

# BASIS-EXCHANGE PROPERTIES OF SPARSE PAVING MATROIDS

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ABSTRACT. It has been conjectured that, asymptotically, almost all matroids are sparse paving matroids. We provide evidence for five long-standing, basis-exchange conjectures by proving them for this large class of matroids.

*To Geoff Whittle on his 60th birthday*

## 1. INTRODUCTION

A matroid is *paving* if the closure of each nonspanning circuit is a hyperplane; it is *sparse paving* if each nonspanning circuit is a hyperplane. Thus, a matroid  $M$  of rank  $r$  is sparse paving if and only if each  $r$ -subset of  $E(M)$  is either a basis or a circuit-hyperplane. It follows that the class of sparse paving matroids is dual-closed. It is easy to show that this class is also minor-closed. Sparse paving matroids can also be characterized as the matroids  $M$  for which both  $M$  and its dual,  $M^*$ , are paving.

While paving and sparse paving matroids have received increasing attention recently (see, e.g., [6, 9, 13, 14, 15]), they have long played important roles in matroid theory. For instance, D. Knuth [12] constructed at least

$$\frac{2^{\binom{n}{\lfloor n/2 \rfloor}}}{n!}$$

nonisomorphic sparse paving matroids of rank  $\lfloor n/2 \rfloor$  on  $n$  elements; with the upper bound by M. Piff [18], it follows that the number  $g_n$  of nonisomorphic simple matroids on  $n$  elements satisfies

$$(1.1) \quad n - \frac{3}{2} \log_2 n + O(\log_2 \log_2 n) \leq \log_2 \log_2 g_n \leq n - \log_2 n + O(\log_2 \log_2 n),$$

with sparse paving matroids accounting for the lower bound. Taking this further, in [13], D. Mayhew, M. Newman, D. Welsh, and G. Whittle have conjectured that, asymptotically, almost all matroids are sparse paving.

The five basis-exchange conjectures treated in this paper, all of which have been open for decades and have been proven for only a few classes of matroids, are part of the circle of ideas that revolve around the well-known symmetric basis-exchange property: for any bases  $B_1, B_2$  of a matroid  $M$ , if  $b_1 \in B_1 - B_2$ , then, for some  $b_2 \in B_2 - B_1$ , both  $(B_1 - b_1) \cup b_2$  and  $(B_2 - b_2) \cup b_1$  are also bases of  $M$ .

The first conjecture concerns the *basis pair graph*,  $G(M)$ , of a matroid  $M$ , which is defined as follows. The vertices of  $G(M)$  are the ordered triples  $(A_1, A_2, A_3)$  of subsets of  $E(M)$  where  $A_1$  and  $A_2$  are disjoint bases of  $M$  and  $A_3$  is  $E(M) - (A_1 \cup A_2)$ . (Thus, the inequality  $|E(M)| \geq 2r(M)$  must hold in order for  $G(M)$  to have any vertices.) Two

vertices, say  $\mathbf{A} = (A_1, A_2, A_3)$  and  $\mathbf{B} = (B_1, B_2, B_3)$ , of  $G(M)$  are adjacent if  $\mathbf{B}$  can be obtained from  $\mathbf{A}$  by switching some pair of elements in two different sets in  $\mathbf{A}$ , that is, if

$$|A_1 - B_1| + |A_2 - B_2| + |A_3 - B_3| = 2.$$

If  $E(M)$  is the disjoint union of two bases of  $M$ , then  $G(M)$  is isomorphic to the basis-cobasis graph studied by R. Cordovil and M. Moreira [2]. The following conjecture was posed by M. Farber [3], who proved it for transversal matroids. (In [4], M. Farber, B. Richter, and H. Shank proved it for graphic and cographic matroids.)

**Conjecture 1.1.** *The basis pair graph of any matroid is connected.*

The second conjecture involves a family of graphs that we can associate with a matroid. Fix an integer  $k \geq 2$ . Let  $M$  be a matroid of rank  $r$  and let  $S$  be a multiset of size  $kr$  with elements in  $E(M)$ . Define the graph  $G_M(S)$  as follows: the vertices of  $G_M(S)$  are all multisets of  $k$  bases of  $M$  whose multiset union is  $S$ ; two vertices are adjacent if one can be obtained from the other by one symmetric exchange among one pair of bases in one of the vertices. Thus, vertices  $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$  and  $\mathcal{B} = \{B_1, B_2, \dots, B_k\}$  are adjacent if, for some bases  $B_i, B_j \in \mathcal{B}$  and elements  $b_i \in B_i - B_j$  and  $b_j \in B_j - B_i$ , we obtain  $\mathcal{A}$  from  $\mathcal{B}$  by replacing  $B_i$  by  $(B_i - b_i) \cup b_j$  and replacing  $B_j$  by  $(B_j - b_j) \cup b_i$ . (This graph may be empty.) The conjecture below is due to N. White [20, Conjecture 12].

**Conjecture 1.2.** *For any matroid  $M$  and multiset  $S$  of size  $kr(M)$  with elements in  $E(M)$  and with  $k \geq 2$ , the graph  $G_M(S)$  is connected.*

Conjecture 1.2 is sometimes cast in terms of toric ideals. A routine argument shows that the conjecture holds for  $M$  if and only if it holds for  $M^*$ . It has been shown for graphic (and so for cographic) matroids by J. Blasiak [1] and for matroids of rank at most three (and so for matroids of nullity at most three) by K. Kashiwabara [11]. J. Herzog and T. Hibi [7] have shown that Conjecture 1.2 is equivalent to its counterpart for discrete polymatroids. J. Schweig [19] has proven the counterpart of the conjecture for certain discrete polymatroids.

