SIMPLE MATROIDS WITH BOUNDED COCIRCUIT SIZE

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ABSTRACT. We examine the specialization to simple matroids of certain problems in extremal matroid theory that are concerned with bounded cocircuit size. Assume that each cocircuit of a simple matroid M has at most d elements. We show that if M has rank 3, then M has at most $d + \lfloor\sqrt{d}\rfloor + 1$ points and we classify the rank-3 simple matroids M that have exactly $d + \lfloor\sqrt{d}\rfloor + 1$ points. We show that if M is a connected matroid of rank 4 and d is q^3 with $q > 1$, then M has at most $q^3 + q^2 + q + 1$ points; this upper bound is strict unless q is a prime power, in which case the only such matroid with exactly $q^3 + q^2 + q + 1$ points is the projective geometry $\mathrm{PG}(3, q)$. We also show that if d is q^4 for a positive integer q and if M has rank 5 and is vertically 5-connected, then M has at most $q^4 + q^3 + q^2 + q + 1$ points; this upper bound is strict unless q is a prime power, in which case $\mathrm{PG}(4, q)$ is the only such matroid that attains this bound.

1. INTRODUCTION

Problems in matroid theory that revolve around bounded cocircuit size have received considerable attention recently. This branch of extremal matroid theory started with the following question posed by Robin Thomas: Is there an upper bound on the number of elements in a connected matroid in which each circuit has at most c elements and each cocircuit has at most d elements? The upper bound of $2^{c+d} - 1$ proven by Lovász, Schrijver, and Seymour (see [8]) answered this question in the affirmative. Recently, Lemos and Oxley [6] have proven the following sharp upper bound for the number of elements in such matroids. (See [6] and the references there for more on this problem.)

Theorem 1.1. *Let M be a connected matroid with at least two elements. If a largest circuit of M has c elements and a largest cocircuit of M has d elements, then M has at most $cd/2$ elements.*

A related problem is this: Find a sharp upper bound on the number of elements in a connected rank-r matroid in which each cocircuit has at most d elements. This problem is addressed in the following theorem from [1].

Theorem 1.2. *Let M be a connected matroid of rank $r > 0$ with no cocircuit having more than d elements.*

(i) *If d is even, then M has at most $\frac{1}{2}(r + 1)d$ elements.*
(ii) *If d is odd, then M has at most $\frac{1}{2}(r + 1)(d - 1) + 1$ elements.*

Since each circuit in a rank-r matroid has at most $r + 1$ elements, part (i) of Theorem 1.2 follows from Theorem 1.1. However, note that part (ii) of Theorem 1.2 gives a stronger bound than can be deduced from Theorem 1.1. Indeed, the bounds in Theorem 1.2 are sharp for all values of r and d (see [1]).

As is true of many examples that show that upper bounds in this area of extremal matroid theory are sharp, the examples given in [1] that meet the bounds in Theorem 1.2 contain many nonsingleton parallel classes. This suggests it may be interesting to study such problems for simple matroids; the following elementary result provides further motivation for this line of research.

Theorem 1.3. *Let $q > 1$ be an integer. Let G be a rank-3 simple matroid in which each cocircuit has at most q^2 elements. Then the number of points in G is at most $q^2 + q + 1$. Furthermore, G has $q^2 + q + 1$ points if and only if G is a projective plane of order q.*

Proof. If G had at least $q^2 + q + 2$ points, then, since cocircuits have at most q^2 points, each line of G would contain at least $q+2$ points. Consider a line ℓ of G and a point x not in ℓ. The lines $\mathrm{cl}(\{x, y\})$, as y ranges over the points of ℓ, would contain at least $1 + (q + 2)q$ points in the cocircuit complementary to ℓ, contrary to the restriction on the cardinalities of cocircuits. Thus G has at most $q^2 + q + 1$ points.

Assume G has $q^2 + q + 1$ points. Arguing as above, we see that each line of G contains exactly $q + 1$ points and each point of G is in exactly $q + 1$ lines. From these observations, it follows that each pair of lines of M has nonempty intersection. Thus G satisfies the axioms for projective planes (see, e.g., Section 12.1 of [9]), so G is a projective plane of order q. □

