CHARACTERIZING COMBINATORIAL GEOMETRIES BY NUMERICAL INVARIANTS

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Abstract. We show that the projective geometry $PG(r - 1, q)$ for $r > 3$ is the only rank-$r$ (combinatorial) geometry with $(q^r - 1)/(q - 1)$ points in which all lines have at least $q + 1$ points. For $r = 3$, these numerical invariants do not distinguish between projective planes of the same order, but they do distinguish projective planes from other rank-3 geometries. We give similar characterizations of affine geometries. In the core of the paper, we investigate the extent to which partition lattices and, more generally, Dowling lattices are characterized by similar information about their flats of small rank. We apply our results to characterizations of affine geometries, partition lattices, and Dowling lattices by Tutte polynomials, and to matroid reconstruction. In particular, we show that any matroid with the same Tutte polynomial as a Dowling lattice is a Dowling lattice.

1. Introduction

There are several results that use basic counting information to characterize classes of simple matroids (combinatorial geometries; henceforth shortened to geometries). One result of this type is the following theorem of Greene [9], which extends earlier work of Basterfield and Kelly.

Proposition 1.1. Every geometry has at least as many copoints as points; equality holds if and only if the geometry is modular.

Since projective geometries are the only connected modular geometries of rank three or more, it follows that any geometry of rank three or more with as many points as copoints and with no 2-point lines is a projective geometry. If there are $(q^r - 1)/(q - 1)$ points (and copoints) and the rank $r$ exceeds three, we can conclude that $q$ is a prime power and the geometry is the unique rank-$r$ projective geometry of order $q$, namely $PG(r - 1, q)$, the geometry constructed from the field of order $q$.

We derive several results of this type. In Section 2 we show that any rank-$r$ geometry with $(q^r - 1)/(q - 1)$ points in which all lines have at least $q + 1$ points is a projective geometry. Since there can be many projective planes of a given order, this counting information distinguishes projective planes from other rank-3 geometries, but it does not generally characterize a given projective plane. Related results for affine geometries are also treated.

Dowling lattices have many properties similar to those of projective geometries, so it is natural to look for analogous results about Dowling lattices. We treat these in Section 3, which is the core of the paper.

In the last two sections, we apply our results to characterizations of affine geometries and Dowling lattices by Tutte polynomials, and to matroid reconstruction. Apart from relatively simple examples like projective geometries and uniform matroids, few classes of matroids are known to be characterized by their Tutte polynomials. We show that any matroid with the same Tutte polynomial as a Dowling lattice is a Dowling lattice.

We assume the reader is familiar with basic matroid theory. Our notation and terminology follow [13] with the following common additions. The colines of a rank-$r$ matroid $M$ are the flats of rank $r - 2$, and the copoints (hyperplanes) are the flats of rank $r - 1$. As is justified by well-known cryptomorphisms, we use the notions of geometry and geometric lattice interchangeably.

2. Characterizations of Affine and Projective Geometries

In this section we give several characterizations of projective and affine geometries by numerical invariants. We use the following result from [11].
Proposition 2.1. Rank-$r$ geometries having no $(q+2)$-point line minors have at most $(q^r-1)/(q-1)$ points. This upper bound is attained only by projective geometries of order $q$.

Theorem 2.2. Let $M$ be a rank-$r$ geometry on $S$ with $|S| = (q^r-1)/(q-1)$ in which all lines have at least $q+1$ points. Then $M$ is a projective geometry of order $q$.

Proof. By Proposition 2.1, we need only show that $M$ has no $(q+2)$-point line minors. Therefore by the Scum Theorem of Higgs (see [13]), it suffices to show that there are $q+1$ copoints over each coline. To show this, we first prove that each rank-$i$ flat has exactly $(q^i-1)/(q-1)$ points. To see this, let $F$ and $F'$ be flats with $F'' \cup x$ for a point $x \in F - F'$. Counting the points on lines through $x$, each of which has at least $q$ points besides $x$, shows that there are at least $1+q|F''|$ points in $F$. It follows that each rank-$i$ flat has at least $(q^i-1)/(q-1)$ points. Similarly, if any rank-$i$ flat $F$ had more points, induction on a saturated chain from $F$ to $S$ would force more than $(q^r-1)/(q-1)$ points in $S$, contrary to the hypothesis. Thus each rank-$i$ flat has precisely $(q^r-1)/(q-1)$ points. Thus there are

\[ \frac{q^r-1}{q-1} - \frac{q^{r-2}-1}{q-1} = \frac{q^r-q^{r-2}}{q-1} \]

points outside each coline, and each copoint over the coline contains

\[ \frac{q^{r-1}-1}{q-1} - \frac{q^{r-2}-1}{q-1} = \frac{q^{r-1}-q^{r-2}}{q-1} \]

decreases of these points. Thus there are $(q^r-q^{r-2})/(q^{r-1}-q^{r-2})$, or $q+1$, copoints over each coline, as needed. \qed

We next present several characterizations of affine geometries. These are based on the following result from [3].

Proposition 2.3. Rank-$r$ geometries having no $(q+2)$-point line minors and no $(q+1)$-point lines have at most $q^{r-1}$ points. This upper bound is attained only by affine geometries of order $q$.

Theorem 2.4. Assume $M$ is a rank-$r$ geometry with $q^{r-1}$ points in which lines have $q$ points and planes have at least $q^2$ points. Then $M$ is an affine geometry of order $q$.

Theorem 2.4, which plays a key role in Section 4, is the case $j = 2$ of the following theorem.