While Conjecture 1.2 is the most well-known of the three parts of [20, Conjecture 12], the next conjecture is the strongest of the three. Consider the graph  $G'_M(S)$  in which  $k$ -tuples of bases replace multisets of bases. Thus, its vertices are all  $k$ -tuples of bases of  $M$  whose multiset union is  $S$ ; vertices  $\mathbf{A} = (A_1, A_2, \dots, A_k)$  and  $\mathbf{B} = (B_1, B_2, \dots, B_k)$  are adjacent if, for some distinct integers  $i$  and  $j$  in  $\{1, 2, \dots, k\}$  and some  $b_i \in B_i - B_j$  and  $b_j \in B_j - B_i$ , we obtain  $\mathbf{A}$  from  $\mathbf{B}$  by replacing  $B_i$  by  $(B_i - b_i) \cup b_j$  and replacing  $B_j$  by  $(B_j - b_j) \cup b_i$ .

**Conjecture 1.3.** *For any matroid  $M$  and multiset  $S$  of size  $kr(M)$  with elements in  $E(M)$  and with  $k \geq 2$ , the graph  $G'_M(S)$  is connected.*

In Theorem 2.5, we show that Conjecture 1.3 holds for a matroid  $M$  if Conjecture 1.2 holds for  $M$  and Conjecture 1.1 holds for all of its minors. Since Conjectures 1.1 and 1.2 have been proven for graphic and cographic matroids, we therefore get the (apparently new) result that Conjecture 1.3 also holds for such matroids. Likewise, since we prove that Conjectures 1.1 and 1.2 hold for sparse paving matroids, Conjecture 1.3 therefore also holds for these matroids. It also holds for matroids of rank or corank at most three (Corollary 2.6). As with Conjecture 1.2, Conjecture 1.3 holds for  $M$  if and only if it holds for  $M^*$ .

The fourth conjecture was made by Y. Kajitani, S. Ueno, and H. Miyano [10]. A matroid  $M$  is *cyclically orderable* if there is a cyclic permutation  $(a_1, a_2, \dots, a_n)$  of  $E(M)$  in which each set of  $r(M)$  cyclically-consecutive elements is a basis of  $M$ .

**Conjecture 1.4.** *A matroid  $M$  is cyclically orderable if and only if, for all nonempty subsets  $A$  of  $E(M)$ ,*

$$(1.2) \quad r(M) |A| \leq r(A) |E(M)|.$$

A counting argument shows that inequality (1.2) holds if  $M$  is cyclically orderable. J. van den Heuvel and S. Thomassé [8] proved Conjecture 1.4 when  $r(M)$  and  $|E(M)|$  are relatively prime.

The fifth conjecture was raised as a problem by H. Gabow [5] and stated as a conjecture by R. Cordovil and M. Moreira [2]. To match our work below, we state it in the case of disjoint bases; it is easy to show that this implies its counterpart for arbitrary bases.

**Conjecture 1.5.** *If  $B_1$  and  $B_2$  are disjoint bases of a rank- $r$  matroid  $M$ , then some cycle  $(b_1, b_2, \dots, b_r, b_{r+1}, \dots, b_{2r})$  has  $B_1 = \{b_1, b_2, \dots, b_r\}$  and  $B_2 = \{b_{r+1}, b_{r+2}, \dots, b_{2r}\}$ , and has each set of  $r$  cyclically-consecutive elements being a basis of  $M$ .*

It is not hard to show that if this conjecture holds for  $M$ , then it holds for  $M^*$  and for all minors of  $M$ . H. Gabow [5] noted that the conjecture holds for transversal matroids. It has also been proven for graphic matroids [2, 10]. A. de Mier [16] observed that this conjecture holds for strongly base-orderable matroids. Recall that  $M$  is *strongly base-orderable* if for each pair of bases  $B_1$  and  $B_2$  of  $M$ , there is a bijection  $\phi : B_1 \rightarrow B_2$  such that for every subset  $X \subseteq B_1$ , both  $(B_1 - X) \cup \phi(X)$  and  $(B_2 - \phi(X)) \cup X$  are bases. If  $M$  is strongly base-orderable, then listing the elements of  $B_1$  in any order followed by their images under  $\phi$ , in the corresponding order, gives the required cycle. The class of strongly base-orderable matroids is both minor-closed and dual-closed, and it strictly contains the class of all gammoids (which include transversal matroids).

As we noted above, Conjecture 1.3 is stronger than Conjecture 1.2. The only other known implications among these conjectures appear to be those mentioned above (namely, Theorem 2.5 and Corollary 2.6).

In this paper we prove Conjectures 1.1–1.5 in the special case of sparse paving matroids.

Our notation follows J. Oxley [17]. The symmetric difference,  $(X - Y) \cup (Y - X)$ , of two sets  $X$  and  $Y$  is denoted by  $X \triangle Y$ . We let  $[n]$  denote the set  $\{1, 2, \dots, n\}$ .

## 2. PROOFS OF CONJECTURES 1.1–1.5 IN THE CASE OF SPARSE PAVING MATROIDS

We will use the lemmas below. The first follows easily from the definition of sparse paving.

**Lemma 2.1.** *Let  $M$  be a sparse paving matroid of rank  $r$ . Let  $H$  and  $B$  be two  $r$ -subsets of  $E(M)$  with  $|H \triangle B| = 2$ . If  $H$  is a circuit-hyperplane of  $M$ , then  $B$  is a basis.*

Although we will not use it, we note that the following strengthening of Lemma 2.1 is easy to prove: a matroid  $M$  of rank  $r$  is sparse paving if and only if whenever  $H$  and  $B$  are two  $r$ -subsets of  $E(M)$  with  $|H \triangle B| = 2$  and  $H$  is not a basis, then  $B$  is a basis. (We remark that the analogous condition on discrete polymatroids winds up being too restrictive to be of interest.)

