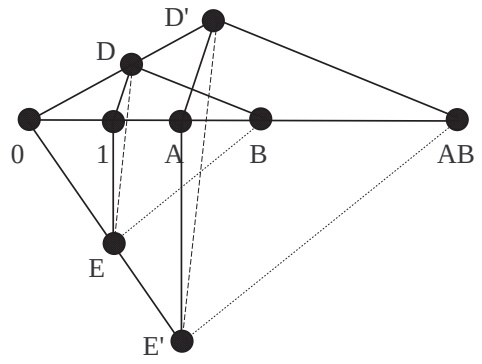

FIGURE 15

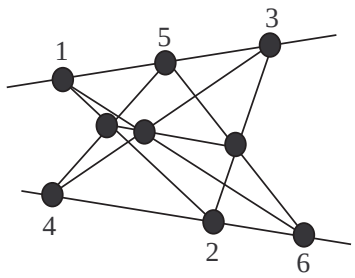

FIGURE 16. Pappus' configuration.

the projective planes that arise from 3-dimensional vector spaces over fields? I.e., how can we geometrically capture the axiom that multiplication is commutative? This is provided by Pappus' theorem, some form of which dates back to around 300 A.D. (See Figure 16.)

Theorem 4.3. *A projective plane is isomorphic to a projective plane arising from a field if and only if it satisfies the following condition:*

> PAPPUS' THEOREM. *If a hexagon is inscribed alternately on two lines, then the three points of intersection of the opposite sides are collinear.*

Like Desargues' theorem, from the modern perspective Pappus' theorem is not a theorem; it is a condition, or axiom, that characterizes the projective planes that arise from fields. It is an elementary exercise to prove Theorem 4.3.

As a corollary of Theorems 4.1 and 4.3, it follows that Pappus' theorem implies Desargues' theorem; it is an interesting elementary exercise to prove this directly, without using Theorems 4.1 and 4.3. In general, the converse of course is not true since there are division rings that are not fields, but there is an important case in which the converse holds. Recall Wedderburn's theorem: Every finite division ring is a field. Thus, for *finite* projective planes, Desargues' theorem implies Pappus' theorem. It seems there should be a geometric proof of this fact that does not

simply geometrically encode the known algebraic proofs of Wedderburn's theorem; presently no such proof is known.

5. Matroids

We now generalize the two classical spaces, projective and affine geometries, that we examined in the first part of this brief introduction to matroid theory.

Definition 5.1. *A* matroid *M is a finite set S and a collection $\mathcal{F}$ of subsets of S, the* flats *of M, such that:*

(F1) *the set S is a flat,*
(F2) *the intersection of any collection of flats is a flat, and*
(F3) *if X is a flat and $X_1, X_2, \ldots, X_t$ are the flats that cover X, then the differences $X_1 - X, X_2 - X, \ldots, X_t - X$ partition $S - X$.*

The set S is often called the ground set of the matroid.

Thus, a matroid is simply a set together with a collection of subsets, the flats, that satisfies the three properties we observed for the flats of projective and affine geometries.

Matroids were introduced by Hassler Whitney, and, although the subject has subsequently developed in more directions than could be imagined in the 1930's, his founding paper [36] is still an excellent entry point into the subject. Although Whitney gave several different equivalent formulations of a matroid, that in Definition 5.1 is not among them. It is partly a reflection of the large number of branches of mathematics in which matroids play a role that there are now over fifty different equivalent formulations of a matroid; about a dozen of these are frequently used while others were devised for very special purposes. When working with matroids it is typically very useful to shift freely between several different approaches. However, what makes for efficiency and insight for researchers in the field may be confusing to those new to the subject, so we will restrict ourselves to the approach in Definition 5.1.

In low ranks, we can draw the types of diagrams we drew for projective and affine planes. For instance, in the matroid $\overline{F}_7$ in Figure 17, the flats are the empty set, the seven points, the nine lines $\{D, A, E\}$, $\{D, B, G\}$, $\{E, C, G\}$, $\{E, F, B\}$, $\{D, F, C\}$, $\{A, F, G\}$, $\{A, B\}$, $\{A, C\}$, and $\{B, C\}$, and the plane consisting of all seven points. Note that when a line such as $\{A, B\}$ contains just two points, we do not bother to draw a line between the two points since it is understood. Such lines are called *trivial lines*.

To give a concrete example of axiom (F3), consider the flats that cover $\{A\}$. These are $\{D, A, E\}$, $\{A, F, G\}$, $\{A, B\}$, and $\{A, C\}$; the resulting differences, namely, $\{D, E\}$, $\{F, G\}$, $\{B\}$, and $\{C\}$, indeed partition $\{B, C, D, E, F, G\}$, the set of points not in $\{A\}$.

While in projective and affine geometries, the flats were specified by the lines together with the line-closure condition, this is not the case in arbitrary matroids; in general we need to list *all* flats. Indeed, if we start with the points and lines of $\overline{F}_7$ and consider the resulting line-closed sets (the sets that satisfy the line-closure condition), these violate axiom (F3). For instance, the smallest line-closed set that contains the line $\{A, B\}$ and the point D is the entire set $\{A, B, C, D, E, F, G\}$ yet $\{A, B, C\}$ is a line-closed set with $\{A, B\} \subset \{A, B, C\} \subset \{A, B, C, D, E, F, G\}$, so the partitioning property fails. This prompts the first of the open problems we will mention.

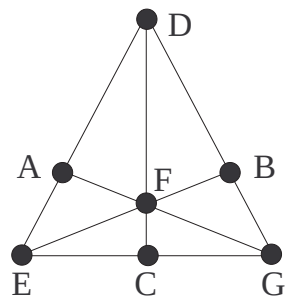

FIGURE 17. The non-Fano matroid, $\overline{F}_7$.

Open Problem 5.2. *Characterize the collections of points and lines for which the resulting line-closed sets are the flats of a matroid.*

Although for most matroids, the line-closed sets are not the same as the flats, when these two collections agree, certain arguments become considerably simpler. (See, e.g., [3, 9]. See [22] for more on this topic.)

As in projective and affine geometries, the flats give rise to the notion of closure.

Definition 5.3. *The* closure $\mathrm{cl}(T)$ *of a set* T *of points in a matroid is given by*

$$\mathrm{cl}(T) = \bigcap_{\substack{\text{flats } X \\ \text{with } T \subseteq X}} X.$$

By axiom (F1), this is the intersection of a nonempty collection of sets. By axiom (F2), $\mathrm{cl}(T)$ is a flat.

Closure, in turn, gives rise to the notion of rank. It will be useful to generalize this a bit, and consider the rank of any set of points, not just flats.