Theorem 2.5. Let $j$ be an integer with $2 \leq j \leq r - 2$. Assume $M$ is a rank-$r$ geometry with $q^{r-1}$ points in which lines have at most $q$ points, rank-$(j-1)$ flats have $q^{j-2}$ points, rank-$j$ flats have $q^{j-1}$ points, and rank-$(j+1)$ flats have at least $q^j$ points. Then $M$ is an affine geometry of order $q$.

Proof. As above, we need only show that there are $q+1$ copoints over each coline. This follows by the same type of counting as at the end of the proof of Theorem 2.2 once we establish that for each $i \geq j$, each rank-$i$ flat has exactly $q^{i-1}$ points. To see that for $i \geq j$ each rank-$i$ flat has at least $q^{i-1}$ points, induct on $i$. Assume $F'$ is a rank-$i$ flat with at least $q^{i-1}$ points and that $F$ is a rank-$(i+1)$ flat with $F = F' \cup x$ for a point $x \in F - F'$. Fix a rank-$(j-1)$ flat $Y$ in $F'$. Since $|Y| = q^{j-2}$ and all rank-$j$ flats have $q^{j-1}$ points, $Y$ is in at least $(q^{j-1} - q^{j-2})/(q^{j-1} - q^{j-2})$ rank-$j$ flats in $F'$. The point $x$ from $F - F'$ together with any rank-$j$ flat in $F'$ containing $Y$ determines a rank-$(j+1)$ flat in $F$. Only points in the rank-$j$ flat $Y \cup x$ are in more than one such rank-$(j+1)$ flat and each such rank-$(j+1)$ flat has at least $q^j - q^{j-1}$ points in $F - (Y \cup x)$. Thus $F$ has at least

\[ q^{j-1} + \frac{q^{j-1} - q^{j-2}}{q^{j-1} - q^{j-2}} (q^j - q^{j-1}) = q^j \]

points. As in the last proof, equality follows since $M$ has $q^{r-1}$ points. \qed

We turn to another result of this type. The proof is valid only for $q$ greater than 2, although it seems likely that the statement is correct also for $q = 2$. 

Theorem 3.2. Assume $q$ exceeds 2 and $M$ is a rank-$r$ geometry with $q^{r-1}$ points in which lines have $q$ points and copoints have $q^{r-2}$ points. Then $M$ is an affine geometry of order $q$.

Proof. Again we need only show that there are at most $q + 1$ copoints over each coline. Assume a coline $C$ is covered by $q + 2$ or more copoints. Since there are $(q^{r-1} - |C|)/(q^{r-2} - |C|)$ copoints over $C$, we get

$$|C| \geq 2q^{r-2}/(q + 1).$$

Selecting a point $x \notin C$ in a hyperplane $H$ over $C$ and counting the points on lines through $x$ and, in turn, the $2q^{r-2}/(q + 1)$ or more points of $C$ shows that there are at least $1 + (q - 1)2q^{r-2}/(q + 1)$ points in $H$. Since this exceeds $q^{r-2}$, this contradiction shows that no coline is covered by $q + 2$ or more copoints, as needed. □

3. A Characterization of Dowling Lattices

In this section we derive a result for Dowling lattices akin to Theorems 2.2 and 2.4. Because Dowling lattices lack analogs of Propositions 2.1 and 2.3, and the number of points in a flat is not constant for each rank, the argument is more involved than those in Section 2. We start by recalling Dowling lattices, which we present via line-closure. For a complete treatment from a different perspective, see Dowling [8].

Let $M$ be a geometry on $S$. A subset $T$ of $S$ is line-closed if for every two points $x, y \in T$, the line $x \lor y$ is contained in $T$. A geometry $M$ is line-closed if the flats of $M$ are precisely the line-closed sets. Dowling lattices are supersolvable [8], and supersolvable geometries are line-closed [10]. Thus Dowling lattices can be defined by specifying their points and lines.

Let $G$ be a finite group, written with multiplicative notation. The rank-$r$ Dowling lattice over $G$, denoted $Q_r(G)$, has the following points and lines. There are two kinds of points: joints $p_1, p_2, \ldots, p_r$, which form a basis for $Q_r(G)$; and internal points $g_{ij}$ for every $g \in G$ and every pair of indices with $1 \leq i < j \leq r$. Hence $Q_r(G)$ has $r + \binom{r}{2}|G|$ points. There are two types of nontrivial lines (i.e., lines with at least three points): coordinate lines $p_i \lor p_j = \{p_i, p_j\} \cup \{g_{ij}|g \in G\}$; and transversal lines $\{g_{ij}, h_{jk}, (gh)_{ik}\}$ for each pair $g, h \in G$ and triple of indices with $1 \leq i < j < k \leq r$. Thus the transversal lines are contained in the coordinate planes $p_i \lor p_j \lor p_k$, and they encode the group operation. For $i > j$ we set $g_{ij} = (g^{-1})_{ji}$, where $g^{-1}$ is the inverse of $g$ in the group $G$. With this convention we can drop the restriction $i < j < k$ in the definition of transversal lines.

Dowling lattices can be defined for infinite groups in the same manner; however, our results concern only the finite case. For $r = 3$, $G$ need only be a quasigroup for line-closure to give rise to a geometry on the points of $Q_3(G)$. If $G$ is the trivial (one-element) group, then $Q_r(G)$ is isomorphic to the rank-$r$ partition lattice $\Pi_{r+1}$. Thus Dowling lattices generalize partition lattices.

We use the following characterization of Dowling lattices from [2]. In [1] it is observed that when $r = 3$, the axioms below characterize Dowling lattices based on quasigroups.