**Lemma 2.2.** *Let  $B$  and  $B'$  be distinct bases of a sparse paving matroid  $M$ . For  $a \in B - B'$  and  $X \subseteq B' - B$ , there are at least  $|X| - 2$  elements  $x \in X$  for which both  $(B - a) \cup x$  and  $(B' - x) \cup a$  are bases of  $M$ .*

*Proof.* The lemma follows since, by Lemma 2.1, at most one set  $(B - a) \cup x$  with  $x \in X$ , and at most one set  $(B' - x') \cup a$  with  $x' \in X$ , is a circuit-hyperplane.  $\square$

We now turn to Conjecture 1.1.

**Theorem 2.3.** *Conjecture 1.1 holds for sparse paving matroids.*

*Proof.* We first prove the result when  $E(M)$  is the disjoint union of two bases; we will then reduce the general case to this one. In this case, vertices have the form  $(B_1, B_2, \emptyset)$ , which we simplify to  $(B_1, B_2)$  in the next two paragraphs. We must show that for each pair  $(A_1, A_2)$  and  $(B_1, B_2)$  of vertices in  $G(M)$  with  $|A_1 \Delta B_1| \geq 4$ , there is a path between them. For this, it suffices to show that there is a path from  $(B_1, B_2)$  to a vertex  $(B'_1, B'_2)$  with  $|A_1 \Delta B'_1| < |A_1 \Delta B_1|$ .

If  $|B_1 - A_1| \geq 3$ , then fix  $x \in B_1 - A_1$  and set  $X = A_1 - B_1$ . We have  $|X| \geq 3$  and  $X \subseteq B_2$ , so, by Lemma 2.2, the pair  $((B_1 - x) \cup y, (B_2 - y) \cup x)$  is a vertex of  $G(M)$  for some  $y \in X$ . Also,  $|A_1 \Delta ((B_1 - x) \cup y)| < |A_1 \Delta B_1|$ , as needed.

In the remaining case,  $|B_1 - A_1| = 2$ , let  $B_1 - A_1 = \{b_1, b_2\}$  and  $A_1 - B_1 = \{a_1, a_2\}$ . Thus,  $a_1, a_2 \in B_2$ . If any of the following four symmetric exchanges yields only bases, it would provide the desired vertex  $(B'_1, B'_2)$  adjacent to  $(B_1, B_2)$ :

- (a)  $(B_1 - b_1) \cup a_1$  and  $(B_2 - a_1) \cup b_1$ ,
- (b)  $(B_1 - b_1) \cup a_2$  and  $(B_2 - a_2) \cup b_1$ ,
- (c)  $(B_1 - b_2) \cup a_1$  and  $(B_2 - a_1) \cup b_2$ ,
- (d)  $(B_1 - b_2) \cup a_2$  and  $(B_2 - a_2) \cup b_2$ .

Thus, we may assume that each pair contains a circuit-hyperplane. By symmetry, we may assume that  $(B_1 - b_1) \cup a_1$  is a circuit-hyperplane; then  $(B_1 - b_1) \cup a_2$  and  $(B_1 - b_2) \cup a_1$  are bases by Lemma 2.1, so  $(B_2 - a_2) \cup b_1$  and  $(B_2 - a_1) \cup b_2$  are circuit-hyperplanes; thus,  $(B_2 - a_2) \cup b_2$  is a basis by Lemma 2.1, so  $(B_1 - b_2) \cup a_2$  is a circuit-hyperplane. For all four sets just identified to be circuit-hyperplanes, we must have  $r(M) \geq 3$ , so there is an element  $x$  in  $B_1 \cap A_1$ . By comparison with the four known circuit-hyperplanes, it follows that each set in the following symmetric exchanges is a basis:

- (e)  $(B_1 - x) \cup a_1$  and  $(B_2 - a_1) \cup x$ ,
- (f)  $B'_1 = (B_1 - \{x, b_2\}) \cup \{a_1, a_2\}$  and  $B'_2 = (B_2 - \{a_1, a_2\}) \cup \{x, b_2\}$ .

Since  $(B'_1, B'_2)$  is adjacent to  $(A_1, A_2)$ , the needed path from  $(B_1, B_2)$  to  $(A_1, A_2)$  exists.

In the general case, for two vertices  $\mathbf{A} = (A_1, A_2, A_3)$  and  $(B_1, B_2, B_3)$  of  $G(M)$ , we will show that there is a path in  $G(M)$  from  $\mathbf{A}$  to a vertex of the form  $(C_1, C_2, B_3)$ ; the theorem then follows by applying the case just treated to the basis pair graph of  $M \setminus B_3$ . (Recall that the third set in these triples need not be a basis.)

Assume  $|A_3 \Delta B_3| \geq 4$ . By symmetry, we may assume  $|A_1 \cap B_3| \geq 1$ ; fix some  $a_1 \in A_1 \cap B_3$ . Since  $M$  is sparse paving, the hyperplane  $\text{cl}(A_1 - a_1)$  contains at most one element in  $A_3 - B_3$ , so  $A'_1 = (A_1 - a_1) \cup a_3$  is a basis for some  $a_3 \in A_3 - B_3$ . The vertex  $(A'_1, A_2, A'_3)$ , where  $A'_3 = (A_3 - a_3) \cup a_1$ , is adjacent to  $\mathbf{A}$  and has  $|A'_3 \Delta B_3| < |A_3 \Delta B_3|$ .

