LATTICE PATH MATROIDS: THE EXCLUDED MINORS

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ABSTRACT. A lattice path matroid is a transversal matroid for which some antichain of intervals in some linear order on the ground set is a presentation. We characterize the minor-closed class of lattice path matroids by its excluded minors.

1. INTRODUCTION

Among transversal matroids, it is natural to consider those for which some presentations have special structure. We consider transversal matroids for which at least one presentation consists of intervals in some linear order on the ground set and no interval contains another; this gives the class $\mathcal{L}$ of lattice path matroids. Unlike the class of all transversal matroids, if a matroid is in $\mathcal{L}$, then so are its minors. A major theme in matroid theory is characterizing minor-closed classes of matroids by their excluded minors, that is, by the minor-minimal matroids that are not in the class; we give such a characterization of $\mathcal{L}$. After a section of background, this result and its proof occupy the rest of the paper.

We briefly sketch this research area. Nested matroids, the minor-closed subclass of $\mathcal{L}$ whose members have presentations that are chains of intervals in linear orders, have been introduced many times and under many names (see [1]), apparently first by H. Crapo [5]. As N. White noted in his review of [1] in Mathematical Reviews, R. Stanley mentioned lattice path matroids (without this name) in [14]; no results were given. Independently, J. Lawrence [9] introduced and studied oriented counterparts of these matroids. Lattice path matroids were independently introduced and studied in depth in [1, 2]; the lattice path perspective used there accounts for the name. They have been studied further by J. Schweig [13] and applied to a problem in enumeration by A. de Mier and M. Noy [7]. A larger minor-closed class of transversal matroids was defined and studied in [3]. In [6], A. de Mier used lattice paths in higher dimensions to define a related type of flag matroid. Following a suggestion by V. Reiner [12] that there should be a type-B counterpart of the Catalan matroid (a certain nested matroid), J. Bonin and A. de Mier defined a class of Lagrangian matroids based on lattice paths; this topic has been studied by A. Gundert, E. Kim, and D. Schymura [8].

2. BACKGROUND

We assume that readers have a general knowledge of matroid theory (see [10, 15] for excellent accounts), including basic results about transversal matroids (see [4, 10, 15]). The results we use to prove the excluded-minor characterization of $\mathcal{L}$ are collected below.

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2.1. **Connected flats.** A flat F of a matroid M is *connected* if the restriction $M|F$ is connected. No connected flat of rank two or more can be trivial, that is, independent. The following lemma, akin to [10, Exercise 2.1.13], is easy to prove.

Lemma 2.1. *A loopless matroid is determined by (a) its ground set and (b) its collection of nontrivial connected flats and their ranks.*

Lemma 2.2. *If F is a flat of M and $x \in E(M) - F$, then $M/x|F = M|F$; thus, if F is connected and $\mathrm{cl}_M(x) = \{x\}$, then $\mathrm{cl}_{M/x}(F)$ is a connected flat of M/x.*

The connected flats in the definition below play major roles in this paper. (All references to incomparability in this paper are with respect to containment.)

Definition 2.3. *The proper nontrivial connected flats of a matroid are its* pnc-flats*. A pnc-flat is* reducible *if it is the intersection of some pair of incomparable pnc-flats. A* fundamental flat *is a pnc-flat F such that, for some spanning circuit C of the matroid, $F \cap C$ is a basis of F.*

We illustrate these definitions with the matroid in Figure 1, part (iii). Its fundamental flats are $\{1,2\}$, $\{1,2,3,4,5\}$, and $\{4,5,6,7\}$; for each of them, the required spanning circuit is $\{1,3,6,7\}$; all three are irreducible. The only other pnc-flat, $\{4,5\}$, is reducible.

2.2. **Lattice path matroids.** Many of the results in this subsection are from [1]; others are extensions or refinements that are tailored to the work in this paper.

Although Definition 2.4 best suits the work in this paper, we start with the point of view used in [1, 2] since it can provide valuable insight. The lattice paths we consider are finite sequences of steps of unit length, each going either north, N, or east, E. Given two lattice paths P and Q from $(0,0)$ to a point (m,r) with P never going above Q, let $\mathcal{P}$ be the set of paths from $(0,0)$ to (m,r) that remain in the region that P and Q bound. (For the diagram in Figure 1, part (i), the bounding paths are $P = EENEENN$ and $Q = NENEENE$.) Label each north step in the diagram with the position it has in the paths that contain it. Let J_i be the set of labels on the north steps in the i-th row of the diagram (indexed from the bottom up). (See Figure 1, part (ii).) Observe that for each path in $\mathcal{P}$, the set of positions that its north steps occupy is a transversal of the set system $\mathcal{A} = (J_1, J_2, \ldots, J_r)$. This map from $\mathcal{P}$ to the set of bases of the transversal matroid on $\{1,2,\ldots,m+r\}$ arising from the set system $\mathcal{A}$ is clearly injective. In this setting, Proposition 2.5 says that this map is surjective. The following definition replaces $\{1,2,\ldots,m+r\}$ with any linearly ordered set and replaces $\mathcal{A}$ by any collection of incomparable intervals in this linear order.

Definition 2.4. *A* lattice path matroid *is a transversal matroid that has a presentation by an antichain of intervals in some linear order on the ground set.*

We elaborate on this definition and thereby establish the notation that we use below. A lattice path matroid M of rank r has a presentation $\mathcal{A} = (J_1, J_2, \ldots, J_r)$ where, relative to some linear order $e_1 < e_2 < \cdots < e_n$ on $E(M)$, the set J_i is an interval $[a_i, b_i]$ (thus, $a_i \leq b_i$) and we have $a_1 < a_2 < \cdots < a_r$ and $b_1 < b_2 < \cdots < b_r$. Such a linear order is a *path order* of M, the elements e_1 and e_n are *terminal elements* of M, and we call $\mathcal{A}$ an *interval presentation* of M. Let $\mathcal{L}$ be the class of lattice path matroids.

A matroid in $\mathcal{L}$ may have many path orders and many terminal elements. For example, for the lattice path matroid in Figure 1, in addition to the natural order on the ground set, the order $2 < 1 < 3 < 5 < 4 < 7 < 6$ is a path order; in this order, the intervals $\{2,1,3\}$, $\{3,5,4,7\}$, $\{7,6\}$ make up the interval presentation. Thus, 1, 2, 6, and 7 are terminal elements (by Corollary 2.14 below, these are the only terminal elements). As a

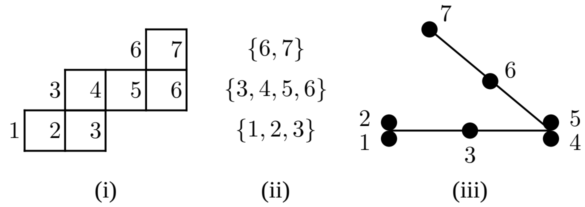

FIGURE 1. (i) A region bounded by two lattice paths. The label on each north step reflects the position of that step in all paths it is in. (ii) The set system that arises from the labels in the rows. (iii) The transversal matroid defined by this set system.

general class of examples, we note that all uniform matroids are in $\mathcal{L}$ and all linear orders on their ground sets are path orders, so all of their elements are terminal. Note that if $e_1 < e_2 < \cdots < e_n$ is a path order of $M \in \mathcal{L}$, then so is $e_1 > e_2 > \cdots > e_n$.

From the next result (which is essentially [2, Theorem 3.3]), it easily follows that the bases of any lattice path matroid can be identified with the paths in a certain region of the plane; this connects the two perspectives on these matroids.

Proposition 2.5. *For $M \in \mathcal{L}$, fix a path order and interval presentation of M. If the elements of a basis of M are $x_1 < x_2 < \cdots < x_r$, then $x_i \in J_i$ for $1 \leq i \leq r$.*

The class $\mathcal{L}$ is easily seen to be closed under direct sums. By the following result [1, Theorem 3.5], connectivity can be determined readily from any interval presentation.

Proposition 2.6. *Given an interval presentation $\mathcal{A}$ of $M \in \mathcal{L}$ as above, M is connected if and only if $a_1 = e_1$, $b_r = e_n$, and $a_{i+1} \leq b_i$ for all i with $1 \leq i < r$.*

The next result [1, Theorem 5.3] identifies the fundamental flats of connected lattice path matroids.

