# CHARACTERIZATIONS OF $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ BY NUMERICAL AND POLYNOMIAL INVARIANTS

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ABSTRACT. We show that the simple matroid $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$, for $n \geq 4$ and $1 \leq k \leq n-2$, is characterized by a variety of numerical and polynomial invariants. In particular, any matroid that has the same Tutte polynomial as $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ is isomorphic to $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$.

*In Memory of Rodica Simion*

## 1. INTRODUCTION

The present paper is motivated in part by Turán's theorem [13], a simple corollary about chromatic polynomials, and the Bose-Burton theorem [3].

Recall that the Turán graph $T_{r-1}(n)$ is the complete $(r-1)$-partite graph on $n$ vertices in which each of the $r-1$ classes has either $\lfloor n/(r-1)\rfloor$ or $\lceil n/(r-1)\rceil$ vertices. The number of edges in $T_{r-1}(n)$ is denoted $t_{r-1}(n)$. Turán's theorem, Theorem 1.1, gives a sharp upper bound on the number of edges in graphs on $n$ vertices that have no complete $r$-vertex subgraph.

**Theorem 1.1.** *The greatest number of edges in any graph on $n$ vertices that has no subgraph isomorphic to $K_r$ is $t_{r-1}(n)$. Furthermore, $T_{r-1}(n)$ is the only such graph that has $t_{r-1}(n)$ edges.*

Note that if the chromatic polynomial $\chi(\Gamma;\lambda)$ of a simple graph $\Gamma$ is equal to $\chi(T_{r-1}(n);\lambda)$, it follows that $\Gamma$ has $n$ vertices, $t_{r-1}(n)$ edges, and (since the chromatic number is $r-1$) no subgraph isomorphic to $K_r$. Thus, one corollary of Turán's theorem is that the Turán graph $T_{r-1}(n)$ is characterized, within the class of simple graphs, by its chromatic polynomial in the following sense.

**Corollary 1.1.** *If $\Gamma$ is a simple graph with $\chi(\Gamma;\lambda)=\chi(T_{r-1}(n);\lambda)$, then $\Gamma$ is isomorphic to $T_{r-1}(n)$.*

The Bose-Burton theorem, Theorem 1.2, is a matroid counterpart of Turán's theorem. In this theorem, projective geometries play the role that complete graphs play in Turán's theorem.

**Theorem 1.2.** *Assume that $M$ is a subgeometry of $\mathrm{PG}(n-1,q)$ that has no subgeometry isomorphic to $\mathrm{PG}(m-1,q)$. Then the number of points in $M$ is at most*

$$\frac{q^n-q^{n-m+1}}{q-1}.$$

*Also, $M$ has $(q^n-q^{n-m+1})/(q-1)$ points if and only if $M$ is isomorphic to $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$, the deletion of $\mathrm{PG}(n-m,q)$ from $\mathrm{PG}(n-1,q)$.*

The goal of this paper is to prove several characterizations of the matroid $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$, or, equivalently, $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$. (The latter notation unclutters many expressions we will encounter.) The matroid invariants used in these characterizations are reviewed in Section 2. Section 3 contains variations on the theme that $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ is characterized, within the class of simple matroids that are representable over $\mathrm{GF}(q)$, by its characteristic polynomial. In Section 4, we prove that $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ is characterized by numerical information about the flats of ranks

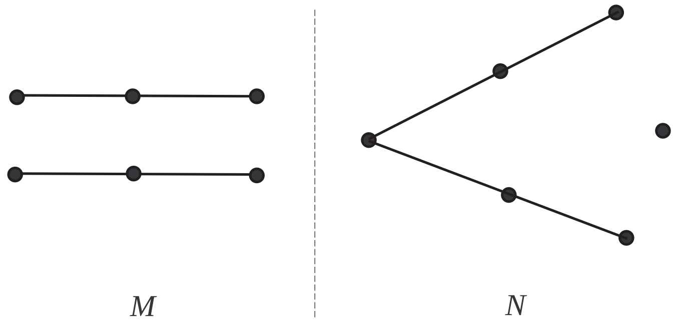


FIGURE 1. Two matroids that have the same Tutte polynomial.

$n-1$, $n-2$, and $n-k+1$. Theorem 5.1 is our main result: the geometry $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ is characterized, within the class of *all* matroids, by its Tutte polynomial. In Section 6, we show that $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ is hyperplane reconstructible and deletion reconstructible.

One striking feature of Theorem 5.1 is that it is rare to be able to deduce representability properties of a matroid from the Tutte polynomial. The key to this is the following result from [1], which plays a crucial role throughout this paper.