Definition 5.4. *The* rank $r(X)$ *of a set* X *in a matroid is given by*

$$r(X) = \min\{|T| : T \subseteq \mathrm{cl}(X) \text{ and } \mathrm{cl}(T) = \mathrm{cl}(X)\}.$$

By axiom (F2) there is a unique smallest flat, the intersection of all flats. (Indeed, by axioms (F1) and (F2), the collection of flats forms a lattice.) This unique smallest flat has rank 0 and it is the only flat of rank 0.

We borrow even more terms from projective and affine geometry: *points* are flats of rank 1, *lines* are flats of rank 2, *planes* are flats of rank 3, and *hyperplanes* are flats of rank $n-1$ in a matroid of rank n.

Nothing in Definition 5.1 forces the empty set to be a flat; it need not be a flat. Likewise, singleton subsets of S need not be a flat. (We will see examples of this arising naturally soon.) For some purposes (e.g., some of the extremal problems we will see later), it is useful to have the points correspond exactly with the elements of S. To capture this, we introduce the notion of a geometry.

Definition 5.5. *A* combinatorial geometry, simple matroid, *or* geometry *is a matroid in which the empty set and all singleton subsets of the ground set are flats.*

We now turn to a construction with which we will be able to produce many matroids.

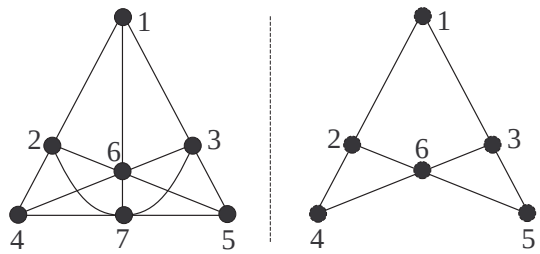

FIGURE 18. The restriction of $PG(2,2)$ to $\{1,2,3,4,5,6\}$ gives $M(K_4)$.

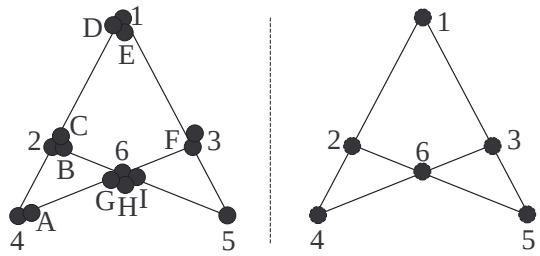

FIGURE 19. The simplification of a matroid.

Definition 5.6. *Let M be a matroid on the set S and let T be a subset of S. The* restriction *$M|T$ of M to T is the matroid on T that has as flats the sets $F \cap T$ as F ranges over the flats of M.*

It is an easy exercise to show that the flats of $M|T$ indeed satisfy axioms (F1)–(F3) of Definition 5.1.

For example, in the restriction of $PG(2,2)$ in Figure 18, the flat $\{1,2,4\}$ of $PG(2,2)$ restricted to the subset $\{1,2,3,4,5,6\}$ gives the same flat, $\{1,2,4\}$, while the flat $\{1,6,7\}$ of $PG(2,2)$ yields the flat $\{1,6\}$ in the restriction. This restriction of $PG(2,2)$ is the matroid $M(K_4)$; the notation comes from a connection (which we will not pursue) with the complete graph K_4.

If we focus instead on what is being removed, the restriction $M|T$ is called the *deletion* $M\backslash(S-T)$. Thus, $M(K_4)$ is $PG(2,2)\backslash\{7\}$.

In Figure 19, the original matroid is one in which rank-1 flats are not all singletons (so this is not a geometry); the rank-1 flats are $\{1,D,E\}$, $\{2,B,C\}$, $\{3,F\}$, $\{4,A\}$, $\{5\}$, and $\{6,G,H,I\}$. However, by restricting this matroid to $\{1,2,3,4,5,6\}$ we obtain a geometry that is intimately related to the original matroid — in some sense it contains the same geometric information without the multiple representatives. What we see in this example is similar to choosing a single representative vector out of each 1-dimensional subspace when forming a projective geometry. These are both instances of the simplification of a matroid.

Definition 5.7. *Let M be a matroid on a set S and let T be a subset of S that contains no elements of the rank-0 flat and precisely one element of each rank-1 flat. The restriction $M|T$ is the* simplification *of M.*

Up to isomorphism, a matroid has a unique simplification, so calling this *the* simplification is appropriate.

Using restriction, we can now give many examples of matroids: $PG(n-1,q)$ and its restrictions (or subgeometries). Note that $AG(n-1,q)$ is one of these restrictions. Such matroids, generalized mildly in the next definition, form a very important class of matroids.

Definition 5.8. *A matroid M is* representable over $GF(q)$ *if the simplification of M is isomorphic to a restriction of $PG(n-1,q)$ for some n.*

We could define matroids representable over any field F, but the case of finite fields will be our chief interest.

The matroid $M(K_4)$ in Figure 18 is representable over $GF(2)$, as is the matroid in Figure 19.

Note that a matroid is representable over $GF(q)$ if it is basically a subgeometry of $PG(n-1,q)$, for some n, perhaps with multiple copies of points added, possibly more copies of a point added than the number of scalar multiples of a vector over $GF(q)$. One might want to assign vectors in $(GF(q))^n$ to the elements of the matroid and represent the elements of the matroid by the columns of a matrix over $GF(q)$; matrices naturally allow for repeated columns. (Indeed, the word "matroid" is intended to suggest a generalization of a matrix.)

The next section examines representable matroids in more detail.

6. Representable Matroids

It is natural to ask the following questions. Which matroids are representable over a given field? Which matroids are representable over every field? For which matroids do there exist fields over which the matroids are representable? Such question are central to matroid theory. They are different aspects of the basic question: How do we capture our motivating examples, projective geometries and matroids easily obtained from them, within the class of all matroids?

Our first issue is this: Are there matroids that are not representable over any field? Desargues' theorem leads us to an example. (See Figure 20.) Since the condition illustrated by the Desargues configuration characterizes the projective planes that arise from 3-dimensional vector spaces over a division ring, the configuration in which all lines are the same except that P_1, P_2, and P_3 are not collinear is not representable over any field (or any division ring). One can easily check that this rank-3 configuration, the non-Desargues matroid, is indeed a matroid. Note that this configuration *cannot* be interpreted as a matroid of rank 4 since the points P_1, P_2, and P_3 would then be in the intersection of two distinct planes, $\mathrm{cl}(\{A,B,C\})$ and $\mathrm{cl}(\{A',B',C'\})$, and therefore collinear.

The same idea gives the non-Pappus matroid, a matroid that is not representable over any field although it is representable over skew fields. (See Figure 21.)

Thus, matroid theory is not limited to representable matroids; many matroids do not arise from projective geometries. It is of great interest to characterize the matroids that are representable over a field. Note that any such characterization must be more "sensitive" than conditions such as Desargues' theorem or Pappus' theorem since the configurations in these theorems are based on having certain lines actually intersecting (or parallel, in the affine cases); there is no assumption that coplanar lines of a matroid intersect or that a counterpart of the parallel postulate holds.