Proposition 2.7. *Let $M \in \mathcal{L}$ be connected. Fix a path order and interval presentation as given after Definition 2.4. The fundamental flats of M are of the following two types.*

(i) *The interval $[e_1, e_h]$ is a fundamental flat if and only if $e_{h+1} = a_{j+1}$ for some j with $1 \leq j < r$ and $\big|[a_j, a_{j+1}]\big| > 2$. In this case, $[e_1, e_h]$ has rank j.*

(ii) *The interval $[e_i, e_n]$ is a fundamental flat if and only if $e_{i-1} = b_k$ for some k with $1 \leq k < r$ and $\big|[b_k, b_{k+1}]\big| > 2$. In this case, $[e_i, e_n]$ has rank $r - k$.*

Since the definition of a fundamental flat does not use a linear order on the ground set, applying this result to each connected component of a lattice path matroid yields the following corollary [1, Theorem 5.6].

Corollary 2.8. *For each path order of $M \in \mathcal{L}$, there is only one interval presentation.*

The following result (which joins [2, Theorem 3.5] and [1, Corollary 5.5]) is transparent from the lattice path perspective.

Proposition 2.9. *If $M \in \mathcal{L}$, then $M^* \in \mathcal{L}$. If M is also connected, then the fundamental flats of M^* are the set complements of the fundamental flats of M.*

The following result [1, Theorem 5.10] plays a key role in our work; it characterizes connected lattice path matroids. We use η for the nullity function: $\eta(X) = |X| - r(X)$.

Proposition 2.10. *A connected matroid M is in $\mathcal{L}$ if and only if the properties below hold.*

(i) *The fundamental flats of M form at most two disjoint chains under inclusion, say $F_1 \subset F_2 \subset \cdots \subset F_h$ and $G_1 \subset G_2 \subset \cdots \subset G_k$.*
(ii) *If $F_i \cap G_j \neq \emptyset$, then $F_i \cup G_j = E(M)$.*
(iii) *The pnc-flats of M other than $F_1, F_2, \dots, F_h, G_1, G_2, \dots, G_k$ are the intersections $F_i \cap G_j$ where $\eta(M) < \eta(F_i) + \eta(G_j)$.*
(iv) *If $F_i \cap G_j$ is a pnc-flat, then $r(F_i \cap G_j) = r(F_i) + r(G_j) - r(M)$.*

Note that property (ii) precludes any inclusion among any fundamental flats F_i and G_j.

Corollary 2.11. *The fundamental flats of a connected matroid in $\mathcal{L}$ are its irreducible pnc-flats.*

Corollary 2.12. *Let F and G be pnc-flats of $M \in \mathcal{L}$ that are not disjoint. If $F \cup G$ spans M, then $F \cup G = E(M)$.*

The following corollary of Lemma 2.1 and Proposition 2.10 is [1, Corollary 5.8].

Corollary 2.13. *The automorphisms of a connected matroid in $\mathcal{L}$ are the permutations of the ground set that induce rank-preserving permutations of the set of fundamental flats.*

With this result, we can identify the terminal elements as follows.

Corollary 2.14. *Assume $M \in \mathcal{L}$ is connected. If M has just one chain of fundamental flats, say $F_1 \subset \cdots \subset F_h$, then its set of terminal elements is $F_1 \cup \big(E(M) - F_h\big)$. If M has two chains of fundamental flats, say $F_1 \subset \cdots \subset F_h$ and $G_1 \subset \cdots \subset G_k$, then its set of terminal elements is $(F_1 - G_k) \cup (G_1 - F_h)$.*

We now turn to the minors of lattice path matroids. The class $\mathcal{L}$ is closed under minors [1, Theorem 3.1]. Furthermore, given a path order of $M \in \mathcal{L}$ and a minor N of M, the induced order on $E(N)$ is a path order. Thus, if $x \in E(N)$ is a terminal element of M, then x is a terminal element of N.

We next give the interval presentation of a single-element contraction. The last assertion in the next result plays several key roles in our work. We first note that if a presentation $\mathcal{A}$ is as given after Definition 2.4, then the sets in $\mathcal{A}$ that contain a particular non-loop y are successive intervals $J_s, J_{s+1}, \dots, J_t$ for some s and t.

Proposition 2.15. *Fix a path order of $M \in \mathcal{L}$; let the corresponding interval presentation be $\mathcal{A}$. Assume $y \in E(M)$ is not a loop and let $J_s, J_{s+1}, \dots, J_t$ be the sets of $\mathcal{A}$ that contain y. The interval presentation of M/y for the induced path order is*

$$\mathcal{A}' = \begin{cases} (J_1, J_2, \dots, J_{s-1}, J_{s+1}, \dots, J_r), & \text{if } s = t, \\ (J_1, J_2, \dots, J_{s-1}, J'_s, J'_{s+1}, \dots, J'_{t-1}, J_{t+1}, \dots J_r), & \text{if } s < t, \end{cases}$$

where $J'_i = (J_i \cup J_{i+1}) - y$. Also, if $x \in E(M) - (J_s \cap J_{s+1} \cap \cdots \cap J_t)$, then x is in the same number of sets in $\mathcal{A}'$ as in $\mathcal{A}$.

Proof. The bases of M/y are the sets $B \subseteq E(M) - y$ for which $B \cup y$ is a basis of M, so we are claiming that $B \cup y$ is a transversal of $\mathcal{A}$ if and only if B is a transversal of $\mathcal{A}'$. The case $s = t$ is immediate, so assume $s < t$. Let $B = \{x_1, x_2, \dots, x_{r-1}\}$ with $x_1 < x_2 < \cdots < x_k < y < x_{k+1} < \cdots < x_{r-1}$. By Proposition 2.5, if $B \cup y$ is a transversal of $\mathcal{A}$, then (a) $x_i \in J_i$ for $1 \leq i \leq k$, (b) $y \in J_{k+1}$, and (c) $x_i \in J_{i+1}$ for $k+1 \leq i \leq r-1$. Thus, $x_i \in J'_i$ for $s \leq i < t$, so B is a transversal of $\mathcal{A}'$. The converse follows with a similar argument upon noting that if $x_i < y$ and $x_i \in J'_h$, then $x_i \in J_h$ (note that $[a_{h+1}, y] \subset [a_h, y]$); likewise, if $y < x_i$ and $x_i \in J'_h$, then $x_i \in J_{h+1}$; thus, y can represent J_{k+1}. The last assertion is immediate. □

The following two corollaries of Propositions 2.6, 2.7, and 2.15 guarantee that certain single-element contractions are connected.

Corollary 2.16. *Assume $M \in \mathcal{L}$ is connected. Let e be a terminal element of M. Let the fundamental flats of M that contain e be $F_1 \subset F_2 \subset \cdots \subset F_h$. If $r(F_1) > 1$, then M/e is connected and $F_1 - e, F_2 - e, \dots, F_h - e$ are fundamental flats of M/e.*

Corollary 2.17. *Assume $M \in \mathcal{L}$ is connected and, in a given path order, x is neither the first nor the last element. The contraction M/x is connected if and only if x is in at least two sets in the interval presentation. In particular, in the notation introduced above, if $1 < i \leq r$, then M/a_i is connected; also, if $1 \leq j < r$, then M/b_j is connected.*

Given a presentation $\mathcal{A}$ of a transversal matroid M, we obtain a presentation of $M\backslash e$ from $\mathcal{A}$ by removing e from all sets. For $M \in \mathcal{L}$, some adjustment may be needed so that the presentation of $M\backslash e$ is an antichain; for our work, it suffices to treat these adjustments when e is not a loop and e is either the least element, e_1, or the greatest element, e_n. If $J_1 = \{e_1\}$, then $(J_2, J_3, \dots, J_r)$ is the interval presentation of $M\backslash e_1$ for the induced path order. If $\{e_1\} \subset J_1$, then $(J_1 - e_1, J_2 - e_2, \dots, J_r - e_r)$ is the interval presentation of $M\backslash e_1$ since, by Proposition 2.5, e_i, the $(i-1)$-st element of $E(M\backslash e_1)$, is not needed in the i-th set; note that, in this case, if $e_i \in J_i$ with $1 < i \leq r$, then e_i is the lower endpoint of $J_{i-1} - e_{i-1}$. The interval presentation of $M\backslash e_n$ is obtained similarly.