**Proposition 1.1.** *For any integer $n$ with $n > 3$ and any prime power $q$, any simple matroid of rank $n$ that has no minor isomorphic to $U_{2,q+2}$, the $(q+2)$-point line, and that has at least $q^{n-1}$ points is representable over $\mathrm{GF}(q)$. Any simple matroid of rank $3$ that has no minor isomorphic to the $(q+2)$-point line and that has at least $q^2$ points is a restriction of some projective plane of order $q$.*

Proposition 1.1 plays a key role in the proofs of several earlier theorems that are generalized in the present paper. This proposition plays an important role in the proof that the affine geometry $\mathrm{AG}(n-1,q)$, for $n$ exceeding 3, is characterized by its Tutte polynomial (see Corollary 4.3, as well as Section 2, of [2]). The proofs of the concluding results in Section 5.6 of [10] use Proposition 1.1; these results foreshadow those we derive in Section 3.

We follow the notation and terminology for matroid theory in [12] with the following additions. An *embedding* of a matroid $M$, on the ground set $S$, into a matroid $N$, on the ground set $T$, is an injection $\phi : S \to T$ such that the map $\phi : S \to \phi(S)$ is an isomorphism of $M$ onto the restriction $N|\phi(S)$. In a matroid of rank $n$, flats that have rank $n-1$ are called *copoints* (or hyperplanes) and flats that have rank $n-2$ are called *colines*. We refer to simple matroids as *geometries* (short for combinatorial geometries). Restrictions of geometries are called *subgeometries*. We draw the reader's attention to the fact that, consistent with [12], we use the term point for a rank-1 flat of a matroid; thus, a matroid has fewer points than elements unless it is simple.

## 2. MATROID INVARIANTS

Several matroid polynomials, and other invariants that can be derived from them, play a key role in this paper. In this section, we review basic facts about the Tutte polynomial, the characteristic polynomial, the weighted characteristic polynomial, and the critical exponent. For more information on these topics, see [6, 7, 10].

Recall that the *Tutte polynomial* $t(M;x,y)$ of a matroid $M$ on the ground set $S$ is given by

$$t(M;x,y) = \sum_{A\subseteq S} (x-1)^{r(M)-r(A)}(y-1)^{|A|-r(A)}.$$

Non-isomorphic matroids may have the same Tutte polynomial; for instance, the matroids shown in Figure 1 have the following Tutte polynomial.

$$(x-1)^3+6(x-1)^2+15(x-1)+18+2(x-1)(y-1)+15(y-1)+6(y-1)^2+(y-1)^3$$

Nevertheless, from the Tutte polynomial one can determine much information about a matroid. For instance, we have the following proposition, due to Brylawski, which we use in a crucial way in Section 5. A considerably stronger form of this result appears as Proposition 5.9 in [6] (see the discussion beginning on p. 195 of [6]).

**Proposition 2.1.** *Let $M$ be a rank-$n$ matroid. For each integer $i$ with $0 \le i \le n$, let $c_i$ be the largest cardinality among rank-$i$ flats of $M$. Then for each integer $i$ with $1 \le i \le n$ and each integer $j$ with $c_{i-1} < j \le c_i$, the number of flats of $M$ that have rank $i$ and cardinality $j$ can be computed from the Tutte polynomial $t(M;x,y)$.*

Our main result, Theorem 5.1, asserts that $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ is characterized by its Tutte polynomial. We use the following proposition from [4] to construct examples to show that this property need not hold for other geometries, indeed, not even for geometries that are formed by deleting relatively few elements from $\mathrm{PG}(n-1,q)$. (The statement below follows from, but does not capture the full power of, Proposition 5.9 in [4].)

**Proposition 2.2.** *Assume that $M$ and $N$ are geometries that are representable over $\mathrm{GF}(q)$ and that have the same Tutte polynomial. Let $S$ and $T$ be subsets of the ground set of $\mathrm{PG}(n-1,q)$ so that the restrictions $\mathrm{PG}(n-1,q)|S$ and $\mathrm{PG}(n-1,q)|T$ are isomorphic to $M$ and $N$, respectively. Then the deletions $\mathrm{PG}(n-1,q)\backslash S$ and $\mathrm{PG}(n-1,q)\backslash T$ have the same Tutte polynomial.*

The matroids $M$ and $N$ of Figure 1 are representable over the finite field $\mathrm{GF}(q)$ for every prime power $q$ with $q > 3$. Since $M$ and $N$ have the same Tutte polynomial, Proposition 2.2 implies that if $\mathrm{PG}(n-1,q)|S$ and $\mathrm{PG}(n-1,q)|T$ are isomorphic to $M$ and $N$, respectively, then the deletions $\mathrm{PG}(n-1,q)\backslash S$ and $\mathrm{PG}(n-1,q)\backslash T$ have the same Tutte polynomial. These deletions are not isomorphic: the two $(q-2)$-point lines of $\mathrm{PG}(n-1,q)\backslash S$ intersect in a point while those of $\mathrm{PG}(n-1,q)\backslash T$ do not. Thus, for every prime power $q$ with $q > 3$ and every integer $n$ with $n \ge 3$, there are at least two non-isomorphic six-element deletions of $\mathrm{PG}(n-1,q)$ that have the same Tutte polynomial.