Proposition 3.1. A geometry $M$ of rank $r \geq 4$ is a Dowling lattice if and only if $M$ has points $p_1, p_2, \ldots, p_r$ satisfying these axioms.

(D1) Each point of $M$ lies on a coordinate line $p_i \lor p_j$.

(D2) No coordinate line $p_i \lor p_j$ is trivial.

(D3) For points $x \in (p_i \lor p_j) - \{p_i, p_j\}$ and $y \in (p_i \lor p_k) - \{p_i, p_k\}$, the line $x \lor y$ is nontrivial.

Note that (D1) and the rank imply that $p_1, p_2, \ldots, p_r$ form a basis of $M$. In the case of a nontrivial group $G$, the basis $p_1, p_2, \ldots, p_r$ of $Q_r(G)$ satisfying (D1)–(D3) is unique. Since $\Pi_{r+1}$ has $r + 1$ such bases, special consideration is needed for $\Pi_{r+1}$; we treat this case first. It is immediate that the counting conditions (1) to (4) in the next theorem hold in $\Pi_{r+1}$. We show that these statistics about flats of the first four ranks characterize $\Pi_{r+1}$.

Theorem 3.2. Assume that a rank-$r$ geometry $M$ has

(1) $\binom{r+1}{2}$ points,
(2) \( \binom{r+1}{3} \) lines with three points,
(3) no 5-point planes, \( \binom{r+1}{3} \) planes with six points, no planes with more than six points, and
(4) no rank-4 flats with more than ten points.

Then \( M \cong \Pi_{r+1} \).

**Proof.** We first prove that \( M \) has several properties that hold for \( \Pi_{r+1} \), namely:

(a) the number of 3-point planes through each point \( x \) is exactly \( r-1 \);
(b) the number of 6-point planes through each point \( x \) is exactly \( \binom{r-1}{2} \); and
(c) all 6-point planes are isomorphic to \( \Pi_4 \).

Let \( \ell_1, \ell_2, \ldots, \ell_r \) be the 3-point lines through \( x \). By (3), no three of these lines are coplanar. Therefore the 3-point lines through \( x \) determine \( \binom{r}{2} \) planes \( \ell_i \cup \ell_j \) through \( x \), each having at least five points. Hence by (3) these are 6-point planes. Let \( x_{ij} = x_{ji} \) be the unique point of \( \ell_i \cup \ell_j \) on neither \( \ell_i \) nor \( \ell_j \).

We claim that there are \( \binom{r-1}{2} \) distinct points \( x_{ij} \). Note that \( x_{ij} \neq x_{ik} \) since \( \ell_i \cup \ell_j \) and \( \ell_i \cup \ell_k \) meet in the line \( \ell_t \). To see that \( x_{ij} \neq x_{ik} \) when \( [i,j,h,k] = 4 \), note that if \( x_{ij} = x_{hk} \), then the planes \( \ell_i \cup \ell_j \) and \( \ell_h \cup \ell_k \) meet in the line \( x \cup x_{ij} \). Therefore \( \ell_i \cup \ell_j \cap \ell_h \cup \ell_k \) has rank 4. However \( \ell_i \cup \ell_j \cap \ell_h \cup \ell_k \) contains at least twelve points, namely the nine points from \( \ell_i, \ell_j, \ell_h, \ell_k \) together with \( x_{ij}, x_{ih} \) and \( x_{ik} \). This contradiction of (4) establishes the claim.

The lines \( \ell_1, \ell_2, \ldots, \ell_r \) through \( x \) and the points \( x_{ij} \) account for \( 2t + 1 + \binom{r}{2} \) points of \( M \). Therefore \( 2t + 1 + \binom{r}{2} \leq \binom{r+1}{3} \), so \( t \leq r-1 \). However, by (1) and (2), the average number of 3-point lines through a point is exactly \( \frac{3\binom{r+1}{3}}{\binom{r+1}{2}} = r-1 \). Hence the number of 3-point planes through each point of \( M \) is exactly \( r-1 \), establishing (a).

To prove (b), note that while each point \( x \) lies in at least \( \binom{r-1}{2} \) planes having six points (the planes \( \ell_i \cup \ell_j \) discussed above), the average number of 6-point planes through \( x \) is also \( \frac{6\binom{r+1}{3}}{\binom{r+1}{2}} = \binom{r-1}{2} \).

We have established that there are \( \binom{r+1}{3} \binom{r-1}{2} \) pairs of 3-point lines intersecting in a point, and each such pair is in exactly one 6-point plane. Since a point in a 6-point plane can be on at most two 3-point lines in that plane, the number of pairs of 3-point lines intersecting in a point is at most \( \binom{r+1}{4} \), which is \( \binom{r+1}{3} \binom{r-1}{2} \). Therefore each point in each 6-point plane is on at least \( \binom{r-1}{2} \) 3-point lines in that plane. From this, (c) follows.

We now select a basis for \( M \) satisfying axioms (D1)--(D3) of Proposition 3.1. Let \( x \) be any point of \( M \), let \( \ell_1, \ell_2, \ldots, \ell_{r-1} \) be the 3-point lines through \( x \), let \( \ell_t = \{x, x_{1t}, x_{2t}, \ldots, x_{rt-1}, x_{rt} \} \), and let \( x_{ij} \) be the unique point in \( \ell_i \cup \ell_j \) on neither \( \ell_i \) nor \( \ell_j \). Define an equivalence relation \( \sim \) on \( \{x_{1t}, x_{2t}, \ldots, x_{rt-1}, x_{rt} \} \) by: \( x_{1t} \sim x_{2t} \) if and only if \( x_{1t} = x_{2t} \). This contradiction of (4) establishes the claim.