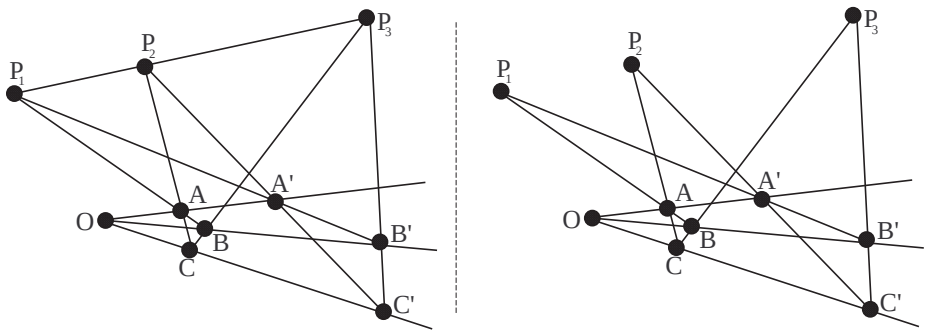

FIGURE 20. The Desargues configuration and the non-Desargues matroid.

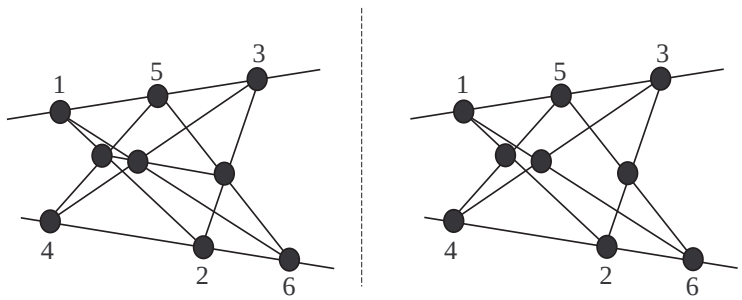

FIGURE 21. The Pappus configuration and the non-Pappus matroid.

One immediate but very useful observation about representability is the following theorem. (Since our interest is chiefly in the case of finite fields, we focus on that case; the result holds for arbitrary fields.)

Theorem 6.1. *If a matroid M is representable over $GF(q)$, then every restriction of M is representable over $GF(q)$.*

This is clear since the simplification of M is a restriction of a projective geometry $PG(n-1,q)$, so the simplification of a restriction of M is just a further restriction of $PG(n-1,q)$.

Recall that lines in $PG(n-1,q)$ have precisely $q+1$ points. In particular, lines in $PG(n-1,2)$ have exactly three points. Thus, if a matroid M has a line with four (or more) points, then M is not representable over $GF(2)$.

The four point line is denoted $U_{2,4}$. (See Figure 22.) The matroid $U_{2,4}$ has four elements, four points, and rank two. The argument in the last paragraph shows that *binary matroids* (those representable over $GF(2)$) cannot have $U_{2,4}$ as a restriction.

The matroid $U_{2,4}$ is one of an infinite family of very basic matroids, the uniform matroids. Let $U_{n,m}$ be the matroid in which the ground set has m elements and the flats are the entire set and all subsets of size less than n. It is easy to check that axioms (F1)–(F3) are satisfied. The matroid $U_{n,m}$ is called the *uniform matroid* of

FIGURE 22. $U_{2,4}$

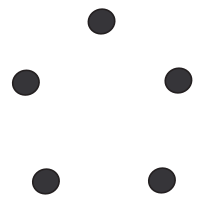

FIGURE 23. $U_{3,5}$

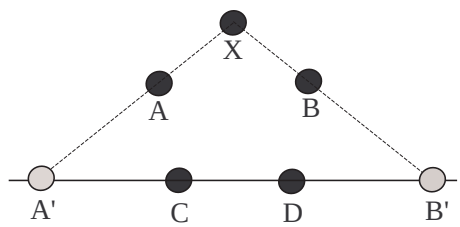

FIGURE 24. $U_{2,4}$ as a projection of $U_{3,5}$

rank n on m elements; $U_{n,m}$ captures the idea of m points in general position in rank n. The matroid $U_{3,5}$ is shown in Figure 23.

Note that $U_{3,5}$ does not have $U_{2,4}$ as a restriction. Is $U_{3,5}$ binary? No. The problem is that the matroid $U_{3,5}$ contains $U_{2,4}$ as a projection or, in matroid terminology, a contraction, as we will now see. (See Figure 24.) Consider the matroid $U_{3,5}$ with ground set $\{A, B, C, D, X\}$. If we could realize $U_{3,5}$ as a restriction of a projective geometry, then the points of $U_{3,5}$ span a plane of this geometry, so it suffices to view $U_{3,5}$ as a restriction of $PG(2, q)$. The lines $\ell(A, X)$ and $\ell(B, X)$ of $PG(2, q)$ must intersect the line $\ell(C, D)$ of $PG(2, q)$ since these are lines of a projective plane. These points of intersection, A' and B', are distinct since A, X, and B are not collinear, and similarly neither A' nor B' can be either C or D. Therefore the line $\ell(C, D)$ of $PG(2, q)$ has at least four points, that is, $PG(2, q)$ has a restriction isomorphic to $U_{2,4}$. Therefore q cannot be 2, so $U_{3,5}$ is not binary.

If we focus on the line $\{A', C, D, B'\}$ in Figure 24 and relabel the points A' and B' with the points A and B that projected onto these points, we get a geometry in which the flats are precisely that flats of $U_{3,5}$ that contained X, but now with X removed. The flats of $U_{3,5}$ that contained X are $\{X\}$, $\{X, A\}$, $\{X, B\}$, $\{X, C\}$, $\{X, D\}$, and $\{X, A, B, C, D\}$; the flats of the projection are $\emptyset$, $\{A\}$, $\{B\}$, $\{C\}$, $\{D\}$, and $\{A, B, C, D\}$. The generalization of this is the following very important operation.

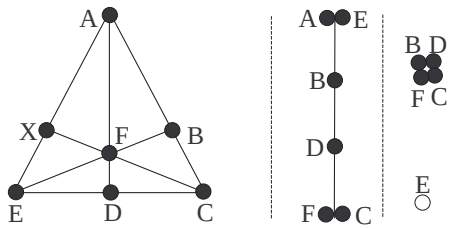

FIGURE 25. The non-Fano matroid $\overline{F}_7$ and its contractions by $\{X\}$ and by $\{X, A\}$.

Definition 6.2. *Let M be a matroid on the set S and let Z be a subset of S. The* contraction M/Z *of M by Z is the matroid on $S - Z$ that has as its collection of flats*

$$\{F - Z \mid F \text{ is a flat of } M \text{ with } Z \subseteq F\}.$$

For more examples of contraction, Figure 25 shows the contractions of the non-Fano matroid $\overline{F}_7$ by $\{X\}$ and by $\{X, A\}$. Note that the contraction $\overline{F}_7/\{X\}$ is not a geometry: the non-singleton sets $\{A, E\}$ and $\{F, C\}$ are points. This shows that applying a basic operation to a geometry does not necessarily produce a geometry; to obtain a geometry, we would need to take the simplification of the contraction, but there are sometimes good reasons for not doing this.