The *characteristic polynomial* $\chi(M;\lambda)$ of $M$ is, up to sign, a special evaluation of the Tutte polynomial of $M$:

$$\begin{aligned}\chi(M;\lambda) &= (-1)^{r(M)}t(M;1-\lambda,0)\\ &= \sum_{X\subseteq S}(-1)^{|X|}\lambda^{r(M)-r(X)}.\end{aligned}$$

Equivalently, $\chi(M;\lambda)$ can be written as the sum

$$\sum_{\text{flats } X}\mu(\emptyset,X)\lambda^{r(M)-r(X)},$$

where $\mu$ is the Möbius function of $M$. In particular, the absolute value of the coefficient of $\lambda^{r(M)-1}$ is the number of points of $M$. It is easy to show that if $M$ has loops, then $\chi(M;\lambda)$ is the zero polynomial. Also, if $M$ has no loops, then $M$ and its simplification have the same characteristic polynomial.

The characteristic polynomial is the matroid counterpart of the chromatic polynomial in graph theory: the chromatic polynomial $\chi(\Gamma;\lambda)$ of a graph $\Gamma$ is $\lambda^{\omega(\Gamma)}$ times the characteristic polynomial $\chi(M(\Gamma);\lambda)$ of the cycle matroid $M(\Gamma)$ of $\Gamma$, where $\omega(\Gamma)$ is the number of components of $\Gamma$.

Brylawski [4] defined the *weighted characteristic polynomial* $\overline{\chi}(M;x,y)$ of a matroid $M$ to be

$$\overline{\chi}(M;x,y) = \sum_{\text{flats } F} x^{|F|}\chi(M/F;y).$$

(This sum could be taken over all subsets $F$ of the ground set of $M$ since if $F$ is not a flat, then the contraction $M/F$ has loops, and so $\chi(M/F;y)$ is 0. The weighted characteristic polynomial is, upon switching the variables, the coboundary polynomial of [8]. See also Section 6.3.F of [7], where the notation $\overline{\chi}(M;x,y)$ is used for the coboundary polynomial.) The following formulas are well-known and easy to prove.

$$t(M;x,y)=\frac{\overline{\chi}(M;y,(x-1)(y-1))}{(y-1)^{r(M)}}$$

$$\overline{\chi}(M;x,y)=(x-1)^{r(M)}t(M;\frac{y}{x-1}+1,x)$$

Thus, one can obtain $t(M;x,y)$ from $\overline{\chi}(M;x,y)$ and conversely.

Assume the geometry $M$ is representable over $\mathrm{GF}(q)$. The *critical exponent of* $M$ *over* $\mathrm{GF}(q)$ is the least positive integer $c$ so that $q^c$ is not a root of the characteristic polynomial $\chi(M;\lambda)$. The critical exponent derives much of its significance from the following theorem, which is a special case of the fundamental theorem on critical exponents due to Crapo and Rota (see [9, Chapter 16]).

**Theorem 2.1.** *Assume that the geometry $M$ is representable over $\mathrm{GF}(q)$ and has critical exponent $c$ over $\mathrm{GF}(q)$. Let $n$ be any integer not less than the rank of $M$. For any embedding of $M$ in $\mathrm{PG}(n-1,q)$, the least codimension of a subspace of $\mathrm{PG}(n-1,q)$ disjoint from the image of $M$ is $c$.*

Corollary 2.1 follows immediately from Theorem 2.1.

**Corollary 2.1.** *The maximum number of points in a subgeometry of $\mathrm{PG}(n-1,q)$ that has critical exponent $n-k$ over $\mathrm{GF}(q)$ is*

$$\frac{q^n-q^k}{q-1}.$$

*Moreover, $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ is the only subgeometry of $\mathrm{PG}(n-1,q)$ that has critical exponent $n-k$ over $\mathrm{GF}(q)$ and has $(q^n-q^k)/(q-1)$ points.*

## 3. Characterizations based on characteristic polynomials

In this section we show that, subject to additional hypotheses, the geometry $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ is determined by its characteristic polynomial. Theorem 5.16 in [10] includes results of this type for projective geometries ($k=0$); the discussion after Theorem 5.16 in [10] treats results of this type for affine geometries ($k=n-1$). Thus, we assume $1\le k\le n-2$.

**Theorem 3.1.** *Assume integers $n$ and $k$ satisfy $n\ge 3$ and $1\le k\le n-2$. Let $M$ be a geometry that has the same characteristic polynomial as the geometry $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ and for which each coline is contained in at most $q+1$ copoints. If $n>3$, then $M$ is isomorphic to $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$. If $n=3$, then $M$ is isomorphic to a single-element deletion of some projective plane of order $q$.*

*Proof.* Since $M$ and $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ have the same characteristic polynomial, it follows that $M$ has $(q^n-q^k)/(q-1)$ points. By the assumption that each coline is contained in at most $q+1$ copoints, together with the Scum Theorem [12, Theorem 3.3.7], it follows that $M$ has no minor isomorphic to the $(q+2)$-point line. Therefore by Proposition 1.1, if $n$ exceeds 3, then $M$ is representable over $\mathrm{GF}(q)$, while if $n$ is 3, then $M$ is isomorphic to a restriction of some projective plane $\Pi$ of order $q$. In the case $n=3$, note that $k$ must be 1, so $M$ is isomorphic to a single-element deletion of $\Pi$. To complete the argument in the case $n>3$, note that from the characteristic polynomial of $M$ we know that the critical exponent of $M$ over $\mathrm{GF}(q)$ is $n-k$. The result now follows from Corollary 2.1. $\square$

In analogy with Corollary 1.1, the geometry $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ is characterized, within the class of $\mathrm{GF}(q)$-representable geometries, by its characteristic polynomial.