To see that \( p_1, p_2, \ldots, p_r \) satisfy axioms (D1)--(D3) of Proposition 3.1, note first that by construction, each \( p_i \cup p_j \) is a 3-point line so (2) holds. Axiom (D1) holds since the union of the lines \( p_i \cup p_j \) contains \( \binom{r}{2} + r = \binom{r+1}{2} \) points. Finally, (D3) holds since each line \( p_i \cup p_j \cup p_k \) is isomorphic to \( \Pi_4 \), and the lines \( p_i \cup p_j, p_i \cup p_k, \) and \( p_j \cup p_k \) are three of the four 3-point lines in \( \Pi_4 \). Hence \( M \) is the rank-\( r \) Dowling lattice \( Q_r(G) \) with \( |G| = 1 \), so \( M \cong \Pi_{r+1} \).

To treat the corresponding result for Dowling lattices over nontrivial groups, we need the following result [8, Theorem 2].
Proposition 3.3. For each flat $F$ in the Dowling lattice $Q_r(G)$, there are integers $s$ and $k$ with $0 \leq s \leq r$ and $k \geq 0$, and integers $n_1, n_2, \ldots, n_k$ with $n_i \geq 2$ such that $F \cong Q_s(G) \oplus \Pi_{n_1} \oplus \Pi_{n_2} \oplus \cdots \oplus \Pi_{n_k}$.

This proposition and the following remarks underlie the counting that is needed for the proof of our characterization of Dowling lattices (Theorem 3.4). Assume $|G| > 1$. In $Q_r(G)$, flats $F$ with

$$F \cong Q_s(G) \oplus \Pi_{n_1} \oplus \Pi_{n_2} \oplus \cdots \oplus \Pi_{n_k}$$

have

$$|F| = s + \binom{s}{2} |G| + \sum_{i=1}^{k} \binom{n_i}{2}$$

points and rank

$$r(F) = s + \sum_{i=1}^{k} (n_i - 1).$$

For $s \geq 2$, such flats are formed in the following way. Let $p_1, p_2, \ldots, p_r$ be the joints of $Q_r(G)$ and let $\{t_1, t_2, \ldots, t_s\}$, $\{i_1, i_2, \ldots, i_{n_1}\}$, $\cdots$, $\{j_1, j_2, \ldots, j_{n_k}\}$ be pairwise disjoint subsets of $\{1, 2, \ldots, r\}$ with $\{t_1, t_2, \ldots, t_s\}$ serving as a “distinguished” subset. For the distinguished subset, we form the flat

$$p_{t_1} \lor p_{t_2} \lor \cdots \lor p_{t_s},$$

which is isomorphic to $Q_s(G)$. For the subset $\{i_1, i_2, \ldots, i_{n_1}\}$, choose one internal point $x_{i_h}$ on each coordinate line $p_{i_1} \lor p_{i_2}$ for $2 \leq h \leq n_1$. The flat $x_{i_2} \lor x_{i_3} \lor \cdots \lor x_{i_{n_1}}$ is isomorphic to the partition lattice $\Pi_{n_1}$. (The points $x_2, x_3, \ldots, x_{n_1}$ serve as a basis for $\Pi_{n_1}$ satisfying axioms (D1)–(D3) of Proposition 3.1.) Doing the same for the remaining subsets and considering the flat $F$ spanned by all these points, we get

$$F \cong Q_s(G) \oplus \Pi_{n_1} \oplus \Pi_{n_2} \oplus \cdots \oplus \Pi_{n_k}.$$
For our work in Section 4 it is important to note that the cardinality of submaximal flats of rank \( i \) exceeds that of maximal flats of rank \( i - 1 \). This holds since the maximal flats of rank \( i - 1 \) are isomorphic to \( Q_{i-1}(G) \), and the flats isomorphic to \( Q_{i-1}(G) \sqcup \Pi_2 \) have rank \( i \) but are not maximal.

The next theorem characterizes Dowling lattices using the cardinalities of the maximal rank-\( i \) flats for \( i \leq 6 \) and the cardinalities of the submaximal rank-\( i \) flats for \( i = 2 \) and 5. Since these statistics are shared by all Dowling lattices of a given rank based on groups of the same order, they do not determine a Dowling lattice uniquely unless the order of the group is prime. However, no geometries other than Dowling lattices can share these statistics about flats.

**Theorem 3.4.** Assume \( M \) is a rank-\( r \) geometry and \( g > 1 \) is an integer such that:

1. \( M \) has \( \binom{g}{2} \) lines with \( g + r \) points,
2. \( M \) has \( \binom{g}{3} \) lines with \( g + 2 \) points, \( \binom{g}{4} \) lines with three points, and no other nontrivial lines,
3. \( M \) has \( \binom{g}{3} \) planes with \( 3g + 3 \) points, and no other planes with \( 2g + 3 \) or more points,
4. \( M \) has \( \binom{g}{4} \) rank-4 flats with \( 6g + 4 \) points, and no larger rank-4 flats,
5. \( M \) has \( \binom{g}{5} \) rank-5 flats with \( 10g + 5 \) points, \( \binom{g}{6} \binom{r-4}{2} g \) rank-5 flats with \( 6g + 5 \) points, and no other rank-5 flats with more than \( 6g + 4 \) points,
6. the rank-6 flats (if any) with most points have \( 15g + 6 \) points, and no other rank-6 flats have \( 14g + 6 \) or more points, and
7. all rank-7 flats (if any) have fewer than \( 22g + 8 \) points.