Note that there are only two flats in the contraction $\overline{F}_7/\{X, A\}$, namely $\{E\}$ as the least flat and $\{E, B, F, C, D\}$ as the sole rank-1 flat. The element E of the rank-0 flat of $\overline{F}_7/\{X, A\}$ is drawn as a hollow dot off to the side since it does not determine a point. The contraction $\overline{F}_7/\{X, A\}$ fails both conditions required of a geometry — the empty set is not a flat and the points are not singletons.

Note that $\overline{F}_7/\{X\}/\{A\} = \overline{F}_7/\{X, A\}$; this is true in general for contractions by a sequence of disjoint sets. This and the corresponding result for deletions by a sequence of disjoint sets follow immediately from the definitions of these operations.

While it was immediate that restrictions of representable matroids are representable, it is perhaps somewhat less transparent that the corresponding statement about contractions is also true.

Theorem 6.3. *If a matroid M is representable over $GF(q)$, then every contraction of M is representable over $GF(q)$.*

This result holds for arbitrary fields, but, again, our chief interest is in finite fields. One way of seeing Theorem 6.3 is to observe that contracting a subspace of a vector space is essentially taking the quotient of the vector space by the subspace; in the process of taking this quotient, a representation of a matroid in the vector space is carried to a representation of the corresponding contraction in the quotient space.

The middle diagram in Figure 25 shows that the non-Fano matroid $\overline{F}_7$ contains the 4-point line $U_{2,4}$ as a restriction of a contraction. Therefore, by Theorem 6.3, the non-Fano matroid $\overline{F}_7$ is not binary. It is often useful to combine restrictions and contractions as illustrated in this example.

Definition 6.4. *A* minor *of a matroid M is any matroid that can be obtained from M by repeatedly applying the operations of restriction and contraction.*

It turns out that the operations of restriction and contraction commute, so every minor of a matroid M amounts to a contraction of a restriction of M.

From Theorems 6.1 and 6.3, we get Theorem 6.5.

Theorem 6.5. *If a matroid M is representable over $GF(q)$, then every minor of M is representable over $GF(q)$.*

We can now state what has so far been the most successful of the several paths that have been pursued in the attempt to characterize representability. Since minors of matroids that are representable over $GF(q)$ are also representable over $GF(q)$, one could characterize representability by finding the minor-minimal matroids that are not representable over $GF(q)$. That is, one will have characterized representability if one can find the minor-minimal obstructions to representability. The *excluded minors* for representability over $GF(q)$ are these the minor-minimal obstructions, the minor-minimal matroids that are not representable over $GF(q)$.

The first excluded minor characterization of representability was proven by Tutte in 1958. We have already seen that the four-point line is not binary; Tutte [32] showed that $U_{2,4}$ is the only excluded minor for representability over $GF(2)$.

Theorem 6.6. *A matroid is binary if and only if it has no minor isomorphic to $U_{2,4}$.*

There are a variety of proofs of Theorem 6.6; while most are algebraic, the proof in [6] is geometric.

In [32], Tutte also gave the following characterization of the matroids that are representable over *every* field.

Theorem 6.7. *A matroid is representable over every field if and only if it has no minor isomorphic to any of $U_{2,4}$, $PG(2,2)$, and a single-element deletion $AG(3,2)\backslash x$ of $AG(3,2)$.*

Affine and projective geometries have transitive automorphism groups, so there is, up to isomorphism, only one single-element deletion $AG(3,2)\backslash x$ of $AG(3,2)$. The matroids that are representable over all fields are called *regular* matroids; they are also known as *unimodular* matroids since they are the matroids that can be represented by totally unimodular matrices (matrices with entries among the integers 0, 1, or -1 in which all subdeterminants are 0, 1, or -1). Unimodular matroids play an important role in optimization.

Since lines in $PG(n-1,q)$ have $q+1$ points, the $(q+2)$-point line $U_{2,q+2}$ is an excluded minor for representability over $GF(q)$, but for $q > 2$, there will be additional excluded minors. Theorem 6.8, the characterization of ternary matroids, or matroids that are representable over $GF(3)$, illustrates this. Ternary matroids were characterized in 1971 in unpublished work of R. Reid. The first published proofs were by Bixby [4] and Seymour [30] in 1979. Seymour's proof introduced many new ideas that revolutionized work on representability and other aspects of matroid theory.

Theorem 6.8. *A matroid is ternary if and only if it has no minor isomorphic to $U_{2,5}$, $U_{3,5}$, $PG(2,2)$, or $AG(3,2)\backslash x$.*

As a corollary of Theorems 6.6, 6.7, and 6.8, we have the following theorem.

Theorem 6.9. *A matroid is representable over every field if and only if it is representable over both $GF(2)$ and $GF(3)$.*

Recent work of Whittle [37] studies matroids that are representable over other pairs of fields of relatively prime characteristic.

The excluded minors for matroids that are representable over $GF(4)$ have been found recently by Geelen, Gerards, and Kapoor [20]. There are seven excluded minors, the largest two of which have eight points and rank four.

One of the outstanding problems in this area is the following conjecture due to Gian-Carlo Rota.

Conjecture 6.10. *For each prime power q, the number of excluded minors for representability over $GF(q)$ is finite.*

Rota made this conjecture in 1971, when there was barely any evidence for it. His motivation for the conjecture came from invariant theory. Recent work of Johnson, Seymour, and Robertson suggests a strategy to prove Rota's conjecture that has grown out of the graph minors project; this will be an exciting area to watch.

Recently Oxley, Semple, and Vertigan [28] gave the following exponential lower bound on the number of excluded minors for representability over $GF(q)$.

Theorem 6.11. *There are at least 2^{q-4} excluded minors for representability over $GF(q)$.*

Note that 2^{q-4} is well below the number of excluded minors mentioned above for $GF(2)$, $GF(3)$, and $GF(4)$. Looking at the strategy in [28] reveals that their argument covers only excluded minors of a relatively simple type; the excluded minors they count are all variations on the $(q+2)$-point line obtained by the exchange operations they develop. This suggests that the actual number of excluded minors is extremely large.

We have seen that, apart from the case $q = 2$, the $(q + 2)$-point line $U_{2,q+2}$ is not the only minor one has to exclude in order to characterize representability over $GF(q)$. However, as the following theorem from [6] shows, if there are enough points relative to the rank, then excluding only $U_{2,q+2}$ as a minor is enough to guarantee representability.