By iterating the argument above, it now suffices to treat the case  $|A_3 \Delta B_3| = 2$ . Let  $A_3 - B_3 = \{a_3\}$  and  $B_3 - A_3 = \{b_3\}$ . We may assume  $b_3 \in A_1$ . If  $(A_1 - b_3) \cup a_3$  is a basis of  $M$ , then the claim holds, so assume instead that this set is a circuit-hyperplane. By symmetrically exchanging any element  $a_1 \in A_1 - b_3$  with some element  $a_2 \in A_2$ , we get a vertex  $((A_1 - a_1) \cup a_2, (A_2 - a_2) \cup a_1, A_3)$  that is adjacent to  $\mathbf{A}$  and in which, by Lemma 2.1, we can exchange  $b_3$  in  $(A_1 - a_1) \cup a_2$  with  $a_3$  in  $A_3$ , which completes the proof of the claim and so of the theorem.  $\square$

We now turn to Conjecture 1.2.

**Theorem 2.4.** *Conjecture 1.2 holds for sparse paving matroids.*

*Proof.* Let  $M$  be a sparse paving matroid. We prove that  $G_M(S)$  is connected by induction on  $k$ , where  $|S| = k r(M)$ . The base case  $k = 1$  is trivial:  $G_M(S)$  is connected since it has at most one vertex. For  $k \geq 2$ , we claim that for any two vertices

$$\mathcal{A} = \{A_1, A_2, \dots, A_k\} \quad \text{and} \quad \mathcal{B} = \{B_1, B_2, \dots, B_k\}$$

of  $G_M(S)$ , there are (possibly trivial) paths from  $\mathcal{A}$  to some vertex  $\{A'_1, A'_2, \dots, A'_k\}$  and from  $\mathcal{B}$  to some vertex  $\{B'_1, B'_2, \dots, B'_k\}$  with  $A'_1 = B'_1$ . Proving this claim gives the result by induction since having a path from  $\mathcal{A}$  to  $\mathcal{B}$  in  $G_M(S)$  follows from having a path from  $\{A'_2, A'_3, \dots, A'_k\}$  to  $\{B'_2, B'_3, \dots, B'_k\}$  in  $G_M(S - A'_1)$ , where  $S - A'_1$  is the multiset difference. List the sets in  $\mathcal{A}$  and  $\mathcal{B}$  so that  $|A_1 \triangle B_1| \leq |A_h \triangle B_j|$  for all  $h, j \in [k]$ . Set  $|A_1 \triangle B_1| = 2i$ . To prove the claim, it suffices to show that if  $i > 0$ , then

(\*) there is a path from  $\mathcal{B}$  to a vertex  $\{B''_1, B''_2, \dots, B''_k\}$  with  $|A_1 \triangle B''_1| < 2i$ .

Set  $A_1 - B_1 = \{a_1, a_2, \dots, a_i\}$  and  $B_1 - A_1 = \{b_1, b_2, \dots, b_i\}$ . By symmetry, we may assume that the sum of the multiplicities of the elements in  $A_1 - B_1$  in  $S$  is at least as large as the corresponding sum for  $B_1 - A_1$ . It follows that some basis in  $\mathcal{B}$ , say  $B_2$ , has more elements from  $A_1 - B_1$  than from  $B_1 - A_1$ . We consider several options for  $B_2$ .

For the case  $i \geq 3$ , first assume  $B_2 \cap (B_1 - A_1) = \emptyset$ . We may assume  $a_1 \in B_2$ . Apply Lemma 2.2 with  $x = a_1$  and  $X = B_1 - A_1$  (so  $|X| \geq 3$ ): for some  $b_h \in B_1 - A_1$ , both  $(B_1 - b_h) \cup a_1$  and  $(B_2 - a_1) \cup b_h$  are bases, so statement (\*) follows.

Now, along with  $i \geq 3$ , assume  $|B_2 \cap (A_1 - B_1)| \geq 3$ . Let  $X = B_2 \cap (A_1 - B_1)$ . Since  $B_2$  has more elements from  $A_1 - B_1$  than from  $B_1 - A_1$ , some element in  $B_1 - A_1$ , say  $b_1$ , is not in  $B_2$ . Apply Lemma 2.2 to  $B_1$  and  $B_2$  with  $x = b_1$  and  $X$ : for some  $a_h \in X$ , both  $(B_1 - b_1) \cup a_h$  and  $(B_2 - a_h) \cup b_1$  are bases. Statement (\*) now follows.

We now address the case with  $B_2 \cap (A_1 \triangle B_1) = \{a_1, a_2, b_3\}$ , thereby completing the argument for  $i \geq 3$ . If we can symmetrically exchange one of  $a_1, a_2$  in  $B_2$  for one of  $b_1, b_2$  in  $B_1$  to get bases, then statement (\*) holds. Assume that none of these four symmetric exchanges yields only bases. An argument like that in the third paragraph of the proof of Theorem 2.3 shows that we may assume that

$$(B_1 - b_1) \cup a_1, \quad (B_2 - a_2) \cup b_1, \quad (B_2 - a_1) \cup b_2, \quad \text{and} \quad (B_1 - b_2) \cup a_2$$

are circuit-hyperplanes. In order to have  $|A_1 \triangle B_1| \leq |A_1 \triangle B_2|$  given that  $B_2 \cap (A_1 \triangle B_1)$  is  $\{a_1, a_2, b_3\}$ , there must be an element, say  $y$ , in  $B_2 - (A_1 \cup B_1)$ . From Lemma 2.1 and the circuit-hyperplanes above, we have that  $(B_1 - b_1) \cup y$  and  $(B_2 - y) \cup b_1$  are bases, as are  $(B_1 - \{b_1, b_2\}) \cup \{y, a_1\}$  and  $(B_2 - \{y, a_1\}) \cup \{b_1, b_2\}$ . Statement (\*) now follows, which completes the argument for  $i \geq 3$ .