Then \( M \) is the Dowling lattice \( Q_r(G) \) for some group (or quasigroup, if \( r = 3 \)) \( G \) of order \( g \).

**Proof.** If \( r = 3 \), there are three maximal lines. Since there are \( 3g + 3 \) points, it follows that each pair of maximal lines intersects in a point, and there are exactly three such points of intersection, say \( p_1, p_2, p_3 \). Using the basis \( p_1, p_2, p_3 \) and the assumption that there are \( g^2 \) submaximal lines, it is easy to check that axioms (D1)–(D3) in Proposition 3.1 hold.

We treat ranks four and higher through a series of deductions about the structure of maximal flats, especially those of ranks four and five.

**Deduction 3.5.** Three maximal planes in a rank-4 flat \( F \) cannot intersect in a line.

**Proof.** If three maximal planes in \( F \) intersect in a line containing \( m \) points, then together these planes contain \( m + 3(3g + 3 - m) \) points. Thus we have \( 9g + 9 - 2m \leq |F| \leq 6g + 4 \). However, since \( m \leq g + 2 \), this is impossible. \( \square \)

**Deduction 3.6.** No point of a maximal rank-4 flat \( F \) can be in more than three maximal planes of \( F \).

**Proof.** Consider a point \( x \) in three maximal planes \( P_1, P_2, P_3 \) of \( F \). Let \( \ell_{ij} = P_i \cap P_j \) and let \( m_{ij} = |\ell_{ij} - \{x\}| \). Thus \( m_{ij} \leq g + 1 \). Note that \( P_1 \cup P_2 \cup P_3 \) is the disjoint union of \( \{x\} \), the three sets \( \ell_{ij} - \{x\} \), and the three sets \( \ell_i - (\ell_{ij} \cup \ell_{jk}) \). Thus \( |P_1 \cup P_2 \cup P_3| \) is

\[
1 + m_{12} + m_{13} + m_{23} + \left(3g + 2 - m_{12} - m_{13}\right) + \left(3g + 2 - m_{12} - m_{23}\right) + \left(3g + 2 - m_{13} - m_{23}\right),
\]

or \( 9g + 7 - m_{12} - m_{13} - m_{23} \). Since \( |F| = 6g + 4 \) and \( m_{ij} \leq g + 1 \), we get \( m_{ij} = g + 1 \), and so \( P_1 \cup P_2 \cup P_3 = F \). Thus \( P_1, P_2, P_3 \) each contain two maximal lines through \( x \) and precisely \( g \) points not on these lines, and \( x \) is on precisely three maximal lines in \( F \). Therefore \( x \) cannot be in a fourth maximal plane in \( F \). \( \square \)

**Deduction 3.7.** Each maximal rank-4 flat contains at most four maximal planes.

**Proof.** Assume the maximal rank-4 flat \( F \) contains \( i \geq 4 \) maximal planes. Consider the set \( \mathcal{P} \) of pairs \((x, P)\) where \( P \) is a maximal plane in \( F \) and \( x \in P \). Since there are \( i \) maximal planes, \( |\mathcal{P}| = i(3g + 3) \). Each point is in at most three maximal planes by 3.6. If each point were in at most two maximal planes, then \( |\mathcal{P}| \leq 2(6g+4) \), or \( i(3g + 3) \leq 2(6g + 4) \), which is impossible. Thus some point is in three maximal planes.
Let \( x \) be such a point. By the proof of 3.6, \( x \) is on three maximal lines, \( \ell_1, \ell_2, \ell_3 \) in \( F \), and the three maximal planes containing \( x \) are \( \ell_1 \lor \ell_2, \ell_1 \lor \ell_3, \ell_2 \lor \ell_3 \). Let \( P \) be a fourth maximal plane in \( F \). By 3.5, \( P \) intersects each of the lines \( \ell_1, \ell_2, \ell_3 \) in at most one point. Since there are \( 3g \) points of \( F \) not on these lines and \( |P| = 3g + 3 \), it follows that \( P \) intersects each of these lines in a single point and consists of these three points of intersection and the \( 3g \) points of \( F \) not on these lines. Since \( g \geq 2 \), the \( 3g \) points of \( P \) not on \( \ell_1, \ell_2, \ell_3 \) are not collinear and hence span \( P \). Thus there is only one maximal plane not containing \( x \).

We can say more about maximal rank-4 flats containing four maximal planes.

**Deduction 3.8.** Let \( F \) be a maximal rank-4 flat containing four maximal planes \( P_1, P_2, P_3, \) and \( P_4 \). Then each of the four intersections \( P_i \cap P_j \cap P_k \) is a single point, \( p_{ijk} \), where \( \{i, j, k, h\} = \{1, 2, 3, 4\} \). The maximal lines of \( F \) are the six lines \( P_i \cap P_j \cap P_k \cap P_h \). The plane \( P_i \) is \( p_j \lor p_k \lor p_h \). Each point of \( F \) is on a maximal line.

Furthermore, each 3-point line of \( F \) is in some plane \( P_i \), and no 3-point line contains any of the points \( p_1, p_2, p_3, p_4 \). Therefore \( F \) contains at most \( 4g^2 \) submaximal lines.

**Proof.** From the work above, only the claims in the second to last sentence require proof. All points not on maximal lines with \( p_i \) are in \( P_i \); thus any 3-point line with \( p_i \) would lie in \( P_i \), contrary to \( F \) having rank 4.