**Corollary 3.1.** *If the geometry $M$ is representable over $\mathrm{GF}(q)$ and has the same characteristic polynomial as $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$, then $M$ is isomorphic to $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$.*

One can replace the assumption about colines in Theorem 3.1 by an assumption about characteristic polynomials of contractions, as in the next theorem.

**Theorem 3.2.** *Assume integers $n$ and $k$ satisfy $n \geq 3$ and $1 \leq k \leq n-2$. Let $G$ be the geometry $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$. Assume that $M$ is a geometry such that $\chi(G;\lambda)=\chi(M;\lambda)$ and*

$$\{\chi(G/Y;\lambda) \mid Y \text{ is a coline of } G\} = \{\chi(M/Z;\lambda) \mid Z \text{ is a coline of } M\}.$$

*If $n>3$, then $M$ is isomorphic to $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$. If $n=3$, then $M$ is isomorphic to a single-element deletion of some projective plane of order $q$.*

*Proof.* The simplification of each contraction $G/Y$, for $Y$ a coline of $G$, is a line with $q+1$ points. Thus, the absolute value of the coefficient of $\lambda$ in $\chi(G/Y;\lambda)$, and hence in $\chi(M/Z;\lambda)$ for any coline $Z$ of $M$, is $q+1$. It follows that each coline of $M$ is contained in $q+1$ copoints of $M$, so the result follows from Theorem 3.1. □

## 4. A CHARACTERIZATION BY NUMERICAL INVARIANTS

In this section we show that a few numerical facts about flats suffice to characterize the geometry $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$. We start by restating the Bose-Burton theorem to reflect our preferred notation.

**Lemma 4.1.** *Let $M$ be a subgeometry of $\mathrm{PG}(n-1,q)$ with no subgeometry isomorphic to $\mathrm{PG}(n-k,q)$. There are at most $(q^n-q^k)/(q-1)$ points in $M$. Furthermore, $M$ has $(q^n-q^k)/(q-1)$ points if and only if $M$ is isomorphic to $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$.*

We turn to a numerical characterization of $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$. In Theorem 4.1, we assume $0<k<n-2$. Section 2 of [2] treats similar results for projective and affine geometries (the cases $k=0$ and $k=n-1$). The case $k=n-2$ would require assumptions beyond the strictly numerical hypotheses of the theorem as stated.

**Theorem 4.1.** *Assume integers $n$ and $k$ satisfy $n \geq 4$ and $1 \leq k \leq n-3$, and that $M$ is a geometry of rank $n$ that satisfies the following conditions.*

(i) *The geometry $M$ has $(q^n-q^k)/(q-1)$ points.*
(ii) *No rank-$(n-k+1)$ flat of $M$ contains $(q^{n-k+1}-1)/(q-1)$ points.*
(iii) *All copoints of $M$ contain either*

$$\frac{q^{n-1}-q^k}{q-1} \quad \text{or} \quad \frac{q^{n-1}-q^{k-1}}{q-1}$$

*points.*

(iv) *All colines of $M$ contain either*

$$\frac{q^{n-2}-q^k}{q-1}, \quad \frac{q^{n-2}-q^{k-1}}{q-1},$$

*or, if $k>1$,*

$$\frac{q^{n-2}-q^{k-2}}{q-1}$$

*points.*

*Then $M$ is isomorphic to $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$.*

*Proof.* Note that it suffices to show that $M$ has no minor isomorphic to $U_{2,q+2}$. Indeed, assumption (i) and the deduction that $M$ has no minor isomorphic to the $(q+2)$-point line imply, by Proposition 1.1, that $M$ is isomorphic to a subgeometry of $\mathrm{PG}(n-1,q)$. Assumption (ii) implies that $M$ has no subgeometry isomorphic to $\mathrm{PG}(n-k,q)$, so the desired conclusion now follows from Lemma 4.1.