Now assume  $i = 2$ . By symmetry, there are two cases:  $B_2 \cap \{b_1, b_2\}$  is either  $\emptyset$  or  $\{b_1\}$ . First assume  $B_2 \cap \{b_1, b_2\} = \emptyset$ . We may assume  $a_1 \in B_2$ . If  $a_1$  in  $B_2$  can be symmetrically exchanged with either  $b_1$  or  $b_2$  in  $B_1$  to yield two bases, then statement (\*) holds, so assume this fails. By symmetry,  $H_1 = (B_1 - b_1) \cup a_1$  and  $H_2 = (B_2 - a_1) \cup b_2$  can be assumed to be circuit-hyperplanes. Since  $|A_1 \triangle B_1| \leq |A_1 \triangle B_2|$ , there are at least two elements, say  $z_2$  and  $z_3$ , in  $B_2 - A_1$ . By Lemma 2.1, either  $(B_2 - z_2) \cup b_1$  or  $(B_2 - z_3) \cup b_1$  is a basis; assume the former is. Comparison with  $H_1$  shows that  $(B_1 - b_1) \cup z_2$  and  $(B_1 - \{b_1, b_2\}) \cup \{z_2, a_1\}$  are bases; similarly,  $(B_2 - \{z_2, a_1\}) \cup \{b_1, b_2\}$  is a basis by comparison with  $H_2$ . Statement (\*) now follows.

We now address the case with  $B_2 \cap \{b_1, b_2\} = \{b_1\}$ , thus completing the argument for  $i = 2$ . Note that  $B_2$  must also contain  $a_1$  and  $a_2$ . Statement (\*) holds if  $b_2$  in  $B_1$  can be symmetrically exchanged with either  $a_1$  or  $a_2$  in  $B_2$  to yield two bases. If neither exchange yields only bases, then, by symmetry, we may assume that  $H_1 = (B_1 - b_2) \cup a_1$  and  $H_2 = (B_2 - a_2) \cup b_2$  are circuit-hyperplanes. At least two elements in  $A_1 \cap B_1$ , say

$x_3$  and  $x_4$ , are not in  $B_2$  since  $|A_1 \triangle B_1| \leq |A_1 \triangle B_2|$ . At least one of  $(B_2 - a_1) \cup x_3$  and  $(B_2 - a_1) \cup x_4$  is a basis by Lemma 2.1; assume the first is. Now  $(B_1 - x_3) \cup a_1$  is a basis by comparison with  $H_1$ . The sets  $(B_1 - \{x_3, b_2\}) \cup \{a_1, a_2\}$  and  $(B_2 - \{a_1, a_2\}) \cup \{x_3, b_2\}$  are also bases by comparison with  $H_1$  and  $H_2$ , respectively. It follows that statement (\*) holds. This completes the argument for  $i = 2$ .

Finally, assume  $i = 1$ , so  $A_1 - B_1 = \{a_1\}$  and  $B_1 - A_1 = \{b_1\}$ . Thus,  $B_2$  contains  $a_1$  and not  $b_1$ . Let  $X = B_2 - a_1$ . If  $X \cup b_1$  is a basis (as it must be if  $k$  is 2), then exchanging  $a_1$  and  $b_1$  in  $B_2$  and  $B_1$  shows that statement (\*) holds. Thus, assume  $k \geq 3$  and

(A)  $X \cup b_1$  is a circuit-hyperplane.

If  $3 \leq h \leq k$  and  $b_1 \notin B_h$ , and if there is an element  $y \in X - B_h$ , then there is a  $z \in B_h - B_2$  for which both  $(B_h - z) \cup y$  and  $(B_2 - y) \cup z$  are bases; from Lemma 2.1 and statement (A), it follows that we can symmetrically exchange  $a_1$  in  $(B_2 - y) \cup z$  with  $b_1$  in  $B_1$  to get two bases, which yields statement (\*). Thus, we may assume

(B) each basis  $B_h$  contains either  $b_1$  or all of  $X$ .

If  $B_h \cap \{a_1, b_1\} = \{b_1\}$  for some  $h$  with  $3 \leq h \leq k$ , then the assumption about the multiplicities of  $a_1$  and  $b_1$  implies that  $B_{h'} \cap \{a_1, b_1\} = \{a_1\}$  for some  $h'$  with  $3 \leq h' \leq k$ . Symmetrically exchange  $a_1$  in  $B_{h'} - B_h$  for some  $z \in B_h - B_{h'}$  to get bases; since  $B_{h'} - a_1$  is  $X$  by statement (B), statement (A) gives  $z \neq b_1$ . Thus, we may assume

(C) for  $3 \leq h \leq k$ , if  $b_1 \in B_h$ , then  $a_1 \in B_h$ .

Assume  $3 \leq h \leq k$  and  $a_1, b_1 \in B_h$ . If  $|B_2 \triangle B_h| \geq 4$ , then for  $x \in (B_h - b_1) - B_2$ , we can symmetrically exchange  $x \in B_h$  with some  $y \in B_2$  (which cannot be  $a_1$ ) to yield two bases; with statement (A), this allows us to exchange  $b_1$  in  $B_1$  with  $a_1$  in  $(B_2 - y) \cup x$  to yield statement (\*). Thus, we may assume

(D) if  $a_1, b_1 \in B_h$ , then  $|B_2 \triangle B_h| = 2$ .