Lemma 2.18. *For $M \in \mathcal{L}$, let x be in an interval I in a given path order of M. Let $\mathcal{A}$ be the corresponding interval presentation of M and let $\mathcal{A}'$ be the induced interval presentation of the restriction $M|I$. Either x is an upper or lower endpoint of some interval in $\mathcal{A}'$ or x is in the same number of intervals in $\mathcal{A}'$ as in $\mathcal{A}$.*

We turn to spanning circuits. Although the following result is part of [1, Theorem 3.6], we give the proof since it is relevant for the remarks below.

Proposition 2.19. *Let $M \in \mathcal{L}$ be connected and nontrivial. Fix an interval presentation $\mathcal{A}$ of M as above. If x is in at least two sets in $\mathcal{A}$ or $x \in \{a_1, b_r\}$, then x is in a spanning circuit of M.*

Proof. The case of $x \in \{a_1, b_r\}$ follows from the proof we give when x is in two sets, say $x \in J_i \cap J_{i+1}$. Since M is connected, $a_h \in J_{h-1} \cap J_h$ if $h > 1$; also, $b_k \in J_k \cap J_{k+1}$ if $k < r$. Therefore each r-subset of $C = \{a_1, a_2, \dots, a_i, x, b_{i+1}, \dots, b_r\}$ is a transversal of $\mathcal{A}$ and hence a basis of M, so C is a spanning circuit. □

Note that $M\backslash x$ is connected if some spanning circuit of M does not contain x. This applies if $x \in F_1 - \{a_1, a_2, \dots, a_r\}$ where F_1 is the smallest fundamental flat that contains the least element e_1. If all pairs of incomparable fundamental flats of M are disjoint, then, by Corollary 2.13, the automorphism group of M is transitive on F_1. These observations give the following result.

Corollary 2.20. *Assume $M \in \mathcal{L}$ is connected and has at least one fundamental flat. If all pairs of incomparable fundamental flats of M are disjoint, then for any element x in a smallest fundamental flat of M, the deletion $M\backslash x$ is connected.*

Proposition 2.19 will sometimes be used with the following lemma about any matroid.

Lemma 2.21. *If M is connected and y is in the spanning circuit C of M, then $C - y$ is a spanning circuit of M/y. Thus, if, in addition,* $\mathrm{cl}(y) = \{y\}$, *then M/y is connected.*

Lemma 2.18 and Proposition 2.19 have the following corollary.

Corollary 2.22. *For $M \in \mathcal{L}$, let x be in an interval I in a given path order of M and let $\mathcal{A}$ be the corresponding interval presentation of M. If $M|I$ is connected and x is either a terminal element of M or in at least two intervals in $\mathcal{A}$, then x is in a spanning circuit of $M|I$.*

This corollary applies, for instance, if I is a pnc-flat since, by Propositions 2.7 and 2.10, such flats are intervals in any path order.

We close this subsection by briefly mentioning a special type of lattice path matroid. Matroids that have interval presentations $\mathcal{A}$ as above where either $\{a_1, a_2, \dots, a_r\}$ or $\{b_1, b_2, \dots, b_r\}$ is an interval in the path order are *nested matroids* (called generalized Catalan matroids in [1]). Let $\mathcal{C}$ be the class of these matroids. A connected matroid in $\mathcal{L}$ is nested if and only if its fundamental flats form a chain. The following related result is essentially Lemma 2 of [11].

Proposition 2.23. *A loopless matroid is in $\mathcal{C}$ if and only if its pnc-flats form a chain.*

Let P_n be $T_n(U_{n-1,n} \oplus U_{n-1,n})$, the truncation to rank n of the direct sum of two n-circuits. Thus, P_n is the rank-n paving matroid whose only pnc-flats are two disjoint circuit-hyperplanes whose union is the ground set. The following result is from [11].

Proposition 2.24. *A matroid is in $\mathcal{C}$ if and only if it has no P_n-minor for any $n \geq 2$.*

2.3. **Parallel connections.** For our purposes, the next result [10, Proposition 7.1.13] can be taken as the definition of the *parallel connection* $P_x(M_1, M_2)$ of matroids M_1 and M_2 using basepoint x. The special case $P_x(M, U_{1,2})$ is the *parallel extension* of M at x.

Proposition 2.25. *Assume that M_1 and M_2 are matroids with $E(M_1) \cap E(M_2) = \{x\}$ and $r_{M_1}(x) + r_{M_2}(x) > 0$.*

(1) *A set $B \subseteq E(M_1) \cup E(M_2)$ with $x \in B$ is a basis of $P_x(M_1, M_2)$ if and only if $B \cap E(M_i)$ is a basis of M_i for both $i \in \{1, 2\}$.*
(2) *A set $B \subseteq E(M_1) \cup E(M_2)$ with $x \notin B$ is a basis of $P_x(M_1, M_2)$ if and only if, for some distinct i and j in $\{1, 2\}$, the set $B \cap E(M_i)$ is a basis of M_i and $\big(B \cap E(M_j)\big) \cup x$ is a basis of M_j.*

It follows that if M_1 and M_2 are simple, then their ground sets are flats of $P_x(M_1, M_2)$.

We will use the result below [10, Proposition 7.1.15] on minors of parallel connections.

Proposition 2.26. *For $y \in E(M_1) - x$, we have $P_x(M_1, M_2)\backslash y = P_x(M_1\backslash y, M_2)$ and $P_x(M_1, M_2)/y = P_x(M_1/y, M_2)$. Also, $P_x(M_1, M_2)/x = (M_1/x) \oplus (M_2/x)$.*

Clearly $P_x(M_1, M_2) = P_x(M_2, M_1)$, so the analogous results hold for $y \in E(M_2) - x$.

The next result [10, Theorem 7.1.16] gives an important link between connectivity and parallel connection.

Proposition 2.27. *Let M be a connected matroid with $x \in E(M)$. If $M/x = M_1 \oplus M_2$, then $M = P_x\big(M\backslash E(M_2), M\backslash E(M_1)\big)$; furthermore, both $M\backslash E(M_2)$ and $M\backslash E(M_1)$ are connected.*

3. THE EXCLUDED MINORS OF LATTICE PATH MATROIDS

Theorem 3.1 gives the excluded minor characterization of $\mathcal{L}$. With four exceptions, the excluded minors fall into five infinite families, one of which is self-dual. The first members of these infinite families, along with the four exceptions, are shown in Figure 2.

In Theorem 3.1 and its proof, we use the following notation. The free extension and the free coextension of M by e are denoted $M + e$ and $M \times e$, respectively. Besides the

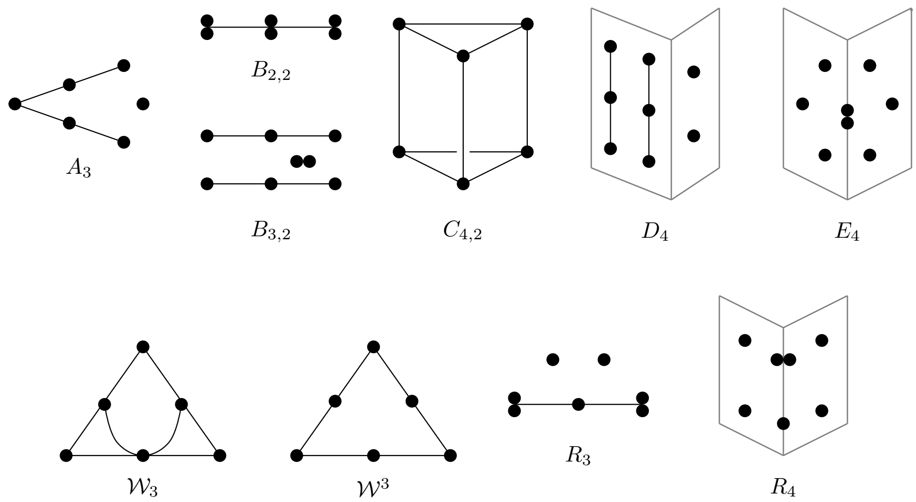

FIGURE 2. Some excluded minors of lattice path matroids. Those in the top row are in infinite families of excluded minors.

matroids P_n in Proposition 2.24, a family of matroids that plays an important role in this work is $P'_n = P_{n-1} \times e$ for $n \geq 3$. Equivalently, $P'_n = T_n\big(P_e(U_{n-1,n}, U_{n-1,n})\big)$, the truncation to rank n of the parallel connection of two n-circuits.