Since the three points on a submaximal line must be on distinct lines \( p_i \lor p_j \), it follows that two of the three points are on \( p_i \lor p_j \) and \( p_i \lor p_k \) respectively for some \( i, j, \) and \( k \). Thus two of the three points are in \( P_i \), so the line is in \( P_i \). (The same argument also shows that maximal lines lie in maximal planes, and hence there are indeed only six maximal lines.)

If \( r \) is 4, by (2) there are precisely \( 4g^2 \) submaximal lines. From 3.8 and Proposition 3.1, it follows that \( M \) is a Dowling lattice. Thus we turn to ranks 5 and greater.

It should cause no confusion to refer to the points \( p_1, p_2, p_3, p_4 \) in 3.8 as joints. We shall adopt the same terminology for the analogous points in higher-rank flats as the need arises.

**Deduction 3.9.** Each rank-5 flat contains at most five maximal rank-4 flats.

**Proof.** Let \( F \) be a rank-5 flat and let \( T_1, T_2, \ldots, T_5 \) be the maximal rank-4 flats in \( F \). Since

\[
|T_i \cup T_j| = |T_i| + |T_j| - |T_i \cap T_j| = 2(6g + 4) - |T_i \cap T_j| \leq |F| \leq 10g + 5,
\]

we get \( |T_i \cap T_j| \geq 2g + 3 \). We conclude that \( T_i \cap T_j \) is a maximal plane. By similar counting, we get that the maximal planes \( T_1 \cap T_2, T_1 \cap T_3, \ldots, T_1 \cap T_5 \) in the rank-4 flat \( T_1 \) are distinct. Thus \( t \leq 5 \) by 3.7.

**Deduction 3.10.** Let \( F \) be a maximal flat of rank 5 containing five maximal rank-4 flats. Then there are five points \( p_1, p_2, p_3, p_4, p_5 \) such that the ten lines \( p_i \lor p_j \) are precisely the maximal lines of \( F \) and these lines contain all points of \( F \). The five maximal rank-4 flats are the flats \( p_i \lor p_j \lor p_k \lor p_h \lor p_l \) for \( \{i, j, k, h, l\} \subseteq \{1, 2, 3, 4, 5\} \). All submaximal lines in \( F \) lie in planes of the form \( p_i \lor p_j \lor p_k \lor p_l \); hence there are at most \( 10g^2 \) submaximal lines in \( F \).

**Proof.** Let \( T_1, T_2, T_3, T_4, T_5 \) be the maximal rank-4 flats in \( F \). Each \( T_i \) contains four maximal planes, namely, the intersections of \( T_i \) with the other four maximal rank-4 flats. Thus 3.8 applies to each \( T_i \). Let \( p_2, p_3, p_4, p_5 \) be the joints of \( T_1 \) and we may assume that \( T_1 \cap T_2 \) is the maximal plane \( p_2 \lor p_4 \lor p_5 \). Thus there is a point \( p_1 \) in \( T_2 \) such that \( p_1, p_3, p_4, p_5 \) are the joints of \( T_2 \). By considering \( T_3 \), we deduce that \( p_1 \lor p_2 \) is also a maximal line. Counting shows that the union of the lines \( p_i \lor p_j \) is \( F \). Since each point of \( F \) is on one of the lines \( p_i \lor p_j \) and two points suffice to span a line, it follows that all lines lie in maximal rank-4 flats. Thus our assertions about submaximal lines, as well as that the lines \( p_i \lor p_j \) are the only maximal lines, follow from 3.8.

If \( r \) is 5, by (2) there are precisely \( 10g^2 \) submaximal lines, so by 3.10 and Proposition 3.1, \( M \) is a Dowling lattice. We now treat all higher ranks.

**Deduction 3.11.** Each maximal rank-4 flat is in exactly \( r - 4 \) maximal rank-5 flats.
Proof. We first show that each maximal rank-4 flat is in at most \( r - 4 \) maximal rank-5 flats. Let \( F_1, F_2, \ldots, F_t \) be the maximal rank-5 flats containing the maximal rank-4 flat \( F \). Since \( F_i \) and \( F_j \) cover \( F_i \cup F_j = F \), the \( F_{ij} = F \lor F_j \) has rank 6. The set \( F_i \cup F_j \) contains \( 6g + 4 + 2(4g + 1) = 14g + 6 \) points. It follows that each \( F_{ij} \) is a maximal flat and there are precisely \( g \) points in \( L_{ij} = F_{ij} - (F_i \cup F_j) \). The sets \( L_{ij} \) are pairwise disjoint. To see this, first note that since \( F_{ij} \) and \( F_{ik} \) are rank-6 flats meeting in the rank-5 flat \( F_i \), it follows that \( L_{ij} \cap L_{ik} = \emptyset \). Next, if \( x \in L_{ij} \cap L_{ik} \), where \( \{i, j, h, k\} = 4 \), then the rank-6 flats \( F_{ij} \) and \( F_{hk} \) cover their intersection \( F \lor x \), and so \( F_{ij} \lor F_{hk} \) has rank 7. However the set \( F_i \cup F_j \cup F_h \lor F_k \) contains \( (6g + 4) + 4(4g + 1) = 22g + 8 \) points, which by (7) is contrary to \( F_{ij} \lor F_{hk} \) having rank 7. Hence the sets \( L_{ij} \) are pairwise disjoint. Thus the union of \( F \) with the flats \( F_1, F_2, \ldots, F_t \) and the \( \binom{t}{2} \) sets \( L_{ij} \) contains \( 6g + 4 + (4g + 1) + \binom{t}{2}g \) points. Since this is at most \( \binom{t}{2}g + r \), the number of points in \( M \), we have \( t \leq r - 4 \).