Thus, we want to show that $M$ has no minor isomorphic to $U_{2,q+2}$, or, equivalently, that each coline of $M$ is contained in at most $q+1$ copoints of $M$. Since copoints have at least $(q^{n-1}-q^k)/(q-1)$ points, and so at least $(q^{n-1}-q^k)/(q-1)-|C|$ points not contained in a given coline $C$, a coline $C$ is contained in at most $q+1$ copoints whenever the inequality

$$|C|+(q+2)\left(\frac{q^{n-1}-q^k}{q-1}-|C|\right)>\frac{q^n-q^k}{q-1} \tag{1}$$

holds. Simple calculations show that for $k$ in the range $1\le k\le n-3$, inequality (1) holds for all colines that have either $(q^{n-2}-q^k)/(q-1)$ or $(q^{n-2}-q^{k-1})/(q-1)$ points; inequality (1) holds for colines that have $(q^{n-2}-q^{k-2})/(q-1)$ points if $k<n-3$ or $q>2$. In the sole remaining case, $k=n-3$ and $q=2$, the colines of interest contain $2^{n-2}-2^{n-5}$ elements. If such a coline is contained in $\alpha$ copoints that have $2^{n-1}-2^{n-4}$ elements and in $\beta$ copoints that have $2^{n-1}-2^{n-3}$ elements, we get the equation

$$\begin{aligned}2^n-2^{n-3}=2^{n-2}-2^{n-5}&+\alpha\big((2^{n-1}-2^{n-4})-(2^{n-2}-2^{n-5})\big)\\&+\beta\big((2^{n-1}-2^{n-3})-(2^{n-2}-2^{n-5})\big).\end{aligned}$$

Dividing each term by $2^{n-5}$ and simplifying yields $21=7\alpha+5\beta$. Thus, the only solution is $\alpha=3$ and $\beta=0$, so a coline that contains $2^{n-2}-2^{n-5}$ elements is contained in at most three copoints, as desired. □

## 5. THE CHARACTERIZATION BY THE TUTTE POLYNOMIAL

In this section we show that, for $n>3$, any matroid that has the same Tutte polynomial as the geometry $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ is isomorphic to $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$; this is extended, as in Section 3, to the case $n=3$. This was proven for projective and affine geometries in [2], so we assume $1\le k\le n-2$. Recall that Theorem 5.1 stands in contrast to the examples that follow Proposition 2.2 above.

**Theorem 5.1.** *Assume integers $n$ and $k$ satisfy $n\ge 3$ and $1\le k\le n-2$. Assume that $M$ is a matroid that has the same Tutte polynomial as the geometry $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$. If $n\ge 4$, then $M$ is isomorphic to $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$. If $n=3$, then $M$ is isomorphic to some single-element deletion of some projective plane of order $q$.*

*Proof.* To prove this result, we show that for $k\le n-3$, the hypotheses of Theorem 4.1 can be verified from the Tutte polynomial of $M$, while for $k=n-2$, the hypotheses of Theorem 3.1 can be verified from this Tutte polynomial.

Since $M$ has the same Tutte polynomial as $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$, and both the number of elements and the number of points can be computed from $t(M;x,y)$, it follows that $M$ is simple, that is, $M$ is a geometry. Since all rank-1 flats of $M$ are singletons, Proposition 2.1 allows us to find the cardinalities of lines: all lines of $M$ have either $q$ or $q+1$ points.

We now show that for $q>2$, we can compute the ranks and cardinalities of all flats of $M$ from $t(M;x,y)$. For this, by Proposition 2.1, it suffices to show that for each rank $i$, the number of elements in any flat of rank $i$ exceeds that in any flat of rank $i-1$. We induct on $i$. The cases $i=1$ and $i=2$ were addressed in the previous paragraph. By the induction hypothesis, we know, in particular, that each flat of rank $i-1$ has between $q^{i-2}$ and $(q^{i-1}-1)/(q-1)$ elements. Let $F$ be a flat of rank $i$, let $F'$ be a flat of rank $i-1$ that is contained in $F$, and let $x$ be a point in $F-F'$. Since the lines $\mathrm{cl}(\{x,y\})$, as $y$ ranges over the $|F'|$ points of $F'$, each contain at least $q-1$ points apart from the common point $x$, we get the inequality $|F|\ge(q-1)|F'|+1$. Since $|F'|\ge q^{i-2}$, we have $|F|\ge(q-1)q^{i-2}+1$. Since $q>2$, we have $(q-1)q^{i-2}+1 > (q^{i-1}-1)/(q-1)$. Thus, $F$ has more elements than any flat of rank $i-1$, as desired.

Thus, for $q>2$ we can compute the cardinality and rank of each flat of $M$ from $t(M;x,y)$. Since $M$ and $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ have the same Tutte polynomial, it follows that we can verify the hypotheses of Theorem 4.1. This completes the proof in the case $q>2$ and $k<n-2$.

We will show that, as for $q>2$, in the case $q=2$ we can compute the rank and cardinality of each flat of $M$ from $t(M;x,y)$. Again by Theorem 4.1, this will complete the proof in the case $q=2$ and $k<n-2$. The key step, which we address next, is to find the isomorphism types of the planes of $M$. In addition to giving the cardinality of each plane of $M$, this will allow us to deduce the cardinalities of all flats of $M$ of higher rank; this will be accomplished with an inductive argument akin to that two paragraphs above, with planes now playing the role lines played before.