Theorem 6.12. *Assume that q is a prime power and n exceeds three. Any rank-n geometry with no $U_{2,q+2}$ minor and with at least q^{n-1} points is representable over $GF(q)$.*

Assume q is an odd prime power and n exceeds three. Any rank-n geometry with no $U_{2,q+2}$ minor and with at least $q^{n-1} - (q^{n-2} - 1)/(q - 1)$ points is representable over $GF(q)$.

This result leads to the next major branch of matroid theory we will glimpse, extremal matroid theory. Theorem 6.12 and related results in [6] strengthen the result of Kung [25] that opens the next section.

7. Extremal Matroid Theory

Theorem 6.12 assumed that q is a prime power because finite fields have prime power order. However, it is possible to consider the effect of excluding the $(q + 2)$-point line minor even if q is not a prime power. A basic result along these lines is the following theorem of Kung [25].

Theorem 7.1. *Rank-n geometries that have no $(q + 2)$-point line minor have at most $(q^n - 1)/(q - 1)$ points. This upper bound is attained only by projective geometries of order q.*

Thus, this upper bound is strict if there is no projective geometry of rank n and order q (e.g., if q is not a prime power and n is at least 4).

Theorem 7.1 captures the spirit of extremal matroid theory: a condition (excluding $U_{2,q+2}$ as a minor) is imposed on matroids and the effect on a parameter (the number of points) is analyzed. Another result of this type is the following theorem from [6] that gives a counterpart of Theorem 7.1 for affine geometries.

Theorem 7.2. *Rank-n geometries that have no $U_{2,q+2}$ minor and have no $(q+1)$-point lines have at most q^{n-1} points. This upper bound is attained only by affine geometries of order q.*

It is natural to ask if the upper bound in Theorem 7.1 can be improved significantly when q is not a prime power. In [6] and [8], a number of such improvements are given, but those upper bounds are not sharp. The only currently known sharp upper bound for a problem of this type is the following theorem from [8]. This addresses the smallest integer that is not a prime power, $q = 6$, so $q + 2$ is 8.

Theorem 7.3. *Let n be greater than 3 and let M be a rank-n geometry with no 8-point line minor. Then the number of points in M is at most $(5^n - 1)/(5 - 1)$. This upper bound is sharp and is attained only by the rank-n projective geometry $PG(n-1, 5)$.*

Theorem 7.3 proves the first case of the following conjecture of Kung [25].

Conjecture 7.4. *Let q_* be the largest prime power not exceeding q. For all sufficiently large n, the greatest number of points a rank-n geometry that has no $U_{2,q+2}$ minor is $(q_*^n - 1)/(q_* - 1)$. This upper bound is sharp and is attained only by the rank-n projective geometry $PG(n-1, q_*)$.*

This conjecture, which appears to be very difficult, asserts that projective geometries play an extremely central role in problems that have $U_{2,q+2}$ as an excluded minor. It would be a solid advance to prove Conjecture 7.4 even for $q = 10$.

To relate these new topics to our motivating examples, we can now characterize classical geometries through extremal matroid theoretic properties. Theorems 7.1 and 7.2 are results of this sort, and they have as further corollaries a number of additional characterizations of projective and affine spaces; the following two theorems, from [9] and [13] respectively, sample a few such results.

Theorem 7.5. *A rank-n geometry with $(q^n - 1)/(q - 1)$ points in which all lines have at least $q + 1$ points is a projective geometry of order q.*

A rank-n geometry with q^{n-1} points in which all lines have q points and all planes have at least q^2 points is an affine geometry of order q.

Assume $q > 2$. A rank-n geometry with q^{n-1} points in which all lines have q points and all hyperplanes have q^{n-2} points is an affine geometry of order q.

Theorem 7.6. *Let $q > 1$ be an integer. Let M be a rank-n geometry in which each hyperplane contains all but at most q^{n-1} elements of M and in which all lines have at least $q + 1$ points. Then the number of points in M is exactly $(q^n - 1)/(q - 1)$ and M is a projective geometry of order q.*

Theorem 7.1 suggests other interesting questions. By excluding $U_{2,q+2}$ as a minor, one gets an upper bound of $(q^n - 1)/(q - 1)$ on the number of points. It follows that there are upper bounds on the number of lines, the number of planes, and, in general, the number of rank-i flats, but what are the optimal upper bounds?

Open Problem 7.7. *Assume that q is a prime power and, if needed, n is sufficiently large. Is the number of rank-i flats of $PG(n-1, q)$ an upper bound on the number of rank-i flats in any rank-n geometry that has no $U_{2,q+2}$ minor? If so, are projective geometries the only geometries that attain these bounds?*

In some extremal problems, one places restrictions on the subgeometries of a geometry, rather than on all minors. An instance of this is the following theorem, which is a corollary of Theorem 7.2.

Theorem 7.8. *Rank-n geometries that are representable over $GF(q)$ and have no lines with $q+1$ points have at most q^{n-1} points. Furthermore, $AG(n-1,q)$ is the only such geometry with q^{n-1} points.*

Note that the $(q+1)$-point line is $PG(1,q)$; note also that $AG(n-1,q)$ is $PG(n-1,q)\backslash PG(n-2,q)$. It would not be surprising if other deletions of the form $PG(n-1,q)\backslash PG(k-1,q)$ arose from excluding other projective geometries as subgeometries. The following theorem of Bose and Burton [14] treats this.

Theorem 7.9. *Subgeometries of $PG(n-1,q)$ that have no subgeometries isomorphic to $PG(m-1,q)$ have at most*

$$\frac{q^n - q^{n-m+1}}{q-1}$$

points. Furthermore, $PG(n-1,q)\backslash PG(n-m,q)$ is the only such geometry with $(q^n - q^{n-m+1})/(q-1)$ points.

Note that Theorem 6.12 allows us to replace the hypothesis of representability over $GF(q)$ in Theorem 7.9 with the weaker hypothesis of no minors isomorphic to $U_{2,q+2}$. Thus, rank-n geometries that have no minors isomorphic to $U_{2,q+2}$ and no subgeometries isomorphic to $PG(m-1,q)$ have at most $(q^n-q^{n-m+1})/(q-1)$ points, with the only example that attains this bound being $PG(n-1,q)\backslash PG(n-m,q)$.

To end this brief look at extremal matroid theory, we mention a theorem and a conjecture of a very different flavor. We first establish some notation for dealing with the types of problem encountered in Theorem 7.9. Let $ex_q(M;n)$ be the maximum number of points in a rank-n geometry that is representable over $GF(q)$ and that has no restriction isomorphic to the geometry M. Thus, Theorem 7.9 determines $ex_q(PG(m-1,q);n)$. The following theorem appears in [11].

Theorem 7.10. *Assume that M a subgeometry of $AG(m-1,2)$. Then*

$$\lim_{n\to\infty} \frac{ex_2(M;n)}{2^n-1} = 0.$$

In other words, for any subgeometry M of $AG(m-1,2)$, the "size function" $ex_2(M;n)$ is an order of magnitude smaller than 2^n-1, the number of points in $PG(n-1,2)$. This is the strongest bit of evidence to date for the following conjecture, which, although never stated, appears between the lines in [25].