To prove equality, let the maximal rank-4 flats be \( S_1, S_2, \ldots, S_{\binom{t}{2}} \), and let \( S_i \) be contained in \( m_i \) maximal rank-5 flats. We just showed that \( m_i \leq r - 4 \) and we are claiming equality holds. By (5), any non-maximal rank-5 flat containing \( S_i \) and a point \( x \notin S_i \) is a submaximal rank-5 flat, namely \( S_i \cup \{x\} \). Thus \( S_i \) is in \( \binom{t}{2}g + r - (6g + 4) - m_i(4g + 1) \) submaximal rank-5 flats. There are \( \binom{t}{2}(r - 4)g \) submaximal rank-5 flats by (5). Note that each submaximal rank-5 flat \( F \) contains at most one maximal rank-4 flat \( F' \) since the singleton \( F - F' \) is an isthmus in \( M/F \), and by the cardinalities of planes, the maximal rank-4 flat \( F' \) can contain no isthmuses of \( M/F' \). Thus there are at most \( \binom{t}{2}(r - 4)g \) maximal rank-4 flats contained in submaximal rank-5 flats. Therefore

\[
\sum_{i=1}^{\binom{t}{2}} \binom{r}{2}g + r - (6g + 4) - m_i(4g + 1) \leq \binom{r}{4}(r - 4)g.
\]

Combining this with \( m_i \leq r - 4 \), we get

\[
\binom{r}{4}(r - 4)g + r - (6g + 4) - (r - 4)(4g + 1) \leq \binom{r}{4}(r - 4)g.
\]

However since equality holds here, we get \( m_i = r - 4 \), as claimed. \( \square \)

Since each maximal rank-5 flat contains at most five maximal rank-4 flats, there are at most \( 5\binom{t}{2} \) pairs of incident maximal rank-4 and maximal rank-5 flats. We have just shown that the number of such pairs is exactly \( \binom{t}{2}(r - 4) \). The equality of these expressions gives the next claim.

**Deduction 3.12.** Each maximal rank-5 flat contains precisely five maximal rank-4 flats.

Thus \( M \) applies to all maximal rank-5 flats. With this we can now prove that \( M \) is a Dowling lattice. Let \( F \) be a maximal rank-4 flat and let \( F_0, F_1, \ldots, F_r \) be the \( r - 4 \) maximal rank-5 flats containing \( F \). Let \( p_1, p_2, p_3, p_4 \) be the joints of \( F \). By 3.10 for each \( i \) with \( 5 \leq i \leq r \), there is a point \( p_i \) such that \( p_1, p_2, p_3, p_4, p_i \) are the joints of \( F_i \). By considering the rank-5 flat \( F \lor p_i \lor p_j \lor p_k \) where \( 5 \leq i < j < k \leq r \), it is maximal since it has at least \( 9g + 5 \) points, we deduce that \( p_i \lor p_j \) is a maximal line. Since each plane \( p_i \lor p_j \lor p_k \) contains \( 6 \leq i < j < k \leq r \) contains three maximal lines, it is a maximal plane. Likewise each \( p_i \lor p_j \lor p_k \lor p_h \) with \( 1 \leq i < j < k < h \leq r \) is a maximal rank-4 flat. Therefore the lines \( p_i \lor p_j \lor p_k \) can intersect only in joints. In particular, the union of all lines \( p_i \lor p_j \lor p_k \) contains \( \binom{r}{2}g + r \) points. Thus, every point of \( M \) is on some maximal line \( p_i \lor p_j \). It follows that each submaximal line is in a maximal plane \( p_i \lor p_j \lor p_k \). Since each maximal plane contains at most \( g^2 \) submaximal lines, and there are \( \binom{r}{2}g^2 \) submaximal lines, each maximal plane contains exactly \( g^2 \) submaximal lines. From these conclusions, it is immediate that axioms (D1)–(D3) of Proposition 3.1 hold, proving that \( M \) is a Dowling lattice. \( \square \)

4. Tutte Polynomials

In this section we apply the results in Sections 2 and 3 to investigate the extent to which affine geometries and Dowling lattices are characterized by their Tutte polynomials. We recall the essential background on Tutte polynomials; for more information, see [6, 7].
The Tutte polynomial is defined for a matroid $M$ on the set $S$ by:

$$t(M; x, y) = \sum_{X : X \subseteq S} (x - 1)^{r(M) - r(X)}(y - 1)^{|X| - r(X)}.$$  

From $t(M; x, y)$, one knows much about $M$, including the number of points, the rank, and whether $M$ is connected. Certain other data, such as the number of copoints, cannot, in general, be determined from the Tutte polynomial (see Example 4.5 in [6]). While non-isomorphic matroids may have the same Tutte polynomial, certain matroids $M$ have the property that $M$ is the only matroid with Tutte polynomial $t(M; x, y)$. For instance, if $t(M; x, y) = t(\text{PG}(r - 1, q); x, y)$, then $M$ is a projective geometry of rank $r$ and order $q$, so if $r > 3$, then $M$ is isomorphic to $\text{PG}(r - 1, q)$. This follows from results on perfect matroid designs in the fifth section of [6]. Alternatively one can argue that if $t(M; x, y) = t(\text{PG}(r - 1, q); x, y)$, then $M$ is a geometry, all lines in $M$ have $q + 1$ points, and $M$ has $(q^r - 1)/(q - 1)$ points, so Theorem 2.2 applies.