The proof is completed by showing that statements (A)–(D) yield a contradiction. Consider the multisets  $\mathcal{A} = \{\{a_1\}, A_2, A_3, \dots, A_k\}$  and  $\mathcal{B} = \{\{b_1\}, B_2, B_3, \dots, B_k\}$  of sets. Their multiset unions,  $\bigcup_{A \in \mathcal{A}} A$  and  $\bigcup_{B \in \mathcal{B}} B$ , are equal. Let  $b_1$  have multiplicity  $t + 1$  in these unions. Statements (B)–(D) imply that the sum of the multiplicities of the elements in  $X$  in the sets in  $\mathcal{B}$  is  $|X|(k - t - 1) + (|X| - 1)t$ , that is,  $|X|(k - 1) - t$ . By statement (A),  $X \cup b_1$  is not in  $\mathcal{A}$ , so the sum of the multiplicities of the elements in  $X$  in the sets in  $\mathcal{A}$  is at most  $|X|(k - t - 2) + (|X| - 1)(t + 1)$ , that is,  $|X|(k - 1) - t - 1$ , which, as desired, contradicts the equality  $\bigcup_{A \in \mathcal{A}} A = \bigcup_{B \in \mathcal{B}} B$ .  $\square$

We now prove a general connection between Conjectures 1.1, 1.2, and 1.3.

**Theorem 2.5.** *Let  $M$  be a matroid for which the basis pair graph of each of its minors is connected. For  $k \geq 2$ , let  $S$  be a multiset of size  $kr(M)$  with elements in  $E(M)$ . If  $G_M(S)$  is connected, then so is  $G'_M(S)$ .*

*Proof.* Since  $G_M(S)$  is connected, to show that  $G'_M(S)$  is connected it suffices to show that for each vertex  $\mathbf{A} = (A_1, A_2, \dots, A_k)$  of  $G'_M(S)$  and each permutation  $\sigma$  of  $[k]$ , there is a path in  $G'_M(S)$  from  $\mathbf{A}$  to  $\mathbf{A}_\sigma = (A_{\sigma(1)}, A_{\sigma(2)}, \dots, A_{\sigma(k)})$ . Since every permutation is a composition of transpositions, we focus on a transposition  $\sigma$ , say permuting  $i$  and  $j$  with  $i < j$ . The desired result follows if we show that there is a path from  $\mathbf{A}$  to  $\mathbf{A}_\sigma$  in which all bases but the  $i$ -th and  $j$ -th are fixed. This follows by noting that the sequence of symmetric exchanges that gives a path from  $(A_i - A_j, A_j - A_i, \emptyset)$  to  $(A_j - A_i, A_i - A_j, \emptyset)$  in the basis pair graph of the minor  $M|(A_i \cup A_j)/(A_i \cap A_j)$  also gives the desired path from  $\mathbf{A}$  to  $\mathbf{A}_\sigma$  in  $G'_M(S)$ .  $\square$

**Corollary 2.6.** *For any minor-closed class of matroids for which Conjecture 1.1 holds, Conjectures 1.2 and 1.3 are equivalent. In particular, Conjecture 1.3 holds for sparse paving, graphic, and cographic matroids, and for matroids of rank or corank at most three.*

*Proof.* The assertions about the first three classes of matroids follow from our results on sparse paving matroids and from the results on graphic and cographic matroids that we mentioned in the introduction. For the remaining two assertions, by duality it suffices to treat matroids of corank at most three and to show that Conjecture 1.1 holds for them. If  $M$  has corank at most three, then  $G(M)$  can be nonempty (and so potentially disconnected) only when  $r(M) \leq 3$ ; thus,  $|E(M)| \leq 6$ . It is routine to show that all matroids that satisfy these conditions are either graphic or transversal; since Conjecture 1.1 is known for those classes of matroids, the result follows.  $\square$

For Conjecture 1.4, we start with a definition and a lemma. A  $k$ -interval in a cycle  $\sigma$  is a set of  $k$  cyclically-consecutive elements, that is,  $\{x, \sigma(x), \sigma^2(x), \dots, \sigma^{k-1}(x)\}$  for some  $x$ .

**Lemma 2.7.** *Let  $M$  be a rank- $r$  sparse paving matroid on  $n$  elements. If  $2r \leq n$ , then, over all cycles on  $E(M)$ , the average number of  $r$ -intervals that are circuit-hyperplanes of  $M$  is less than two.*

*Proof.* Let  $b(M)$  and  $\text{ch}(M)$  be, respectively, the numbers of bases and circuit-hyperplanes of  $M$ . By focusing on circuit-hyperplanes, it follows that the average of interest is

$$\frac{\text{ch}(M) r! (n - r)!}{(n - 1)!}.$$

The desired result follows easily from this expression, the assumed inequality,  $2r \leq n$ , and the inequality

$$(2.1) \quad \text{ch}(M) \leq \frac{1}{n - r + 1} \binom{n}{r},$$

which is a consequence of Theorem 4.8 in [15]. (Alternatively, to get inequality (2.1), consider the pairs  $(H, B)$  consisting of a circuit-hyperplane  $H$  and basis  $B$  of  $M$  with  $|H \triangle B| = 2$ ; the inequality follows by noting that each circuit-hyperplane is in  $r(n - r)$  such pairs, each basis is in at most  $r$  such pairs, and  $b(M) + \text{ch}(M) = \binom{n}{r}$ .)  $\square$

**Theorem 2.8.** *Conjecture 1.4 holds for sparse paving matroids.*

*Proof.* As noted after Conjecture 1.4, inequality (1.2) holds in every cyclically orderable matroid. The conjecture is easy to verify for all sparse paving matroids that have rank or nullity at most two (this includes all disconnected sparse paving matroids, i.e.,  $U_{0,n}$ ,  $U_{n,n}$ ,  $U_{n-1,n} \oplus U_{1,1}$ ,  $U_{1,n} \oplus U_{0,1}$ , and  $U_{1,2} \oplus U_{1,2}$ ; this also includes all cases in which inequality (1.2) fails), so below we assume that  $M$  has rank and nullity at least three.