Conjecture 7.11. *If the geometry M is representable over $GF(q)$, then*

$$\lim_{n\to\infty} \frac{ex_q(M;n)}{(q^n-1)/(q-1)} = 1 - q^{-c+1}$$

where c is the minimum number of subgeometries of $AG(m-1,q)$ into which M can be partitioned.

In general, it is extremely difficult to compute $ex_q(M;n)$. (See [11] for all currently known results on $ex_q(M;n)$.) Conjecture 7.11 is an attempt to describe the asymptotic behavior of $ex_q(M;n)$.

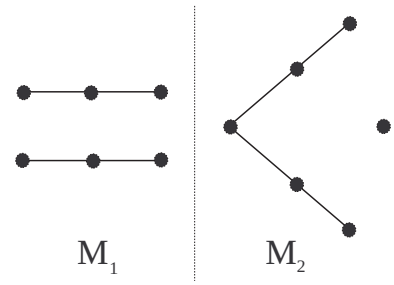

FIGURE 26. Two matroids with the same Tutte polynomial.

8. Matroid Invariants

The final branch of matroid theory we will briefly discuss is the theory of Tutte polynomials and other invariants of matroids. The Tutte polynomial is a two-variable polynomial associated with a matroid. For a matroid M on the ground set S, the Tutte polynomial $t(M; x, y)$ is defined as follows:

$$t(M; x, y) = \sum_{A \subseteq S} (x-1)^{r(S)-r(A)} (y-1)^{|A|-r(A)}. \tag{2}$$

This polynomial generalizes many important polynomials in mathematics, including the chromatic and flow polynomials of graph theory, weight enumerators in coding theory, the Jones polynomial for alternating knots, and, in some cases, the partition function of the Ising model in statistical physics.

The Tutte polynomial contains much information about the matroid; for instance, from the Tutte polynomial, one can determine the rank of the matroid, the cardinality of the ground set, the cardinalities of sufficiently large flats of each rank, and many more parameters that we have not discussed. Still, nonisomorphic matroids can have the same Tutte polynomial. This is illustrated by the matroids in Figure 26. For these matroids, we have

$t(M_1; x, y) = t(M_2; x, y) =$	$(x-1)^3$	the empty set
	$+\,6(x-1)^2$	the six singleton sets
	$+\,15(x-1)$	the fifteen pairs
	$+\,18$	eighteen of the 3-subsets have rank 3
	$+\,2(x-1)(y-1)$	the other two have rank 2
	$+\,15(y-1)$	the 4-element subsets
	$+\,6(y-1)^2$	the 5-element subsets
	$+\,(y-1)^3$	the entire set.

There are several general techniques for producing more examples of matroids with the same Tutte polynomial (see, e.g., [7, 12, 16]). Before turning to some complementary results (matroids that are determined by their Tutte polynomial), we sketch some of the reasons that Tutte polynomials are so important.

A number of frequently-studied invariants satisfy "deletion-contraction" rules. One of the oldest invariants of this type is the chromatic polynomial of a graph.

The chromatic polynomial, $\chi(G;x)$, of a graph G assigns to each positive integer x the number of proper colorings of the vertices of G with x "colors" (which can be taken to be the integers $1, 2, \ldots, x$); *proper* means that vertices that are joined by an edge must be assigned different colors. Notice that for an edge e of a graph G, we have

$$\chi(G-e;x) = \chi(G;x) + \chi(G/e;) \tag{3}$$

where $G-e$ is the deletion, G with the edge e removed, and G/e is the contraction, the graph formed from G by removing e and identifying the vertices that e joined. (These operations are intimately linked with the like-named matroid operations.) Equation (3) holds since any proper coloring of $G-e$ either assigns the endpoints of e different colors (giving a proper coloring of G) or the same colors (giving a proper coloring of G/e). It is useful to write Equation (3) in the form

$$\chi(G;x) = \chi(G-e;x) - \chi(G/e;). \tag{4}$$

This is a recursive expression for $\chi(G;x)$. From this, one can induct to show that $\chi(G;x)$ is indeed a polynomial function of x.

One can show that the Tutte polynomial also satisfies a deletion-contraction rule. Indeed, an alternative way of defining the Tutte polynomial is as follows. The Tutte polynomial of matroid whose ground set is the empty set is 1. If e is an element of a matroid M, then

$$t(M;x,y) = \begin{cases} t(M\backslash\{e\};x,y) + t(M/\{e\};x,y) & \text{if } r(\{e\}) > 0 \text{ and } r(M\backslash\{e\}) = r(M), \\ y\,t(M/\{e\};x,y) & \text{if } r(\{e\}) = 0, \\ x\,t(M\backslash\{e\};x,y) & \text{if } r(M\backslash\{e\}) < r(M). \end{cases}$$

From this formulation, it is not immediately clear that the Tutte polynomial is well-defined; conceivably it could depend on the order in which the elements of M are deleted and contracted. However, since this is equivalent to Equation (2), the Tutte polynomial is indeed well-defined.

The Tutte polynomial is important in part because it satisfies the deletion-contraction rule but more so because it is the universal invariant for all invariants that satisfy a deletion-contraction rule; all other invariants of matroids that satisfy a deletion-contraction rule are evaluations of the Tutte polynomial. The following theorem makes this precise.

Theorem 8.1. *Let R be a commutative ring that has unity. For each choice of elements u, v, σ, τ of R, there is a unique function T from the class of all matroids into R that has these properties.*

(i) *If M is the matroid on the empty set, then $T(M) = 1$.*
(ii) *If e is an element of M and $r(\{e\}) > 0$ and $r(M\backslash\{e\}) = r(M)$, then*

$$T(M) = \sigma\,T(M\backslash\{e\}) + \tau\,T(M/\{e\}).$$

(iii) *If e is an element of M and $r(\{e\}) = 0$, then $T(M) = v\,T(M/\{e\})$.*
(iv) *If e is an element of M and $r(M\backslash\{e\}) < r(M)$, then $T(M) = u\,T(M\backslash\{e\})$.*

Furthermore, T is the following evaluation of the Tutte polynomial $t(M;x,y)$: for a matroid M with a k-element ground set, we have

$$T(M) = \sigma^{k-r(M)}\tau^{r(M)}t(M;u/\tau,v/\sigma).$$

Brylawski [15] proved this in the case of $\sigma = \tau = 1$; Oxley and Welsh [29] observed that the same argument yields the general case. That this result is very useful makes it all the more striking that the proof is a simple induction based on the deletion-contraction rule.