The characteristic polynomial, which plays a prominent role in many enumerative questions (see [7]), is related to the Tutte polynomial by

$$\chi(M; x) = (-1)^r t(M; 1 - x, 0).$$

The Tutte polynomial can be expressed in terms of characteristic polynomials via the weighted characteristic polynomial of a matroid $M$:

$$\chi(M; x, y) = \sum_{X} x^{|X|} \chi(M/X; y)$$

where $M/X$ is the contraction of $M$ by the flat $X$ (see [4]). (Note that the sum could also be taken over all sets $X$, rather than just flats $X$, since the characteristic polynomial of a matroid with loops is zero.) In terms of the weighted characteristic polynomial, the Tutte polynomial is given by

$$t(M; x, y) = \frac{1}{(y - 1)^r(M)} \chi(M; y, (x - 1)(y - 1)).$$

Thus for a matroid $M$ on the set $S$, if one knows the characteristic polynomial of each upper interval $[X, S]$ in the lattice of flats as well as the cardinality of $X$, then one can compute the Tutte polynomial of $M$.

Dowling [8] proved that the characteristic polynomial of $Q_r(G)$ is given by

$$\chi(Q_r(G); x) = \prod_{i=0}^{r-1} (x - i|G| - 1).$$

Thus, the characteristic polynomial of $Q_r(G)$ depends only upon the rank $r$ and the order $|G|$ of $G$. Dowling ([8, Theorem 2]) showed that the contraction $Q_r(G)/X$ of $Q_r(G)$ by a flat $X$ is isomorphic to $Q_i(G)$, where $i$ is the rank of $Q_r(G)/X$. It follows that the characteristic polynomial of the contraction $Q_r(G)/X$ by a flat $X$ depends only upon $|G|, r$, and the rank of $X$. From the description of the flats in [8], it is immediate that the number of flats of each rank $i$ and cardinality $j$ also depends only upon $|G|, r, i, j$. Combining these results, we get the following proposition.

**Proposition 4.1.** If $|G| = |G'|$, then $\chi(Q_r(G); x, y) = \chi(Q_r(G'); x, y)$, so $t(Q_r(G); x, y) = t(Q_r(G'); x, y)$.

A considerably stronger form of the following result appears as Proposition 5.9 in [6] (in particular, see the discussion beginning on p. 195 of [6]).

**Proposition 4.2.** For a rank-$r$ matroid $M$ and any integer $i$ with $0 \leq i \leq r$, let $c_i$ be the largest cardinality among rank-$i$ flats of $M$. Then for each $i$ with $1 \leq i \leq r$ and each $j$ with $c_{i-1} < j \leq c_i$, we can express the number of flats of $M$ having rank $i$ and cardinality $j$ as a linear combination of the coefficients of the Tutte polynomial.

Thus, the validity of all hypotheses in Theorems 2.4, 3.2, and 3.4 can be deduced from the Tutte polynomials $t(\text{AG}(r - 1, q); x, y)$, $t(\Pi_{r+1}; x, y)$, and $t(Q_r(G); x, y)$ respectively. This gives the following corollary.
Corollary 4.3. If \( t(M; x, y) = t(AG(r - 1, q); x, y) \), then \( M \) is an affine geometry of rank \( r \) and order \( q \). Thus, if \( r > 3 \), then \( M \) is isomorphic to \( AG(r - 1, q) \).

If \( t(M; x, y) = t(Q_{r+1}; x, y) \), then \( M \) is isomorphic to \( Q_{r+1} \).

If \( t(M; x, y) = t(Q_r(G); x, y) \), then \( M \) is a Dowling lattice \( Q_r(G') \) for some group (or quasigroup, if \( r = 3 \)) \( G' \) of order \( |G| \). Thus, if \( |G| \) is a prime \( p \) and \( r > 3 \), then \( M \) is isomorphic to \( Q_r(Z_p) \) where \( Z_p \) is the cyclic group of order \( p \).

Proposition 4.1 shows that no more can be said about \( G' \) in the third case.

5. Matroid Reconstruction

There are several matroid problems analogous to the graph reconstruction problems (see [5, 12] and the references given there). We are concerned with reconstruction from hyperplanes. The deck of hyperplanes of a matroid \( M \) is the multiset of its unlabeled hyperplanes. That is, for each isomorphism type \( H \) of rank \( r(M) - 1 \), we know how many hyperplanes of \( M \) are isomorphic to \( H \). A matroid \( M \) is hyperplane reconstructible if any matroid with the same deck of hyperplanes as \( M \) is isomorphic to \( M \).

It is immediate that projective geometries of rank greater than three are hyperplane reconstructible since from the deck of hyperplanes, we can deduce the number of points and the number of copoints, and that there are no trivial lines. Projective planes of order \( q \) are hyperplane reconstructible if and only if there is a unique projective plane of order \( q \).

Brylawski [5] has shown that the Tutte polynomial of a matroid can be reconstructed from the deck of hyperplanes. From this and Corollary 4.3, we get the following corollary.

Corollary 5.1. If \( r > 3 \), then \( AG(r - 1, q) \) and \( Q_r(G) \) are hyperplane reconstructible. The partition lattice \( \Pi_{r+1} \) is hyperplane reconstructible for all ranks \( r > 1 \).

Proof. The results for affine geometries and partition lattices are clear. Let \( M \) be a matroid of rank greater than three with the same deck of hyperplanes as \( Q_r(G) \). From Brylawski’s result, we know the Tutte polynomial of \( M \), and from Corollary 4.3 we therefore know that \( M \) is a Dowling lattice. Since for \( r > 3 \), the only Dowling lattice having a hyperplane isomorphic to \( Q_{r-1}(G) \) is \( Q_r(G) \), the result follows.

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References