Theorem 3.1. *A matroid is a lattice path matroid if and only if it has none of the following matroids as minors:*

1. $A_n = P'_n + x$, *for* $n \geq 3$,
2. $B_{n,k} = T_n(U_{n-1,n} \oplus U_{n-1,n} \oplus U_{k-1,k})$ *and its dual* $C_{n+k,k}$, *for* $n \geq k \geq 2$,
3. $D_n = (P_{n-1} \oplus U_{1,1}) + x$ *and its dual* E_n, *for* $n \geq 4$,
4. *the rank-*3 *wheel,* $\mathcal{W}_3$*, the rank-*3 *whirl,* $\mathcal{W}^3$*, and*
5. *the matroid* R_3 *and its dual* R_4 *(see Figure 2).*

Note that A_n is self-dual. The matroid $C_{n+k,k}$ is a paving matroid of rank $n + k$; its ground set can be partitioned into sets X, Y, Z with $|X| = |Y| = n$ and $|Z| = k$ so that the only nontrivial hyperplanes are $X \cup Y$, $X \cup Z$, and $Y \cup Z$, two (or all, if $n = k$) of which are circuits. In E_n, the element x is in a 2-circuit and $E_n \backslash x = P'_n$

With Proposition 2.10, it is not hard to show that the matroids in items (1)–(5) of the theorem are excluded minors for $\mathcal{L}$; this was addressed in [1]. Our contribution is the proof that the list of excluded minors is complete; this occupies the rest of this paper.

Let $\mathcal{E}_\mathcal{L}$ be the set of excluded minors of $\mathcal{L}$ and let $\mathcal{E}$ be the set of those in items (1)–(5) of the theorem. Our goal is to prove $\mathcal{E}_\mathcal{L} - \mathcal{E} = \emptyset$. We attain this goal through a sequence of lemmas in which we deduce properties that any matroid in $\mathcal{E}_\mathcal{L} - \mathcal{E}$ must have. The argument culminates in the proof that any such matroid satisfies the conditions in Proposition 2.10 and so is in $\mathcal{L}$; this contradiction gives the desired conclusion, that $\mathcal{E}_\mathcal{L} = \mathcal{E}$.

Some properties of matroids in $\mathcal{E}_\mathcal{L}$ are transparent and will be used freely. Specifically, since $\mathcal{L}$ is closed under direct sums, all matroids in $\mathcal{E}_\mathcal{L}$ are connected. Also, since $\mathcal{L}$ is closed under duality, $M \in \mathcal{E}_\mathcal{L}$ if and only if $M^* \in \mathcal{E}_\mathcal{L}$.

To help readers keep the big picture in mind while examining the details, we start with an outline of the proof. (The labels below corresponds to the subsections into which we divide the proof.)

(3.1) We first treat parallel connections of lattice path matroids. For $M_1, M_2 \in \mathcal{L}$ with $E(M_1) \cap E(M_2) = \{x\}$, we determine whether $P_x(M_1, M_2)$ is in $\mathcal{L}$; we show that unless some simple sufficient conditions are met, $P_x(M_1, M_2)$ has one of $B_{n,2}$, $C_{4,2}$, E_n, R_3, or R_4 as a minor.

(3.2) We then turn to connectivity properties that any matroid $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$ has. With the results in Subsections 2.3 and 3.1, routine arguments show that for any $x \in E(M)$, the contraction M/x is connected. By duality, $M\backslash x$ is also connected; duality, applied to the results in Subsection 3.1, is how $C_{n+2,2}$ and D_n enter the argument. We also prove a lemma that requires using results that are particular to the lattice path setting: for all $x, y \in E(M)$, the minor $M\backslash x/y$ is connected. The matroids A_3, $\mathcal{W}_3$, and $\mathcal{W}^3$ arise in the proof of this lemma.

(3.3) The theme of the third subsection is fundamental flats. We show that for any matroid $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$, the fundamental flats of M are the set complements of those of M^*; also, the fundamental flats of M are its irreducible pnc-flats. The proofs of these results use their counterparts for lattice path matroids (Proposition 2.9 and Corollary 2.11), the connectivity results, and several lemmas that relate the fundamental flats and pnc-flats of M to such flats in its single-element deletions and contractions. These results play roles in the proof of the main result of the subsection: $\mathcal{W}_3$, $\mathcal{W}^3$, $B_{n,k}$, and $C_{n+k,k}$ are the only excluded minors for $\mathcal{L}$ that have three or more mutually incomparable fundamental flats.

(3.4) With the result just mentioned, it is not hard to show that the fundamental flats of any matroid $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$ form at most two chains under inclusion. This is a major part of our goal, which we achieve in the final subsection: using the earlier lemmas, we show that the conditions in Proposition 2.10 hold for M, thus giving the contradiction that shows $\mathcal{E}_{\mathcal{L}} = \mathcal{E}$. The matroid A_n appears in the argument.

3.1. **Parallel connections of lattice path matroids.** We start by giving simple conditions that guarantee that certain parallel connections of matroids in $\mathcal{L}$ are also in $\mathcal{L}$.

Lemma 3.2. *Let $M_1, M_2 \in \mathcal{L}$ be nontrivial and connected matroids of positive rank with $E(M_1) \cap E(M_2) = \{x\}$.*

(i) *If x is a terminal element of both M_1 and M_2, then $P_x(M_1, M_2) \in \mathcal{L}$.*
(ii) *If M_1 is a parallel connection using basepoint x, then $P_x(M_1, U_{1,2}) \in \mathcal{L}$.*

Proof. To prove assertion (i), fix path orders

$$e_1 < e_2 < \cdots < e_m < x \qquad \text{and} \qquad x < f_1 < f_2 < \cdots < f_n$$

of M_1 and M_2, respectively, with the corresponding interval presentations

$$\mathcal{A}_1 = (J_1, J_2, \ldots, J_{r_1}) \qquad \text{and} \qquad \mathcal{A}_2 = (J'_1, J'_2, \ldots, J'_{r_2}).$$

Thus, x is the upper endpoint of J_{r_1} and the lower endpoint of J'_1. It follows from Proposition 2.25 that $(J_1, \ldots, J_{r_1-1}, J_{r_1} \cup J'_1, J'_2, \ldots, J'_{r_2})$ is the interval presentation of $P_x(M_1, M_2)$ for the path order $e_1 < \cdots < e_m < x < f_1 < \cdots < f_n$.

Assertion (i) gives assertion (ii) if x is terminal in M_1, so assume x is not terminal. By Proposition 2.26, M_1/x is disconnected, so, by Corollary 2.17, x is in just one set in an interval presentation $\mathcal{A}$ of M_1. To obtain an interval presentation of the parallel extension of M_1 by y, insert y immediately after x in the path order of M_1 and adjoin y to the only interval in $\mathcal{A}$ that contains x. □

The next lemma is the converse of the previous lemma.

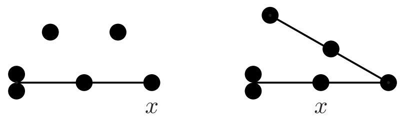

FIGURE 3. Two minors that arise in the proof of Lemma 3.4.