To sketch (very incompletely) one relevant application of Theorem 8.1, note that in Equation (4) we have $\sigma = 1$ and $\tau = -1$. The chromatic polynomial of a graph that consists of a single edge is $x(x-1)$ since one vertex can be colored with any of x colors and the other vertex can be colored with any of the $x-1$ other colors. A graph with an edge that is incident with only one vertex (a loop) has no proper colorings. With a graph G, there is an associated matroid, the cycle matroid $M(G)$. Skipping many steps, one can argue from this that $\chi(G;x)$ is $x^{\omega(G)}(-1)^{r(M(G))}t(M(G);(x-1)/(-1),0/1)$, or

$$x^{\omega(G)}(-1)^{r(M(G))}t(M(G);1-x,0) \tag{5}$$

where $\omega(G)$ is the number of components of the graph G.

Before pursuing such specializations of the Tutte polynomial further, we mention more results about Tutte polynomials. The example at the beginning of this section shows that several matroids can have the same Tutte polynomial; we mentioned that there are general constructions for producing more such examples. At the opposite end of the spectrum, some matroids are determined by their Tutte polynomials. The next theorem is from [9].

Theorem 8.2. *If $n \geq 4$ and $t(M;x,y) = t(AG(n-1,q);x,y)$, then M is isomorphic to $AG(n-1,q)$. If $t(M;x,y) = t(AG(2,q);x,y)$, then M is an affine plane of order q.*

There are extremely few results of this type known. The corresponding result for $PG(n-1,q)$ was known earlier (see [16]) and has many simple proofs. To prove Theorem 8.2, one can show that the hypotheses of the first characterization of affine geometries in Theorem 7.5 can be verified from the Tutte polynomial of a matroid. Uniform matroids are also determined by their Tutte polynomials, as are the cycle matroids of complete graphs and certain generalizations of such matroids (see [9]).

Recall that $AG(n-1,q)$ is $PG(n-1,q)\backslash PG(n-2,q)$. With this in mind, we see that the following result from [1] in a natural follow-up to Theorem 8.2.

Theorem 8.3. *Assume n and k are integers with $n \geq 3$ and $1 \leq k \leq n-2$. Assume the matroid M has the same Tutte polynomial as $PG(n-1,q)\backslash PG(k-1,q)$. If $n \geq 4$, then M is isomorphic to $PG(n-1,q)\backslash PG(k-1,q)$. If $n = 3$, then M is isomorphic to some single-element deletion of some projective plane of order q.*

One key element of the proof of Theorem 8.3 and related results in [1] is that the geometries $PG(n-1,q)\backslash PG(k-1,q)$ are the unique geometries that attain the bound in a result of extremal matroid theory, specifically, Theorem 7.9.

In contrast to such results, huge collections of representable matroids can have the same Tutte polynomial. In [7], techniques are developed to produce many families of such collections; the following theorem is the tip of the iceberg in this area.

Theorem 8.4. *Assume that q is the t-th power of a prime, that q exceeds 5, and that $q-1$ is not a Mersenne prime. Let d be the largest proper divisor of $q-1$. For each integer n with $n \geq 3$, there are at least*

$$\frac{(d-1)^{n-1}(q-1-d)(q-2-d)}{t}$$

non-isomorphic 3*-connected geometries that are representable over* $GF(q)$*, that have rank* $2n+2$*, that contain*

$$\frac{q^{2n+2}-1}{q-1} - 5n - 5$$

points, and that have the same Tutte polynomial. In particular, if q *is odd, then there are at least*

$$\left(\frac{q-3}{2}\right)^n \frac{q-1}{2t}$$

such geometries.

We mention two open problems along these lines.

Open Problem 8.5. *Find more matroids that are uniquely determined by their Tutte polynomials.*

Open Problem 8.6. *Find more constructions that produce nonisomorphic matroids that have the same Tutte polynomial.*

The particular evaluation of the Tutte polynomial in Equation (5) arises so frequently that it is singled out. The *characteristic polynomial* of a matroid M is defined by

$$p(M;x) = (-1)^{r(M)} t(M; 1-x, 0).$$

Thus, up to a power of x, the characteristic polynomial $p(M;x)$ is a generalization of the chromatic polynomial of a graph.

The chromatic polynomial of a graph has the following simple property: the first so many positive integers are roots of the chromatic polynomial and then there is never another positive integer root of this polynomial. This is simply because if G cannot be colored with k colors, then G cannot be colored with $k-1$ colors. One can show that matroids that are representable over $GF(q)$ have an analogous property with respect to powers of q; the first so many powers of q are roots of the characteristic polynomial and then there are no more powers of q that are roots of this polynomial. Thus, in analogy with the chromatic number of a graph, the smallest integer k which is not a root of the chromatic number (equivalently, the smallest integer k such that the graph can be colored with k colors), we define the critical exponent of a geometry that is representable over $GF(q)$. (We could define the critical exponent of a matroid in general but the critical exponent depends only on the simplification of the matroid. Therefore the theory of critical exponents generally focuses on geometries.)

Definition 8.7. *Assume that* M *is a geometry that is representable over* $GF(q)$*. The* critical exponent of M over $GF(q)$ *is the least positive integer* k *such that* $p(M;q^k) \neq 0$*.*

The following theorem of Crapo and Rota [18] begins to hint at the importance of the critical exponent.

Theorem 8.8. *Assume that the rank-*m *geometry* M *is representable over* $GF(q)$ *and has critical exponent* c *over* $GF(q)$*. For any* $n \geq m$ *and any embedding of* M *in* $PG(n-1,q)$*, the least codimension of a subspace of* $PG(n-1,q)$ *disjoint from* M *is* c*.*

It is striking that this codimension does not depend on the embedding or even the dimension in which the matroid is embedded! One can argue from this that the

critical exponent is also the smallest number of subgeometries of $AG(n-1,q)$ into which M can be partitioned. Thus, the c in Conjecture 7.11 is the critical exponent of M.

We are approaching the critical problem of matroid theory, a huge area of matroid theory that connects with many important topics. For instance, if we understood critical exponents well enough, we would have a solution to the fundamental problem of linear coding theory [19]. While it would be fascinating to pursue this topic here, we refer the reader to Kung [26] for a state-of-the-art survey.

In Theorems 8.2 and 8.3, and the related results mentioned above, we saw that some matroids are uniquely determined by their Tutte polynomials. We close this section with a few analogous results about characteristic polynomials from [1].

Theorem 8.9. *Assume that the geometry M is representable over $GF(q)$ and that $p(M;x) = p(PG(n-1,q)\backslash PG(n-m,q);x)$. Then M is isomorphic to the geometry $PG(n-1,q)\backslash PG(n-m,q)$.*

Theorem 8.10. *Assume $n \geq 4$. If $p(M;x) = p(PG(n-1,q)\backslash PG(n-m,q);x)$ and the multiset*

$$\{p(M/X;x) \mid X \text{ a flat of } M \text{ of rank } n-2\}$$

is equal to the corresponding multiset constructed from $PG(n-1,q)\backslash PG(n-m,q)$, then M is isomorphic to $PG(n-1,q)\backslash PG(n-m,q)$.

