EVERY GROUP IS THE AUTOMORPHISM GROUP OF A RANK-3 MATROID

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ABSTRACT. Every group is the automorphism group of a rank-3 extension of a rank-3 Dowling geometry.

By modifying Cayley graphs, Frucht [5, 6] showed that every group is the automorphism group of a graph. Using this result, Harary, Piff, and Welsh ([7]; see also [11, §17.7]) showed that every group is the automorphism group of a graphic matroid, possibly having infinite rank. Mendelsohn used Frucht's theorem to show that every group is the automorphism group of an infinite projective plane ([8]; see also [9, 10]). In this note, we use Dowling geometries ([4]; see also [1]) to give an elementary proof of the following theorem.

Theorem 1. *Let G be a group. If G is finite, then G is the automorphism group of a finite rank-3 geometry (simple matroid). If G is infinite, then G is the automorphism group of a rank-3 geometry of the same cardinality.*

In the case that G is infinite, our proof uses the axiom of choice. Most of this paper is self-contained.

A rank-3 geometry M on the point set S can be specified by a collection of subsets of S called *lines* such that every line contains at least two points and any pair of distinct points is contained in a unique line. A line is said to be *long* if it contains at least three points. To specify a rank-3 geometry M, it suffices to specify its long lines. An *automorphism* or *collineation* of M is a bijection on S sending lines to lines.

We begin the proof of Theorem 1 by constructing two rank-3 geometries. Let G be a group. The *rank*-3 *Dowling geometry* $Q_3(G)$ is constructed in the following way: Start with a basis of three points, p_1, p_2, and p_3, called the *joints*. On each of the three lines $p_i \vee p_j$, $i < j$, put the points a_{ij} for $a \in G$ so that the points a_{12}, b_{23}, c_{13} are collinear whenever $ab = c$ in the group G. The points a_{ij} are called *internal* points. The geometry $Q_3(G)$ can be thought of as a geometrical coding of the multiplication table of G. Lemma 1 shows that the automorphism group of $Q_3(G)$ contains a subgroup isomorphic to G. (See [2] for a description of the full automorphism group.)

Lemma 1. *Let Φ be an automorphism of $Q_3(G)$ fixing every point on the line $p_2 \vee p_3$. Then $\Phi(p_1) = p_1$ and there exists an element $g \in G$ such that for all $a \in G$,*

$$\Phi(a_{12}) = (ga)_{12} \text{ and } \Phi(a_{13}) = (ga)_{13}.$$

In particular, the subgroup of the automorphism group of $Q_3(G)$ stabilizing $p_2 \vee p_3$ is isomorphic to G.

Proof. It is easy to check that the lemma holds when G is the group of order one. Suppose that $|G| \geq 2$. Because p_1 is the unique point not on $p_2 \vee p_3$ that is on exactly two long lines, Φ fixes p_1. Hence, Φ sends an internal point on $p_1 \vee p_2$ to an internal point on the same line. Let g be the group element such that $\Phi(e_{12}) = g_{12}$, where e is the identity of G. Because e_{12}, a_{23}, and a_{13} are collinear and $\Phi(a_{23}) = a_{23}$, we have $\Phi(a_{13}) = (ga)_{13}$. Likewise from the collinearity of a_{12}, e_{23}, and a_{13}, we conclude that $\Phi(a_{12}) = (ga)_{12}$. □

The second geometry is a geometry with trivial automorphism group. Let S be an initial segment of the ordinal numbers. The geometry T_S is constructed as follows (see Figure 1). Start with three basis points p_2, p_3 and q, and two copies $\{a : a \in S\}$ and $\{a' : a \in S\}$ of S. Let $\ell = \{p_2, p_3\} \cup \{a : a \in S\}$, $\ell' = \{a' : a \in S\}$, and $\{q, a, a'\}$ be lines. For each $a \in S$ and $b > a$, add the point b_a on $q \vee b$ and $p_3 \vee a'$. Thus the sets $\{a', p_3\} \cup \{b_a : b > a\}$ and $\{b, q, b'\} \cup \{b_a : a < b\}$ are lines. The geometry T_S is representable over every sufficiently large field.