We start by collecting the information about flats of low rank that can be deduced from Proposition 2.1. As is true of $\mathrm{PG}(n-1,2)\backslash\mathrm{PG}(k-1,2)$, the geometry $M$ has $2^n-2^k$ points. There are $(2^n-2^k)(2^k-1)/2$ lines that contain two points and $(2^n-2^k)(2^n-2^{k+1})/(3\cdot 2)$ lines that contain three points; there are no other lines of $M$. The numbers $N_4$, $N_6$, and $N_7$ of 4-, 6-, and 7-point planes of $M$, respectively, are given as follows.

(2) $$N_4 = \frac{(2^k-1)(2^k-2)(2^n-2^k)}{3\cdot 2\cdot 4}$$

(3) $$N_6 = \frac{(2^k-1)(2^n-2^k)(2^n-2^{k+1})}{6\cdot 4}$$

(4) $$N_7 = \frac{(2^n-2^k)(2^n-2^{k+1})(2^n-2^{k+2})}{7\cdot 6\cdot 4}$$

There are no 5-point planes in $M$. We cannot deduce the number of 3-point planes directly from Proposition 2.1.

We first deduce that all 6-point planes of $M$ are isomorphic to $M(K_4)$, the cycle matroid of the complete graph on four vertices, and that all 7-point planes are isomorphic to $\mathrm{PG}(2,2)$. Let $p_1,p_2,\ldots,p_t$ be the points of $M$, where $t=2^n-2^k$. Let $a_i$ be the number of 3-point lines of $M$ that contain the point $p_i$. Thus,

(5) $$\sum_{i=1}^{t} a_i = \frac{(2^n-2^k)(2^n-2^{k+1})}{2}$$

since this is three times the number of 3-point lines. Let $c_i$ be the number of 6-point planes of $M$ that contain $p_i$ and that contain two 3-point lines through $p_i$. Thus,

(6) $$\sum_{i=1}^{t} c_i \le 6\cdot N_6$$

with equality if and only if every point in every 6-point plane is on two 3-point lines in the plane, that is, if and only if every 6-point plane is isomorphic to $M(K_4)$. Let $d_i$ be the number of 7-point planes of $M$ that contain $p_i$ and that contain three 3-point lines through $p_i$. Thus,

(7) $$\sum_{i=1}^{t} d_i \le 7\cdot N_7$$

with equality if and only if every point in every 7-point plane is on three 3-point lines in the plane, i.e., if and only if every 7-point plane is isomorphic to $\mathrm{PG}(2,2)$. Finally, let $e_i$ be the number of 7-point planes that contain $p_i$ and that contain exactly two 3-point lines through $p_i$. (There may be other 6- or 7-point planes that contain $p_i$.) Now $\sum_{i=1}^{t}(d_i+e_i) \le 7\cdot N_7$, so

$$\sum_{i=1}^{t} d_i \le 7\cdot N_7 - \sum_{i=1}^{t} e_i. \tag{8}$$

Note that we have

$$\binom{a_i}{2} = c_i + 3d_i + e_i$$

since the planes counted by $d_i$ give rise to three pairs of 3-point lines through $p_i$, while those counted by $c_i$ and $e_i$ give rise to only one pair of 3-point lines through $p_i$. Since $\binom{a_i}{2} = a_i^2/2 - a_i/2$, we get

$$\sum_{i=1}^{t} a_i^2 = 2\sum_{i=1}^{t}(c_i+3d_i+e_i) + \sum_{i=1}^{t} a_i.$$

By using inequalities (6) and (8), together with equation (5), we get

$$\sum_{i=1}^{t} a_i^2 \le 12\cdot N_6 + 2\cdot 3\left(7\cdot N_7 - \sum_{i=1}^{t} e_i\right) + 2\left(\sum_{i=1}^{t} e_i\right) + \frac{(2^n-2^k)(2^n-2^{k+1})}{2},$$

or

$$\sum_{i=1}^{t} a_i^2 \le 12\cdot N_6 + 42\cdot N_7 - 4\left(\sum_{i=1}^{t} e_i\right) + \frac{(2^n-2^k)(2^n-2^{k+1})}{2}.$$

A calculation using equations (3) and (4) reduces the last inequality to the following inequality.

$$\sum_{i=1}^{t} a_i^2 \le \frac{(2^n-2^k)(2^n-2^{k+1})^2}{4} - 4\left(\sum_{i=1}^{t} e_i\right) \tag{9}$$

By the Cauchy-Schwarz inequality, we have

$$\frac{1}{t}\left(\sum_{i=1}^{t} a_i\right)^2 \le \sum_{i=1}^{t} a_i^2.$$

Using this inequality, equation (5), and the fact that $t$ is $2^n-2^k$, we get

$$\frac{(2^n-2^k)(2^n-2^{k+1})^2}{4} \le \sum_{i=1}^{t} a_i^2. \tag{10}$$

By comparing inequalities (9) and (10), we deduce that $\sum e_i$ is zero, so all $e_i$ are zero. Furthermore, equality must hold in inequality (9), and hence in inequalities (6) and (7). Thus, as claimed, all 6-point planes of $M$ are isomorphic to $M(K_4)$ and all 7-point planes of $M$ are isomorphic to $\mathrm{PG}(2,2)$.