We may assume  $E(M) = [n]$ . For a cycle  $\sigma$  on  $E(M)$ , all  $r(M)$ -intervals in  $\sigma$  are bases of  $M$  if and only if their complements, all  $r(M^*)$ -intervals in  $\sigma$ , are bases of  $M^*$ , so, by replacing  $M$  by  $M^*$  if needed, we may assume that  $2r \leq n$  where  $r = r(M)$ . By Lemma 2.7, for some cycle, say  $\sigma_1 = (1, 2, \dots, n)$ , on  $E(M)$ , at most one of its  $r$ -intervals is a circuit-hyperplane. We may assume there is such an interval, say

$$H_1 = \{4, 5, \dots, r + 3\},$$

otherwise the desired conclusion holds.

Consider  $\sigma_2 = (1, 2, \underline{4}, \underline{3}, 5, \dots, n)$ . (To aid the reader, we underline the entries that differ from  $\sigma_1$ .) Only two of its  $r$ -intervals differ from their counterparts in  $\sigma_1$ , namely,  $\{3, 5, 6, \dots, r+3\}$ , which is a basis (use Lemma 2.1 with  $H_1$ ), and

$$H_2 = \{n-r+4, \dots, n, 1, 2, 4\}.$$

If  $H_2$  is a basis, then  $\sigma_2$  is the cycle we want. Thus, assume that  $H_2$  is a circuit-hyperplane.

We repeatedly apply this type of argument below. For brevity, for each cycle we list its  $r$ -intervals that differ from their counterparts in  $\sigma_1$  and, when possible, the circuit-hyperplanes that, with Lemma 2.1, show that these intervals are bases. For brevity, we omit the  $r$ -interval  $\{i, 5, 6, \dots, r+3\}$ , with  $i \neq 4$ , which is a basis (compare it to  $H_1$ ). Since the permutations  $\sigma_i$  below differ from  $\sigma_1$  in at most four consecutive places, the assumption that the nullity of  $M$  is at least three implies that an  $r$ -interval in  $\sigma_i$  cannot differ from its counterpart in  $\sigma_1$  at both ends.

Consider  $\sigma_3 = (1, \underline{3}, \underline{4}, 2, 5, \dots, n)$ . The relevant intervals are

- ◊  $\{4, 2, 5, 6, \dots, r+2\}$  (compare to  $H_1$ ),
- ◊  $\{n-r+4, \dots, n, 1, 3, 4\}$  (compare to  $H_2$ ), and

$$H_3 = \{n-r+3, \dots, n, 1, 3\}.$$

Thus,  $\sigma_3$  has the desired properties unless  $H_3$  is a circuit-hyperplane, so we assume it is.

Consider  $\sigma_4 = (1, \underline{4}, 3, \underline{2}, 5, \dots, n)$ . The relevant intervals are

- ◊  $\{n-r+4, \dots, n, 1, 4, 3\}$  and  $\{n-r+3, \dots, n, 1, 4\}$  (compare to  $H_3$ ), and

$$H_4 = \{3, 2, 5, 6, \dots, r+2\}.$$

Thus,  $\sigma_4$  has the desired properties unless  $H_4$  is a circuit-hyperplane, so we assume it is.

Consider  $\sigma_5 = (\underline{3}, \underline{4}, \underline{1}, 2, 5, \dots, n)$ . The relevant intervals are

- ◊  $\{1, 2, 5, 6, \dots, r+2\}$  (compare to  $H_4$ ),
- ◊  $\{n-r+4, \dots, n, 3, 4, 1\}$  (compare to  $H_2$ ),
- ◊  $\{n-r+3, \dots, n, 3, 4\}$  and  $\{n-r+2, \dots, n, 3\}$  (compare to  $H_3$ ), and

$$H_5 = \{4, 1, 2, 5, 6, \dots, r+1\}.$$

Thus,  $\sigma_5$  has the desired properties unless  $H_5$  is a circuit-hyperplane, so we assume it is.

Consider  $\sigma_6 = (\underline{4}, \underline{3}, \underline{1}, 2, 5, \dots, n)$ . The relevant intervals are

- ◊  $\{1, 2, 5, 6, \dots, r+2\}$  (compare to  $H_4$ ),
- ◊  $\{3, 1, 2, 5, 6, \dots, r+1\}$  (compare to  $H_5$ ),
- ◊  $\{n-r+4, \dots, n, 4, 3, 1\}$  (compare to  $H_2$ ),
- ◊  $\{n-r+3, \dots, n, 4, 3\}$  (compare to  $H_3$ ), and

$$H_6 = \{n-r+2, \dots, n, 4\}.$$

Thus,  $\sigma_6$  has the desired properties unless  $H_6$  is a circuit-hyperplane, so we assume it is.

Finally, consider  $\sigma = (2, \underline{3}, \underline{4}, \underline{1}, 5, \dots, n)$ . The relevant intervals are

- ◊  $\{4, 1, 5, 6, \dots, r+2\}$  (compare to  $H_1$ ),
- ◊  $\{3, 4, 1, 5, 6, \dots, r+1\}$  (compare to  $H_5$ ),
- ◊  $\{n-r+4, \dots, n, 2, 3, 4\}$  (compare to  $H_2$ ),
- ◊  $\{n-r+3, \dots, n, 2, 3\}$  (compare to  $H_3$ ), and
- ◊  $\{n-r+2, \dots, n, 2\}$  (compare to  $H_6$ ).

Thus,  $\sigma$  has the desired properties, which completes the proof. □

We now turn to Conjecture 1.5.