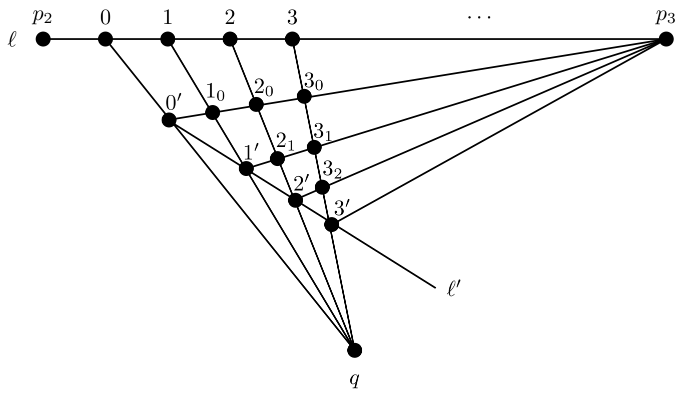

FIGURE 1. The matroid T_S where $S = \{0, 1, 2, \ldots\}$ is an initial segment of the ordinal numbers.

Lemma 2. *When $|S| \geq 3$, the automorphism group of T_S is trivial.*

Proof. Let Φ be an automorphism of T_S. Suppose that S is infinite. (The argument when S is finite is similar.) Observe that p_2 is the only point on exactly one long line, namely ℓ. Therefore $\Phi(p_2) = p_2$ and $\Phi(\ell) = \ell$. Furthermore since p_3 and q are the only points on an infinite number of long lines, and $p_3 \in \ell$ while $q \notin \ell$, it follows that Φ fixes both p_3 and q.

To see that Φ fixes ℓ pointwise, consider the sets $I_a = \{b : (q \vee b) \cap (p_3 \vee a') \neq \emptyset\}$. By construction, I_a equals the final segment $\{b : b \geq a\}$. Since Φ preserves point-line incidences and fixes q and p_3, we have $I_{\Phi(a)} = \Phi(I_a)$. Hence, $I_a \subseteq I_b$ implies that $I_{\Phi(a)} \subseteq I_{\Phi(b)}$. Because $a \leq b$ if and only if $I_b \subseteq I_a$, Φ is an order-preserving bijection on the points $a \in S$ on the line ℓ. Since the only order-preserving bijection on a well-ordered set is the identity, we conclude that Φ fixes every point on ℓ.

Finally note that $\Phi(\ell') = \ell'$ since ℓ' is the only infinite line containing neither p_3 nor q. From $a' = \ell' \cap (q \vee a)$, it follows that Φ fixes ℓ' pointwise, and from $b_a = (q \vee b) \cap (p_3 \vee a')$, that Φ fixes every point in T_S. $\square$

To finish the proof of Theorem 1, we "glue" the geometries $Q_3(G)$ and T_G along the line $p_2 \vee p_3$. More precisely, well-order the elements of the group G and construct the geometry T_G. Let $Q_3(G) \cup T_G$ be the rank-3 geometry on the union of the point sets of $Q_3(G)$ and T_G which has as long lines all the long lines of $Q_3(G)$ and T_G.

Lemma 3. *Let G be a group of order at least 2. The automorphism group of $Q_3(G) \cup T_G$ is isomorphic to G.*

Proof. The case $|G| = 2$ is easy. Suppose $|G| \geq 3$. If Φ is an automorphism of $Q_3(G) \cup T_G$, then its restriction to T_G is the identity. In particular, Φ fixes every point on the line $p_2 \vee p_3$. Hence, Φ restricted to $Q_3(G)$ has the form given in Lemma1. Conversely, every automorphism of $Q_3(G)$ fixing every point on $p_2 \vee p_3$ can be extended to an automorphism of $Q_3(G) \cup T_G$ by defining it to be the identity on T_G. $\square$

Theorem 1 now follows from Lemma 3. We end with a more technical result.

Proposition 1. *Let M be a geometry, ℓ be a modular line, and G be the subgroup of automorphisms of M stabilizing ℓ. Then there exists a rank-preserving extension E of M such that G is the automorphism group of E.*

Sketch of Proof. Let S be an initial segment of the ordinals such that the number of points in M is strictly less than the cardinal $|S|$. Choose a line in T_S having the same cardinality as ℓ and label it bijectively with the points in ℓ. Then the geometry E obtained by truncating the generalized parallel connection (see [3]) of M and T_S at the line ℓ is a rank-preserving extension of M with automorphism group G. $\square$

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