We next deduce that all 4-point planes of $M$ are isomorphic to the uniform matroid $U_{3,4}$, that is, to $\mathrm{AG}(2,2)$. Let $\ell_1,\ell_2,\ldots,\ell_s$ be the 3-point lines of $M$, so $s=(2^n-2^k)(2^n-2^{k+1})/(3\cdot 2)$. Assume that the line $\ell_i$ is in $f_i$ planes that have four points, $g_i$ planes that have six points, and $h_i$ planes that have seven points. To show that all 4-point planes of $M$ are isomorphic to $\mathrm{AG}(2,2)$, it suffices to show that $f_1,f_2,\ldots,f_s$ are zero. The 4-, 6-, and 7-point planes that contain $\ell_i$ partition the points of $M\backslash\ell_i$ into sets of size one, three, and four, respectively, so

$$f_i + 3g_i + 4h_i = 2^n - 2^k - 3.$$

Therefore

$$\sum_{i=1}^{s}(f_i+3g_i+4h_i) = \frac{(2^n-2^k)(2^n-2^{k+1})}{3\cdot 2}(2^n-2^k-3). \tag{11}$$

All 6-point planes of $M$ are isomorphic to $M(K_4)$, which has four 3-point lines, so $\sum g_i = 4 \cdot N_6$; similarly, since all 7-point planes of $M$ are isomorphic to $\mathrm{PG}(2,2)$, which has seven 3-point lines, we have $\sum h_i = 7 \cdot N_7$. Using these sums and equations (3) and (4), one can evaluate the second and third terms on the left side of equation (11); from this calculation, it follows that $\sum f_i$ is zero, so all $f_i$ are zero. This completes the proof that all 4-point planes of $M$ are isomorphic to $\mathrm{AG}(2,2)$.

Since there are $2^n - 2^k$ points in $M$ and $(2^n-2^k)(2^n-2^{k+1})/(3\cdot 2)$ lines that have three points, there are

$$\binom{2^n-2^k}{3} - \frac{(2^n-2^k)(2^n-2^{k+1})}{3\cdot 2}$$

three-element independent sets in $M$. By what we know about 4-, 6-, and 7-point planes, the number of three-element independent sets in $M$ is also $4\cdot N_4 + 16\cdot N_6 + 28\cdot N_7$ plus the number of three-point planes of $M$. A simple calculation using equations (2)–(4) gives the equality

$$\binom{2^n-2^k}{3} - \frac{(2^n-2^k)(2^n-2^{k+1})}{3\cdot 2} = 4\cdot N_4 + 16\cdot N_6 + 28\cdot N_7.$$

It follows that $M$ has no three-point planes.

We have deduced, from $t(M;x,y)$, the isomorphism type of each plane of $M$. We next show that we can compute the rank and cardinality of each flat of $M$ from $t(M;x,y)$. For this, by Proposition 2.1, it suffices to show that for each rank $i$, the number of elements in any flat of rank $i$ exceeds that in any flat of rank $i-1$. We induct on $i$. The cases $i \le 3$ have been established. By the induction hypothesis, we know, in particular, that each flat of rank $i-1$ has between $2^{i-2}$ and $2^{i-1}-1$ elements. Let $F$ be a flat of rank $i$, let $F'$ be a flat of rank $i-1$ that is contained in $F$, and let $x$ be a point in $F-F'$. If each line $\mathrm{cl}(\{x,y\})$, for $y \in F'$, contains three points, then we have

$$|F| \ge 1 + 2|F'| \ge 1 + 2\cdot 2^{i-2},$$

so $F$ contains more points than any flat of rank $i-1$, as desired. Thus, we may assume that $y$ is a point of $F'$ for which the line $\mathrm{cl}(\{x,y\})$ contains just two points. Let the numbers of 2- and 3-point lines in $F'$ that contain $y$ be $\alpha$ and $\beta$, respectively. Thus, $|F'| = 1+\alpha+2\beta$. Since $\mathrm{cl}(\{x,y\})$ is a 2-point line, it follows from the structure of the planes of $M$ that if $\ell$ is a 2-point line of $F'$ that contains $y$, then the plane $\mathrm{cl}(\ell\cup\{x\})$ is isomorphic to $\mathrm{AG}(2,2)$, and so contains two points not in the line $\mathrm{cl}(\{x,y\})$. Likewise, if $\ell$ is a 3-point line of $F'$ that contains $y$, then the plane $\mathrm{cl}(\ell\cup\{x\})$ is isomorphic to $M(K_4)$, and so contains four points not in the line $\mathrm{cl}(\{x,y\})$. Thus, $|F| \ge 2+2\alpha+4\beta$. Since $\alpha+2\beta = |F'|-1 \ge 2^{i-2}-1$, we get the inequality $|F| \ge 2^{i-1}$, so, as needed, $F$ contains more points than any flat of rank $i-1$. As observed earlier, this completes the proof of the theorem in the case $q=2$ and $k<n-2$.