Lemma 3.3. *Let M be connected. Assume $M = P_x(M|S_1, M|S_2)$ where S_1 and S_2 are proper subsets of $E(M)$ with $S_1 \cap S_2 = \{x\}$. If $M \in \mathcal{L}$, then either x is in a 2-circuit of M or x is a terminal element of both $M|S_1$ and $M|S_2$.*

Proof. Assume x is not in a 2-circuit of M. Fix a path order $e_1 < e_2 < \cdots < e_n$ of M. Since $\mathrm{cl}(x) = \{x\}$, both S_1 and S_2 are pnc-flats of M. Since $S_1 \cup S_2 = E(M)$ and $S_1 \cap S_2 = \{x\}$, the description of pnc-flats given in Propositions 2.7 and 2.10 implies that S_1 and S_2 are, in some order, $[e_1, x]$ and $[x, e_n]$, so x is terminal in $M|S_1$ and $M|S_2$. $\square$

The following lemma is the main result of this subsection; it shows that known excluded minors arise from parallel connections of pairs of matroids in $\mathcal{L}$ that do not satisfy the conditions in Lemma 3.2.

Lemma 3.4. *Assume $E(M_1) \cap E(M_2) = \{x\}$ for nontrivial connected matroids M_1, M_2 in $\mathcal{L}$ of positive rank. Assume x is nonterminal in M_1; if M_1/x is disconnected, then also assume $r(M_2) > 1$. At least one of $B_{n,2}$, $C_{4,2}$, E_n, R_3, R_4 is a minor of $P_x(M_1, M_2)$.*

Proof. If $\{x, y\}$ is a circuit of M_1, then since x is nonterminal in M_1, it is nonterminal in $M_1 \backslash y$. Thus, it suffices to prove the result when $\mathrm{cl}_{M_1}(x) = \{x\}$.

Assume M_1/x is disconnected. By Proposition 2.27, M_1 is the parallel connection, at x, of two connected matroids, each of rank at least two since $\mathrm{cl}_{M_1}(x) = \{x\}$, so, by Proposition 2.26, M_1 has a P'_3-minor with x in both 3-circuits. Now $r(M_2) > 1$, so $P_x(M_1, M_2)$ has, as a minor, the parallel connection of three 3-circuits with the basepoint x; deleting x from this minor yields $C_{4,2}$.

Now assume M_1/x is connected. Fix a path order of M_1. Since M_1 has a nonterminal element, it is not uniform. Thus, M_1 has a chain $F_1 \subset \cdots \subset F_h$ of fundamental flats; also, if $M_1 \notin \mathcal{C}$, then it has a second such chain, say $G_1 \subset \cdots \subset G_k$. By Corollary 2.14 and the observation that if $M_1 \notin \mathcal{C}$, then its two chains of fundamental flats play similar roles, we may assume one of the following options holds:

(i) $x \in F_i - F_{i-1}$ for some i with $1 < i \leq h$,
(ii) $x \in F_1 \cap G_1$, or
(iii) $x \notin F_h \cup G_k$.

Assume $x \in F_i - F_{i-1}$. Among all minors of M_1 that meet the following conditions, let N be one for which $|E(N)|$ is minimal:

(a) $x \in E(N)$,
(b) N and N/x are connected, and
(c) for at least one of the chains $F'_1 \subset \cdots \subset F'_t$ of fundamental flats of N, we have $x \in F'_s - F'_{s-1}$ for some s with $1 < s \leq t$.

We claim that N is one of the two matroids in Figure 3. To see this, first note that, by Corollary 2.17, since N/x is connected, x is in at least two intervals in the induced interval presentation $\mathcal{A}'$ of N. We may assume F'_1 contains the least element, a, of $E(N)$. If

$r(F_1') > 1$, then, by Corollary 2.16, N/a would contradict the minimality of $|E(N)|$ (note that since a is terminal in N/x, Corollary 2.16 implies that $N/a, x$ would be connected); thus, $r(F_1') = 1$. If either $|F_1'| > 2$ or $s > 2$, then $N\backslash a$ would contradict the minimality of $|E(N)|$, so F_1' is a 2-circuit and $s = 2$. We claim $r(F_2') = 2$. Assume, to the contrary, $r(F_2') > 2$. Consider the intervals $J_2 = [a_2, b_2]$ and $J_3 = [a_3, b_3]$ of $\mathcal{A}'$. If $a_2 \neq x$, then, since N is connected, either $J_1 \cap \{a_2, x\} = \{a_2\}$ or x is in at least three intervals; thus, by Proposition 2.15, x is in at least two intervals in the presentation of N/a_2, which is connected by Corollary 2.17; these conclusions contradict the minimality of $|E(N)|$, so $a_2 = x$. Thus $x \notin J_3$, so N/a_3 is connected and has x in at least two presentation intervals, which contradicts the minimality of $|E(N)|$. Thus, F_2' is a line. The minimality of $|E(N)|$ also gives $|F_2'| = 4$. If $r(N) > 3$, then N/a_k, where $J_k = [a_k, b_k]$ is the last interval in $\mathcal{A}'$, would contradict the minimality of $|E(N)|$, so $r(N) = 3$. Similar arguments show that $E(N) - F_2'$ is an independent set of size two. Since N/x is connected, $(E(N) - F_2') \cup x$ is not a line. Thus, N is either N_1 or N_2 of Figure 3. If $N = N_1$, then, by Lemma 2.26, any parallel connection with M_1 at x has an R_3-minor; if $N = N_2$, then any such parallel connection has a $B_{2,2}$-minor.

Now assume $x \in F_1 \cap G_1$. Among all minors of M_1 that meet the following conditions, let N be one for which $|E(N)|$ is minimal:

(a) $x \in E(N)$,
(b) N and N/x are connected,
(c) not all fundamental flats of N are comparable, and
(d) x is in all fundamental flats of N.

We claim that N is either P_n', for some $n \geq 4$, or the simplification of R_4. Let F (resp., G) be the smallest fundamental flat that contains the least (resp., greatest) element of $E(N)$. Property (d) implies that x is in all pnc-flats, so, by property (b), N has no 2-circuits. If $r(F) \leq r(N) - 2$, then N/b, where b is the greatest element of $E(N)$, would contradict the minimality of $|E(N)|$, so F (and likewise G) is a hyperplane of N. If $r(F) = r(G) = 2$, then N would be the parallel connection, at x, of two lines, so N/x would be disconnected; this contradiction implies $r(N) \geq 4$. Since F and G are the only fundamental flats of N, by Proposition 2.10, the only possible pnc-flat of N besides F and G is $F \cap G$. Thus, $N|F$ (and likewise $N|G$) is a nested matroid since its pnc-flats form a chain. By Corollary 2.17 and Proposition 2.19, there is a spanning circuit C of N with $x \in C$. If $F \cap G$ were a pnc-flat, then $F \cap G \not\subseteq C$ and the nested matroid $N|F$ and $N|G$ would not be uniform; it follows that $N\backslash y$, for any $y \in (F \cap G) - C$, would contradict the minimality of $|E(N)|$. Thus, F and G are the only pnc-flats of N. Since $N|F$ and $N|G$ are uniform, by the minimality of $|E(N)|$, both F and G are circuits. Assume first $|F \cap G| = r(N) - 2$, so $|F - G| = 2 = |G - F|$. If $r(N) > 4$, then N/y, for any $y \in (F \cap G) - x$, would contradict the minimality of $|E(N)|$. Thus, $r(N) = 4$ and N is the simplification of R_4, with x in both 4-circuits; therefore any parallel connection using M_1 with x as the basepoint has an R_4-minor. (To prepare for the next paragraph, note that the dual of this minor N is a line with four points, two of which are 2-circuits, and x is not in a 2-circuit.) Now assume $|F \cap G| < r(N) - 2$, so $|F - G| \geq 3$ and $|G - F| \geq 3$. The minimality of $|E(N)|$ forces $F \cap G = \{x\}$, so $N = P_n'$ for some $n \geq 4$, with x being common to the two nonspanning circuits. In this case, any parallel connection using M_1 with x as the basepoint has an E_n-minor for some $n \geq 4$.

Finally, assume $x \notin F_h \cup G_k$. Using Proposition 2.9, it follows that x is in all fundamental flats of M_1^*. Therefore, by the results in the last paragraph, M_1 has, as a minor, either (a) a 4-point line with two 2-circuits, neither of which contains x or (b) the dual of P'_n for some $n \geq 4$, with x in neither circuit-hyperplane. It follows that any parallel connection using M_1 with x as the basepoint has, in the first case, a $B_{2,2}$-minor and, in the second case, a $B_{n,2}$-minor with $n \geq 3$. $\square$

3.2. **Connectivity.** Recall that $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$ if and only if M is an excluded minor of $\mathcal{L}$ that is not in items (1)–(5) of Theorem 3.1.