**Theorem 2.9.** *Conjecture 1.5 holds for sparse paving matroids.*

*Proof.* Consider disjoint bases  $B = \{b_1, b_2, \dots, b_r\}$  and  $C = \{c_1, c_2, \dots, c_r\}$  of a sparse paving matroid  $M$ . By the basis-exchange property, we may assume that in the cycle

$$\sigma = (b_1, b_2, \dots, b_r, c_1, c_2, \dots, c_r),$$

every  $r$ -interval of the form  $\{b_i, b_{i+1}, \dots, b_r, c_1, \dots, c_{i-1}\}$  is a basis; such cycles are said to *start properly*. We say that a *problem occurs at  $c_i$*  if  $\{c_i, c_{i+1}, \dots, c_r, b_1, \dots, b_{i-1}\}$  is not a basis; clearly,  $i > 1$ . We will show how, if a problem occurs at  $c_i$ , then we can switch a few elements so that the number of problems decreases and the cycle starts properly; iterating this procedure produces the desired cycle.

First assume  $1 < i < r$ . We will show that one of the following cycles starts properly and has fewer problems (we underline the few elements that are permuted):

$$\begin{aligned} \sigma_1 &= (b_1, b_2, \dots, b_r, c_1, c_2, \dots, \underline{c_i}, \underline{c_{i-1}}, \dots, c_r), \\ \sigma_2 &= (b_1, b_2, \dots, \underline{b_i}, \underline{b_{i-1}}, \dots, b_r, c_1, c_2, \dots, c_r), \\ \sigma_3 &= (b_1, b_2, \dots, b_r, c_1, c_2, \dots, \underline{c_{i+1}}, \underline{c_{i-1}}, \underline{c_i}, \dots, c_r). \end{aligned}$$

Since  $S_0 = \{c_i, c_{i+1}, \dots, c_r, b_1, \dots, b_{i-1}\}$  is a circuit-hyperplane, Lemma 2.1 implies that  $\{c_{i-1}, c_{i+1}, \dots, c_r, b_1, \dots, b_{i-1}\}$  is a basis. Only one other  $r$ -interval in  $\sigma_1$  differs from its counterpart in  $\sigma$ , namely,  $S_1 = \{b_i, \dots, b_r, c_1, \dots, c_{i-2}, c_i\}$ , so it follows that  $\sigma_1$  starts properly and has fewer problems than  $\sigma$  unless  $S_1$  is a circuit-hyperplane. Assume  $S_1$  is a circuit-hyperplane. Only two  $r$ -intervals in  $\sigma_2$  differ from their counterparts in  $\sigma$ ; of these, the set  $\{c_i, c_{i+1}, \dots, c_r, b_1, \dots, b_{i-2}, b_i\}$  is a basis by Lemma 2.1 (compare it to  $S_0$ ); if its complement,  $S_2 = \{b_{i-1}, b_{i+1}, \dots, b_r, c_1, \dots, c_{i-1}\}$ , is a basis, then  $\sigma_2$  starts properly and has fewer problems than  $\sigma$ , so we may assume that  $S_2$  is also a circuit-hyperplane. Four  $r$ -intervals in  $\sigma_3$  differ from their counterparts in  $\sigma$ , namely,

$$T_1 = \{c_{i-1}, c_i, c_{i+2}, \dots, c_r, b_1, \dots, b_{i-1}\}, \quad T_2 = \{c_i, c_{i+2}, \dots, c_r, b_1, \dots, b_i\},$$

and their complements. Each of these sets is a basis by Lemma 2.1 since each symmetric difference  $T_1 \triangle S_0$ ,  $T_2 \triangle S_0$ ,  $(E(M) - T_1) \triangle S_1$ , and  $(E(M) - T_2) \triangle S_2$  has two elements, so  $\sigma_3$  starts properly and has fewer problems than  $\sigma$ .

Now assume  $i = r$ , so  $S_0 = \{c_r, b_1, \dots, b_{r-1}\}$  is a circuit-hyperplane. Consider

$$\begin{aligned} \sigma_1 &= (b_1, b_2, \dots, b_r, c_1, c_2, \dots, \underline{c_r}, \underline{c_{r-1}}), \\ \sigma_2 &= (b_1, b_2, \dots, \underline{b_r}, \underline{b_{r-1}}, c_1, c_2, \dots, c_r), \\ \sigma_3 &= (b_1, b_2, \dots, b_r, c_1, c_2, \dots, \underline{c_{r-1}}, \underline{c_r}, \underline{c_{r-2}}). \end{aligned}$$

An argument similar to that above shows that  $\sigma_1$  starts properly and has fewer problems than  $\sigma$  unless  $S_1 = \{b_r, c_1, c_2, \dots, c_{r-2}, c_r\}$  is a circuit-hyperplane; likewise,  $\sigma_2$  starts properly and has fewer problems than  $\sigma$  unless  $S_2 = \{b_{r-1}, c_1, c_2, \dots, c_{r-1}\}$  is a circuit-hyperplane. Assume both  $S_1$  and  $S_2$  are circuit-hyperplanes. Only four  $r$ -intervals in  $\sigma_3$  differ from their counterparts in  $\sigma$ , namely:

$$T_1 = \{c_r, c_{r-2}, b_1, \dots, b_{r-2}\}, \quad T_2 = \{c_{r-2}, b_1, \dots, b_{r-1}\},$$

and their complements. These sets are bases since  $T_1 \triangle S_0$ ,  $T_2 \triangle S_0$ ,  $(E(M) - T_1) \triangle S_2$ , and  $(E(M) - T_2) \triangle S_1$  each have two elements, so  $\sigma_3$  is the desired cycle on  $B \cup C$ .  $\square$

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