We now address the case $k=n-2$ for all $q$. In this case, colines contain either $q^{n-3}$ or $q^{n-3}+q^{n-4}$ elements. Since no flats other than colines have these cardinalities, the coefficients of $x^{q^{n-3}}$ and $x^{q^{n-3}+q^{n-4}}$ in the weighted characteristic polynomial of $M$ are the sums of the characteristic polynomials of the contractions of $M$ by colines that contain $q^{n-3}$ and $q^{n-3}+q^{n-4}$ elements, respectively. Since $M$ has the same weighted characteristic polynomial as $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-3,q)$, we know these sums of characteristic polynomials. In particular, since each coline of $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-3,q)$ is contained in exactly $q+1$ copoints, it follows that the coefficients of $x^{q^{n-3}}y$ and $x^{q^{n-3}+q^{n-4}}y$ in the weighted characteristic polynomial of $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-3,q)$, and therefore in the weighted characteristic polynomial of $M$, are $q+1$ times the number of colines that have $q^{n-3}$ and $q^{n-3}+q^{n-4}$ points, respectively. In effect, we know that the average number of copoints of $M$ that contain a given coline of $M$ is $q+1$. Therefore to show that each coline of $M$ is contained in at most $q+1$ copoints, it suffices to show that each coline of $M$ is contained in at least $q+1$ copoints. This is what we address next.

Assume that a coline that has $q^{n-3}$ elements is contained in $\alpha$ copoints that contain $q^{n-2}+q^{n-3}$ elements and in $\beta$ copoints that contain $q^{n-2}$ elements. Our assumptions lead to the equation

$$q^{n-1}+q^{n-2}=q^{n-3}+\alpha\big((q^{n-2}+q^{n-3})-q^{n-3}\big)+\beta\big(q^{n-2}-q^{n-3}\big). \tag{12}$$

Dividing each term by $q^{n-3}$ and simplifying yields

$$q^2+q=1+\alpha q+\beta(q-1). \tag{13}$$

We want to show that $\alpha+\beta$ is at least $q+1$. The minimum value of $\alpha+\beta$ occurs when $\alpha$ is as large as possible and $\beta$ is as small as possible. There is no solution with $\beta=0$; however, $\beta=1$ gives the solution $\alpha=q$. Since this solution minimizes $\alpha+\beta$, all solutions $\alpha$, $\beta$ satisfy the inequality $\alpha+\beta\geq q+1$, as desired. The same type of argument, starting with a coline that contains $q^{n-3}+q^{n-4}$ elements, leads to the following equation in place of equation (12).

$$\begin{aligned} q^{n-1}+q^{n-2}=q^{n-3}+q^{n-4}&+\alpha\big((q^{n-2}+q^{n-3})-(q^{n-3}+q^{n-4})\big)\\ &+\beta\big(q^{n-2}-(q^{n-3}+q^{n-4})\big)\end{aligned}$$

Similarly, we get the following equation in place of equation (13).

$$q^3+q^2=q+1+\alpha(q^2-1)+\beta(q^2-q-1)$$

The solution with $\alpha+\beta$ minimal is $\beta=0$ and $\alpha=q+1$, so the inequality $\alpha+\beta\geq q+1$ holds for all solutions, as desired.

Since each coline of $M$ is contained in at most $q+1$ hyperplanes of $M$, and since $M$ has the same characteristic polynomial as the geometry $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-3,q)$, the result follows from Theorem 3.1. $\square$

## 6. An application to matroid reconstruction

Matroid counterparts of graph reconstruction problems have proven to be of interest (see [5, 11] and the references there). We are concerned with reconstruction from hyperplanes and from single-element deletions. The *deck of hyperplanes* of a matroid $M$ is the multiset of unlabeled hyperplanes. That is, for each isomorphism type $H$ of rank $r(M)-1$, we know the number of hyperplanes of $M$ that are isomorphic to $H$. A matroid $M$ is *hyperplane reconstructible* if any matroid that has the same deck of hyperplanes as $M$ is isomorphic to $M$. Similarly, the *deck of single-element deletions* of a matroid $M$ is the multiset of unlabeled single-element deletions. A matroid $M$ is *deletion reconstructible* if any matroid that has the same deck of single-element deletions as $M$ is isomorphic to $M$. Matroids that are hyperplane reconstructible are also deletion reconstructible (see [11]). Projective and affine geometries of rank four or more are known to be hyperplane reconstructible, as are the cycle matroids of complete graphs, and, more generally, Dowling lattices of rank four or more (see [2]). Brylawski [5] showed that the Tutte polynomial of a matroid can be computed from the deck of hyperplanes. From this and Theorem 5.1, we get the following corollary.

**Corollary 6.1.** *The geometry* $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ *is both hyperplane reconstructible and deletion reconstructible if* $n>3$ *and* $1\leq k\leq n-2$.

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