Lemma 3.5. *For $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$, both $M\backslash x$ and M/x are connected for all $x \in E(M)$.*

Proof. We argue by contradiction. Assume M/x is disconnected. By Proposition 2.27, there are subsets $S_1, S_2 \subset E(M)$ with $M = P_x(M|S_1, M|S_2)$ where $M|S_1$ and $M|S_2$ are connected. By Lemma 3.2, since $M|S_1, M|S_2 \in \mathcal{L}$ yet $M \notin \mathcal{L}$, we may assume x is not terminal in $M|S_1$; also, if $M|S_1/x$ is disconnected, then we may assume $r(S_2) > 1$. From Lemma 3.4, some minor of M is in $\mathcal{E}$, contrary to $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$. Thus, M/x is connected. That $M\backslash x$ is connected follows since $M\backslash x = (M^*/x)^*$ and $M^* \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$. $\square$

Corollary 3.6. *Matroids in $\mathcal{E}_{\mathcal{L}} - \mathcal{E}$ have no 2-circuits and no 2-cocircuits.*

In contrast to the proof of Lemma 3.5, the proof of the next connectivity result requires more particular information about lattice path matroids.

Lemma 3.7. *If $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$, then $M\backslash x/y$ is connected for all $x, y \in E(M)$.*

Proof. Assume $M\backslash x/y$ is disconnected; we will get the contradiction that M is A_3, $\mathcal{W}_3$, or $\mathcal{W}^3$. Since $M\backslash x$ is connected, $M\backslash x = P_y(M|S_1, M|S_2)$ for some proper subsets S_1, S_2 of $E(M) - x$ where $M|S_1$ and $M|S_2$ are connected. Lemma 3.3 and Corollary 3.6 imply that y is terminal in $M|S_1$ and $M|S_2$. Now $M/y\backslash x = (M|S_1/y) \oplus (M|S_2/y)$ yet M/y is connected, so $x \notin \mathrm{cl}_M(S_1) \cup \mathrm{cl}_M(S_2)$. Since $M|S_1$ and $M|S_2$ are connected and in $\mathcal{L}$, and since y is terminal in both, some spanning circuits C_1 of $M|S_1$ and C_2 of $M|S_2$ contain y. Now $|C_1| \geq 3$ and $|C_2| \geq 3$ by Corollary 3.6. Since $C_1 \cup C_2$ spans M but $x \notin \mathrm{cl}_M(C_1) \cup \mathrm{cl}_M(C_2)$, if there were an element w in $E(M) - (C_1 \cup C_2 \cup x)$, then applying Corollary 2.12 to $M\backslash w$ with the pnc-flats $\mathrm{cl}_{M\backslash w}(C_1)$ and $\mathrm{cl}_{M\backslash w}(C_2)$, and noting that x in neither set, gives the contradiction $M\backslash w \notin \mathcal{L}$. Thus, $E(M) = C_1 \cup C_2 \cup x$, so M is a single-element extension of $P_y(M|C_1, M|C_2)$. Therefore, since M is simple and $x \notin \mathrm{cl}_M(C_1) \cup \mathrm{cl}_M(C_2)$, both C_1 and C_2 are flats of M. Semimodularity applied to C_1 and $\mathrm{cl}_M(C_2 \cup x)$ gives $r\big(C_1 \cap \mathrm{cl}_M(C_2 \cup x)\big) \leq 2$. If $|C_1| > 3$, then M/z, for z in $C_1 - \mathrm{cl}_M(C_2 \cup x)$, would be an extension, by x, of $P_y(M|C_1/z, M|C_2)$; both $C_1 - z$ and C_2 would be pnc-flats of M/z yet $x \notin (C_1 - z) \cup C_2$, which would contradict Corollary 2.12 since $M/z \in \mathcal{L}$. Thus, $M\backslash x = P'_3$. It follows that, as claimed, M is A_3, $\mathcal{W}_3$, or $\mathcal{W}^3$. $\square$

3.3. **Fundamental flats.** Certain properties are not hard to show from the definition of a fundamental flat; others are not hard to show from the definition of an irreducible pnc-flat. Thus, we gain much by proving the following counterpart of Corollary 2.11: the fundamental flats of any matroid in $\mathcal{E}_{\mathcal{L}} - \mathcal{E}$ are its irreducible pnc-flats. This is Lemma 3.12. The following four lemmas, which include some of the properties of interest, enter into the proof of Lemma 3.12.

Lemma 3.8. *For a connected matroid M and connected deletion $M\backslash x$, if F is a fundamental flat of $M\backslash x$, then $\mathrm{cl}_M(F)$, which is F or $F \cup x$, is a fundamental flat of M.*

Proof. The spanning circuit C of $M\backslash x$ that shows that F is a fundamental flat of $M\backslash x$ also shows that $\mathrm{cl}_M(F)$ is a fundamental flat of M. $\square$

Lemma 3.9. *Assume x is not a loop of M. If F is a pnc-flat of $M\backslash x$, then exactly one of F and $F \cup x$ is a pnc-flat of M. The same conclusion holds if F is a pnc-flat of M/x.*

Proof. The first assertion is evident since $\mathrm{cl}_M(F)$ is either F or $F \cup x$ and x is not a loop. For the second assertion, note that $F \cup x$ is a flat of M since F is a flat of M/x. If $\mathrm{cl}_M(F) = F$, then x is an isthmus of $M|F \cup x$; thus, $M|F = M/x|F$, so F is a pnc-flat of M. Assume $\mathrm{cl}_M(F) = F \cup x$. Now x is not a component of $M|F \cup x$ but $M|(F \cup x)/x = M/x|F$, which we assumed is connected. Thus, $M|F \cup x$ is connected, so $F \cup x$ is a pnc-flat. □

Lemma 3.10. *For $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$, if F is a pnc-flat of M and $y \in F$, then $F - y$ is a pnc-flat of M/y. Furthermore, $F - y$ is reducible in M/y if and only if F is reducible in M.*

Proof. For the first part, we need to show that $M|F/y$ is connected. Fix $x \in E(M) - F$. Take a path order of $M\backslash x$ and the corresponding interval presentation $\mathcal{A}$. By Lemma 3.7, $M\backslash x/y$ is connected, so, by Corollary 2.17, y is either a terminal element or in at least two sets in $\mathcal{A}$. Thus, y is in a spanning circuit of $M|F$ by Corollary 2.22, so $M|F/y$ is connected.

For the second assertion, first assume F is reducible in M, so $F = G \cap H$ for some incomparable pnc-flats G and H of M. As just shown, $G - y$ and $H - y$ are pnc-flats of M/y, so their intersection, $F - y$, is reducible in M/y. Now assume $F - y$ is reducible in M/y, so $F - y = G \cap H$ for some incomparable pnc-flats G and H of M/y. Since $y \in \mathrm{cl}_M(F - y)$, by Lemma 3.9 both $G \cup y$ and $H \cup y$ are pnc-flats of M, so their intersection, F, is reducible. □

The same argument proves the next lemma.

Lemma 3.11. *Fix $y \in E(M)$ where $M \in \mathcal{L}$ and both M and M/y are connected. If F is a reducible pnc-flat of M with $y \in F$, then $F - y$ is a reducible pnc-flat of M/y.*

Lemma 3.12. *A pnc-flat F of $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$ is fundamental if and only if it is irreducible.*

Proof. Assume F is fundamental in M. Thus, M has a spanning circuit C so that $F \cap C$ is a basis of F. Fix $y \in F \cap C$. By Lemma 3.10, $F - y$ is a pnc-flat of M/y. Now $C - y$ is a spanning circuit of M/y and $(C - y) \cap (F - y)$ is a basis of $F - y$ in M/y, so $F - y$ is a fundamental flat of M/y. Since $M/y \in \mathcal{L}$, it follows that $F - y$ is irreducible in M/y. Therefore, by Lemma 3.10, F is irreducible in M.