The proof of Theorem 8.9 rests on the observation, following from Theorem 8.8, that $PG(n-1,q)\backslash PG(n-m,q)$ is the unique largest geometry of rank n that has critical exponent $n-m+1$. Theorem 8.10 uses this observation along with Theorem 6.12.

9. Conclusion

This introduction to matroid theory has scratched the surface of only a few of the many areas to be explored in the subject. We have not at all addressed maps between matroids (strong maps, weak maps, comaps) [33], constructions [27, 33], basis exchange properties [33], structure theory and connectivity [10, 27], the splitter theorem and its consequences [10, 27], enumeration [18, 31], and the entire field of oriented matroids [5]. We have also omitted the numerous applications, which include optimization [34], coding theory [17, 19, 26, 31, 34], connections with differential geometry, structural rigidity [10, 21, 35], arrangements of hyperplanes, and much more. It is hoped that this introduction whets your appetite to pursue matroids further.

References

[1] R. Ankney and J. Bonin, Characterizations of $PG(n-1,q)\backslash PG(k-1,q)$ by numerical and polynomial invariants, *Adv. Appl. Math.*, to appear.

[2] M. K. Bennett, *Affine and Projective Geometry*, (Wiley Interscience, 1995).

[3] M. K. Bennett, K. P. Bogart, and J. Bonin, The geometry of Dowling lattices, *Adv. Math.* **103** (1994) 131–161.

[4] R. Bixby, On Reid's characterization of ternary matroids, *J. Combin. Theory Ser. B* **26** (1979) 174–204.

[5] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler, *Oriented Matroids*, (Cambridge University Press, Cambridge, 1993).

[6] J. Bonin, Matroids with no $(q+2)$-point line minors, *Adv. Appl. Math.* **17** (1996) 460–476.

[7] J. Bonin, Strongly inequivalent representations and Tutte polynomials of matroids, *Algebra Universalis*, to appear.

[8] J. Bonin and J. P. S. Kung, The number of points in a combinatorial geometry with no 8-point-line minor, in: *Mathematical Essays in Honor of Gian-Carlo Rota*, B. Sagan and R. P. Stanley, eds., (Birkhauser, Boston, 1998) 271–284

[9] J. Bonin and W. Miller, Characterizing combinatorial geometries by numerical invariants, *European J. Combin.,* **20** (1999) 713–724.

[10] J. Bonin, J. G. Oxley, and B. Servatius, eds. *Matroid Theory*, (Amer. Math. Soc., Providence RI, 1996).

[11] J. Bonin and H. Qin, Size functions of subgeometry-closed classes of representable combinatorial geometries, *Discrete Math.,* **224** (2000) 37–60.

[12] J. Bonin and H. Qin, Tutte polynomials of q-cones, *Discrete Math.,* **232** (2001) 95–103.

[13] J. Bonin and T. J. Reid, Simple matroids with bounded cocircuit size, *Combin. Probab. Comput.,* **9** (2000) 407–419.

[14] R. C. Bose and R. C. Burton, A characterization of flat spaces in finite geometry and the uniqueness of the Hamming and the MacDonald codes, *J. Combin. Theory* **1** (1966) 244–257.

[15] T. H. Brylawski, A decomposition for combinatorial geometries, *Trans. Amer. Math. Soc.* **171** (1972) 235–282.

[16] T. H. Brylawski, The Tutte polynomial. Part I: General theory, in: *Matroid Theory and Its Applications*, A. Barlotti, ed. (C.I.M.E., Liguori, Naples, 1980) 125–175.

[17] T. H. Brylawski and J. G. Oxley, The Tutte polynomial and its applications, in: *Matroid Applications*, N. White, ed. (Cambridge University Press, Cambridge, 1992) 123–225.

[18] H. H. Crapo and G.-C. Rota, *On the Foundations of Combinatorial Theory: Combinatorial Geometries*, (Preliminary edition, M. I. T. Press, Cambridge MA, 1970).

[19] T. A. Dowling, Codes, packings and the critical problem, in: *Atti del Convegno di Geometria Combinatoria e sue Applicazioni*, A. Barlotti, ed. (Perugia, Italy, 1971) 209–224.

[20] J. F. Geelen, A. M. H. Gerards, and A. Kapoor, The excluded minors for $GF(4)$-representable matroids, preprint.

[21] J. Graver, B. Servatius, H. Servatius, *Combinatorial rigidity*, (Amer. Math. Soc., Providence, RI, 1993).

[22] M. D. Halsey, Line-closed combinatorial geometries, *Discrete Math.* **65** (1987) 245–248.

[23] D. Hilbert, *Foundations of Geometry* (english translation), (Open Court, 1971).

[24] D. R. Hughes and F. C. Piper, *Projective Planes*, (Springer, Graduate Texts in Mathematics **6**, 1973).

[25] J. P. S. Kung, Extremal matroid theory, in: *Graph Structure Theory*, N. Robertson and P. D. Seymour, eds. (Amer. Math. Soc., Providence, RI, 1993) 21–61.

[26] J. P. S. Kung, Critical problems, in: *Matroid Theory*, J. Bonin, J. G. Oxley, and B. Servatius, eds. (Amer. Math. Soc., Providence RI, 1996) 1–127.

[27] J. G. Oxley, *Matroid Theory*, (Oxford University Press, Oxford, 1992).

[28] J. G. Oxley, C. Semple, and D. Vertigan, Generalized $\Delta - Y$ exchange and k-regular matroids, *J. Combin. Theory Ser. B*, to appear.

[29] J. G. Oxley and D. J. A. Welsh, The Tutte polynomial and percolation. in: *Graph Theory and Related Topics* J. A. Bondy and U. S. R. Murty, eds. (Academic Press, New York-London, 1979) 329–339.

[30] P. Seymour, Matroid representation over $GF(3)$, *J. Combin. Theory Ser. B* **26** (1979) 159–173.

[31] D. J. A. Welsh, *Matroid Theory*, (Academic Press, London, 1976).

[32] W. T. Tutte, A homotopy theorem for matroids, I, II, *Trans. Amer. Math. Soc.* **88** (1958) 144–174.

[33] N. White, ed. *Theory of Matroids*, (Cambridge University Press, Cambridge, 1986).

[34] N. White, ed. *Combinatorial Geometries*, (Cambridge University Press, Cambridge, 1987).

[35] N. White, ed. *Matroid Applications*, (Cambridge University Press, Cambridge, 1992).

[36] H. Whitney, On the abstract properties of linear dependence, *Amer. J. Math.* **57** (1935) 509–533.

[37] G. Whittle, On matroids representable over $GF(3)$ and other fields, *Trans. Amer. Math. Soc.* **349** (1997) 579–603.

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