Now assume F is irreducible in M. Fix $y \in F$. The irreducible pnc-flat $F - y$ of M/y is fundamental by Corollary 2.11. By Proposition 2.7, we may assume the first element, e_1, in a given path order of M/y is in $F - y$. Fix $x \notin F$. Since the pnc-flat $F - y$ of $M/y\backslash x$ contains e_1, it is fundamental in $M/y\backslash x$ by Proposition 2.10. Thus, $F - y$ is irreducible in $M\backslash x/y$, so by Lemma 3.11, the pnc-flat F of $M\backslash x$ is irreducible and so fundamental. Thus, by Lemma 3.8, F is fundamental in M. □

The next result is a corollary of Lemmas 3.10 and 3.12.

Corollary 3.13. *For $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$, if F is a fundamental flat of M, then, for all $y \in F$, the set $F - y$ is a fundamental flat of M/y.*

The following counterpart of the second assertion in Proposition 2.9 allows us to deduce dual versions of some of the results above.

Lemma 3.14. *For $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$, a proper nonempty subset F of $E(M)$ is a fundamental flat of M if and only if $E(M) - F$ is a fundamental flat of M^*.*

Proof. Let F be a fundamental flat of M. Since F is a pnc-flat, it is a union of circuits of M. Thus, $E(M) - F$ is an intersection of hyperplanes of M^*, so it is a flat of M^*.

Fix $y \in F$. By Corollary 3.13, $F - y$ is a fundamental flat of M/y. Since $M/y \in \mathcal{L}$, using Proposition 2.9, $E(M) - F$ is a fundamental flat of $(M/y)^*$, that is, $M^*\backslash y$. By Lemma 3.8, $\mathrm{cl}_{M^*}\big(E(M) - F\big)$, which is $E(M) - F$, is a fundamental flat of M^*. The other implication follows by duality. □

The next result follows from Proposition 2.9 and Lemmas 3.8 and 3.14.

Corollary 3.15. *For $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$, if F is a fundamental flat of M/x, then exactly one of F and $F \cup x$ is a fundamental flat of M.*

Via Proposition 2.9 and Lemma 3.14, we get the following dual of Corollary 3.13.

Corollary 3.16. *For $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$, if F is a fundamental flat of M, then, for all $z \notin F$, the set F is a fundamental flat of $M\backslash z$.*

Corollaries 3.13 and 3.16 yield a near-counterpart of property (ii) of Proposition 2.10.

Lemma 3.17. *For $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$, if F and G are incomparable fundamental flats of M with $F \cap G \neq \emptyset$, then $|E(M) - (F \cup G)| \leq 1$; also, if $|F \cap G| \geq 2$, then $E(M) = F \cup G$.*

Proof. The inequality holds since if $y, z \in E(M) - (F \cup G)$, then $M\backslash z$ and its fundamental flats F and G would contradict property (ii) of Proposition 2.10. Similarly, property (ii) applied to M/x, for $x \in F \cap G$, gives the second assertion. □

The next lemma follows easily from the perspective of irreducibility.

Lemma 3.18. *Let F be a fundamental flat of $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$. If C is a spanning circuit of $M|F$ and $u \in F - C$, then $F - u$ is a fundamental flat of $M\backslash u$.*

We now treat the main result of this subsection.

Lemma 3.19. *No three fundamental flats of $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$ are mutually incomparable.*

Proof. To the contrary, assume F_1, F_2, F_3 are mutually incomparable fundamental flats of M. We will derive the contradiction $M \in \mathcal{E}$.

If F_1, F_2, and F_3 are mutually disjoint, we could work instead with M^*, in which, by Lemma 3.14, the complements of these sets are (non-disjoint) fundamental flats. Thus, we may assume $F_1 \cap F_2 \neq \emptyset$. By Lemma 3.17, $|E(M) - (F_1 \cup F_2)| \leq 1$, so $F_3 \cap F_1 \neq \emptyset$ and $F_3 \cap F_2 \neq \emptyset$. Corollary 3.16 gives $F_1 \cup F_2 \cup F_3 = E(M)$, for otherwise deleting an element not in $F_1 \cup F_2 \cup F_3$ would give a matroid in $\mathcal{L}$ with three incomparable fundamental flats, which is impossible. Similarly, $F_1 \cap F_2 \cap F_3 = \emptyset$ by Corollary 3.13.

Assume $|F_1 \cap F_2| = 1$. Lemma 3.17 gives $|F_1 - (F_2 \cup F_3)| \leq 1$, but the connected flat F_1 is not the union of the flat $F_1 \cap F_3$ and the singleton $F_1 \cap F_2$, so we get $|F_1 - (F_2 \cup F_3)| = 1$. Similarly, $|F_2 - (F_1 \cup F_3)| = 1$. We get $|F_1 \cap F_3| = |F_2 \cap F_3| = 1$ from these conclusions and the second part of Lemma 3.17, so F_1, F_2, and F_3 are 3-circuits. It follows that M is either $\mathcal{W}_3$ or $\mathcal{W}^3$, contrary to $M \notin \mathcal{E}$.

Assume $|F_i \cap F_j| \geq 2$ whenever $\{i, j, k\} = \{1, 2, 3\}$, so $E(M) = F_i \cup F_j$ and

$$F_i = (F_i \cap F_j) \cup (F_i \cap F_k) = E(M) - (F_j \cap F_k).$$

We claim that none of F_1, F_2, F_3 is properly contained in a fundamental flat, so none of them is properly contained in any pnc-flat. To see this, assume, for instance, $F_1 \subseteq F_1'$ where F_1' is a fundamental flat. Since $F_1 \cup F_i = E(M)$ for $i \in \{2, 3\}$, any inclusion between F_1' and either F_2 or F_3 would give the contradiction that the larger of the two comparable sets is $E(M)$. Thus, F_1', F_2, F_3 are mutually incomparable, so the arguments above apply to F_1', F_2, F_3; however, this gives $F_1' = E(M) - (F_2 \cap F_3) = F_1$.

We claim that F_1 is a hyperplane. To see this, fix $x \in F_2 \cap F_3$. Both $F_2 - x$ and $F_3 - x$ are fundamental flats of M/x by Corollary 3.13. If F_1 were not a hyperplane, then $\mathrm{cl}_{M/x}(F_1)$ would be a pnc-flat of M/x; furthermore, $\mathrm{cl}_{M/x}(F_1)$ is not properly contained in any pnc-flat of M/x, so it would be a fundamental flat of M/x. However, $M/x \in \mathcal{L}$ cannot have three incomparable fundamental flats, so we may assume $F_2 - x \subseteq \mathrm{cl}_{M/x}(F_1)$. Since $E(M) = F_1 \cup F_2$, we get $\mathrm{cl}_M(F_1 \cup x) = E(M)$, so F_1 actually is a hyperplane of M. By symmetry, F_2 and F_3 are also hyperplanes.

We claim that F_1, F_2, F_3 are the only pnc-flats of M. We first consider fundamental flats since each pnc-flat is contained in some fundamental flat. If such exists, consider a fundamental flat $F \notin \{F_1, F_2, F_3\}$. If F were incomparable to two of F_1, F_2, F_3, say to F_2 and F_3, then applying the arguments above to the triple F, F_2, F_3 would give the contradiction $F = E(M) - (F_2 \cap F_3) = F_1$. Thus, any fundamental flat (and so any pnc-flat) of M other than F_1, F_2, F_3 is a subset of two of these, so assume $F \subset F_1 \cap F_2$. If such exists, let $F' \notin \{F_1, F_2, F_3\}$ be a fundamental flat of M with F' incomparable to F. If $F' \not\subseteq F_1 \cap F_2$, then $F \cap F' = \emptyset$; if $F' \subset F_1 \cap F_2$, then again $F \cap F' = \emptyset$ since F and F' must be fundamental flats (more easily seen here as irreducible pnc-flats) of $M|F_1$ yet $F \cup F' \neq F_1$. Therefore, by Corollary 2.20 applied to $M|F_1$ and to $M|F_2$, for any x in a smallest fundamental flat in $F_1 \cap F_2$, both $M|F_1\backslash x$ and $M|F_2\backslash x$ are connected; thus, $M\backslash x$ would have three incomparable fundamental flats ($F_1 - x$, $F_2 - x$, and F_3), which is impossible since $M\backslash x \in \mathcal{L}$. Thus, F_1, F_2, F_3 are the only fundamental flats of M. To see that they are the only pnc-flats of M, note that if, say, $F_1 \cap F_2$ were connected, then a similar application of Corollary 2.20 shows that for any $x \in F_1 \cap F_2$, both $M|F_1\backslash x$ and $M|F_2\backslash x$ would be connected, leading to the same contradiction.

Thus, $M|F_1$, $M|F_2$, and $M|F_3$ are uniform matroids. Note that at least two of F_1, F_2, F_3 are circuits; indeed, if, say, F_1 and F_2 were not circuits, then, for any $x \in F_1 \cap F_2$, the sets $F_1 - x$, $F_2 - x$, and F_3 would be fundamental flats of $M\backslash x$, which is impossible. It follows that $M = C_{n+k,k}$ where n and k are, respectively, the largest and smallest of $|F_1 \cap F_2|$, $|F_1 \cap F_3|$, $|F_2 \cap F_3|$, contrary to $M \notin \mathcal{E}$. □

3.4. **The last step.**

Lemma 3.20. *All excluded minors of $\mathcal{L}$ are in $\mathcal{E}$.*

Proof. Assume, to the contrary, $M \in \mathcal{E}_{\mathcal{L}} - \mathcal{E}$. We will derive the contradiction $M \in \mathcal{L}$ by showing that M satisfies properties (i)–(iv) in Proposition 2.10.

Not all fundamental flats of M are comparable, for otherwise M would have no other pnc-flats and Proposition 2.23 would give the contradiction $M \in \mathcal{C}$. Fix a fundamental flat F of M. By Lemma 3.19, the fundamental flats of M that are incomparable to F form a chain, say $G_1 \subset G_2 \subset \cdots \subset G_k$. Considering M/y with $y \in F$ shows that the fundamental flats that contain F form a chain; considering $M\backslash x$ with $x \notin F$ shows that those that are contained in F form a chain; together, these give the chain of fundamental flats that are comparable to F, say $F_1 \subset F_2 \subset \cdots \subset F_h$. Thus, property (i) of Proposition 2.10 holds. Note that no F_i is comparable to any G_j, for otherwise the same argument starting with F_i would have the incomparable fundamental flats F and G_j in the chain of those that are comparable to F_i.

To prove property (ii), by Lemma 3.17 it suffices to show that having $F_i \cap G_j = \{x\}$ and $E(M) - (F_i \cup G_j) = \{y\}$ yields a contradiction. Since $F_i - x$ is connected in M/x, by Corollary 2.17 and Proposition 2.19 some spanning circuit C of F_i contains x. If $u \in F_i - C$, then, using Lemma 3.18, $M\backslash u$ with the fundamental flats $F_i - u$ and G_j would contradict property (ii). It follows that F_i, and likewise G_j, is a circuit. We claim that both are also hyperplanes. If F_i were not a hyperplane, then $\mathrm{cl}_M(F_i \cup y) \neq E(M)$, so there would be a $z \in G_j - \mathrm{cl}_M(F_i \cup y)$. Thus, $y \notin \mathrm{cl}_M(F_i \cup z)$. Thus, in M/z, the pnc-flats $G_j - z$ and $\mathrm{cl}_{M/z}(F_i)$ would be incomparable and not disjoint, yet their union would contain all elements except y, contrary to Corollary 2.12. Since no pnc-flat is comparable to either F_i or G_j (they are circuit-hyperplanes) and since no three fundamental flats are incomparable, there are no other fundamental flats and so no other pnc-flats. Thus, y is in no pnc-flat and $M\backslash y = P'_n$, which gives the contradiction $M = A_n$, so property (ii) holds.

To prove properties (iii) and (iv), first note that since the fundamental flats of M form two chains, the other (i.e., reducible) pnc-flats are among the nonempty sets $F_i \cap G_j$. First assume $F_i \cap G_j$ is a pnc-flat of M. Fix $x \in F_i \cap G_j$. Now $(F_i \cap G_j) - x$ is a pnc-flat of M/x; also, $F_i - x$ and $G_j - x$ are fundamental flats in M/x. Since $M/x \in \mathcal{L}$, we have $\eta(M/x) < \eta_{M/x}(F_i - x) + \eta_{M/x}(G_j - x)$, which gives $\eta(M) < \eta_M(F_i) + \eta_M(G_j)$. Property (iv) for F_i and G_j in M follows from this property for $F_i - x$ and $G_j - x$ in M/x.

Now assume $F_i \cap G_j \neq \emptyset$ and $\eta(M) < \eta(F_i) + \eta(G_j)$. Since $F_i \cup G_j = E(M)$, this inequality can be recast as $|F_i \cup G_j| - r(F_i \cup G_j) < |F_i| - r(F_i) + |G_j| - r(G_j)$. Since $F_i \cap G_j \neq \emptyset$, semimodularity gives $r(F_i) + r(G_j) - r(F_i \cup G_j) \geq 1$. The last two inequalities give $|F_i \cap G_j| \geq 2$. Fix $x \in F_i \cap G_j$. Now $F_i - x$ and $G_j - x$ are incomparable fundamental flats of M/x that are not disjoint; also, the assumed inequality about nullity gives $\eta(M/x) < \eta_{M/x}(F_i - x) + \eta_{M/x}(G_j - x)$. Therefore $(F_i - x) \cap (G_j - x)$ is a pnc-flat of M/x, so either $F_i \cap G_j$ or $(F_i \cap G_j) - x$ is a pnc-flat of M. If $F_i \cap G_j$ were not a pnc-flat of M, then the same argument using some $y \in (F_i \cap G_j) - x$ would give both $(F_i \cap G_j) - x$ and $(F_i \cap G_j) - y$ being pnc-flats of M, which is impossible since both x and y would need to be isthmuses of $M|F_i \cap G_j$ for both sets to be flats. Thus, $F_i \cap G_j$ is a pnc-flat of M. The rank assertion follows as above. This completes the proof that M satisfies the properties in Proposition 2.10 and so, contrary to the assumption, $M \in \mathcal{L}$. $\square$

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REFERENCES

[1] J. Bonin and A. de Mier, Lattice path matroids: structural properties, *European J. Combin.* **27** (2006) 701–738.

[2] J. Bonin, A. de Mier, and M. Noy, Lattice path matroids: enumerative aspects and Tutte polynomials, *J. Combin. Theory Ser. A* **104** (2003) 63–94.

[3] J. Bonin and O. Giménez, Multi-path matroids, *Combin. Probab. Comput.* **16** (2007) 193–217.

[4] R. A. Brualdi, Transversal matroids, in: *Combinatorial Geometries*, N. White, ed. (Cambridge Univ. Press, Cambridge, 1987) 72–97.

[5] H. H. Crapo, Single-element extensions of matroids, *J. Res. Nat. Bur. Standards Sect. B* **69B** (1965) 55–65.

[6] A. de Mier, A natural family of flag matroids, *SIAM J. Discrete Math.* **21** (2007) 130–140.

[7] A. de Mier and M. Noy, A solution to the tennis ball problem, *Theoret. Comput. Sci.* **346** (2005) 254–264.

[8] A. Gundert, E. Kim, and D. Schymura, Lattice path and Lagrangian matroids (CRM report on research projects from the 2009 i-MATH Winter School DocCourse on Discrete and Computational Geometry).

[9] J. Lawrence, Some dual pairs of simple oriented matroids by concatenation (unpublished manuscript, 1984).

[10] J. G. Oxley, *Matroid Theory*, (Oxford University Press, Oxford, 1992).

[11] J. G. Oxley, K. Prendergast, and D. Row, Matroids whose ground sets are domains of functions, *J. Austral. Math. Soc. Ser. A* **32** (1982) 380–387.

[12] V. Reiner, personal communication, 2008.

[13] J. Schweig, On the h-vector of a lattice path matroid, *Electron. J. Combin.* **17** (2010) Note 3, 6 pp.

[14] R. Stanley, Some combinatorial aspects of the Schubert calculus, in: *Combinatoire et représentation du groupe symétrique* (Lecture Notes in Math., Vol. 579, Springer, Berlin, 1977) 217–251.

[15] D. J. A. Welsh, *Matroid Theory*, (Academic Press, London, 1976).

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