This paper studies problems in the spirit of Theorems 1.2 and 1.3. In the following outline of the paper, d is an upper bound on the number of elements in each cocircuit of the matroids under consideration. Theorem 2.3 strengthens the conclusion of Theorem 1.2 in the case of simple matroids by giving the following upper bounds on the number of elements in connected simple matroids: $\frac{1}{2}rd+1$ if d is even and $\frac{1}{2}r(d - 1) + 2$ if d is odd. The main parts of the paper, Sections 3 through 5, provide further strengthenings of these bounds for simple matroids of small rank. Theorem 3.1 generalizes Theorem 1.3 by removing the assumption that d is a square: it is shown that rank-3 simple matroids have at most $d + \lfloor\sqrt{d}\rfloor + 1$ points; the rank-3 simple matroids with $d + \lfloor\sqrt{d}\rfloor + 1$ points are classified. Theorem 4.3 provides a counterpart to Theorem 1.3 for rank 4 by showing that rank-4 connected simple matroids for which d is a perfect cube, say q^3, have at most $q^3 + q^2 + q + 1$ points; furthermore, the projective geometry $\mathrm{PG}(3, q)$ is the only such matroid with exactly $q^3 + q^2 + q + 1$ points. Theorem 4.4 carries this further by showing that the number of points in rank-4 connected simple matroids is strictly less than $d + d^{2/3} + d^{1/3} + 1$ if d is not a perfect cube. Section 5 shows that in higher ranks, 3-connectivity does not suffice to yield a counterpart to Theorem 1.3; however, Theorem 5.2 provides a counterpart to Theorem 1.3 in rank 5 under the additional hypothesis that the matroids are vertically 5-connected. It is currently an open problem whether such results hold for higher ranks. These results suggest that it is probably quite difficult to prove the sharper results one might expect when problems about bounded cocircuit size are specialized to simple matroids.

Our terminology and notation follow [7] with the following common additions: we refer to simple matroids as geometries (short for combinatorial geometries) and we refer to flats of rank $r - 2$ in a rank-r matroid as colines.

2. Stronger General Results

In this section, we give several upper bounds on the number of points in a rank-r geometry in which each cocircuit contains at most d elements. These results are stronger than those given in [1] for matroids. These results are of interest in part for comparison with the stronger results we derive in the remaining sections of this paper for geometries of small rank.

Theorem 2.2 gives stronger bounds than Theorem 1.2 by taking into account an additional parameter, namely girth. The *girth* of a matroid M that is not free is the minimum cardinality of a circuit of M. To prove Theorem 2.2, we use the fact that each connected geometry M has at least one single-element contraction M/e that is connected. This is a consequence of the following result of Oxley [7, Corollary 10.2.3], which generalizes an earlier result of Seymour.

Lemma 2.1. *Assume M is a connected matroid of rank $r \geq 2$ and every single-element contraction M/e of M is disconnected. Then M has at least $r + 1$ nonsingleton parallel classes.*

Theorem 2.2. *Let $g \geq 2$ be an integer. Let M be a connected rank-r matroid with girth at least g and with no cocircuit having more than d elements.*

(i) *If d is even, then M has at most $\frac{1}{2}(r+3-g)d+g-2$ elements.*
(ii) *If d is odd, then M has at most $\frac{1}{2}(r+3-g)(d-1)+g-1$ elements.*

Proof. The proof is by induction on g. The base case $g = 2$ is Theorem 1.2, so assume g exceeds 2. Thus M is simple, so Lemma 2.1 implies that there is an element e of M for which the contraction M/e is connected. Note that M/e has rank $r-1$, girth at least $g-1$, and that cocircuits of M/e have at most d elements. The inductive hypothesis applied to M/e gives

$$|E(M)| = |E(M/e)| + 1 \leq \frac{1}{2}\big((r-1)+3-(g-1)\big)d + \big((g-1)-2\big) + 1$$

if d is even, and

$$|E(M)| = |E(M/e)| + 1 \leq \frac{1}{2}\big((r-1)+3-(g-1)\big)(d-1) + \big((g-1)-1\big) + 1$$

if d is odd, as needed. □

The instance of Theorem 2.2 of most interest for the present paper is the case of girth at least three; this is singled out in the following theorem.

Theorem 2.3. *Let M be a connected geometry of rank $r > 1$ with no cocircuit having more than d elements.*
(i) *If d is even, then M has at most $\frac{1}{2}rd+1$ elements.*
(ii) *If d is odd, then M has at most $\frac{1}{2}r(d-1)+2$ elements.*

The results in Sections 3 and 4 show that the upper bounds given in Theorem 2.3 are not sharp in ranks 3 and 4. Indeed, they are probably not sharp for any rank other than two.

The following result was stated in the introduction of [1]. Note that there is no connectivity assumption in Theorems 2.4 and 2.5.

Theorem 2.4. *If M is a geometry of rank r with no cocircuit having more than d elements, then M has at most $r(d+1)/2$ elements.*

This is the case $g = 3$ of the next theorem, which gives sharper results for matroids of large girth.

Theorem 2.5. *If M is a matroid of rank r with girth $g \geq 2$ and with no cocircuit having more than d elements, then M has at most $r + r(d-1)/(g-1)$ elements.*

Proof. Assume M has n elements. Let B be a basis of M. Consider the B-fundamental-circuit incidence matrix D of M (see Section 6.4 of [7]). Since M has girth g, each of the $n-r$ columns of D has at least $g-1$ entries that are 1. Since cocircuits have at most d elements, each of the r rows of D has at most $d-1$ entries that are 1. Thus $(g-1)(n-r) \leq r(d-1)$, which gives $n \leq r + r(d-1)/(g-1)$. □

Whether Theorem 2.2 or Theorem 2.5 yields a sharper bound depends on the particular values of the parameters r, d, and g, although for "most" values of the parameters, the bounds in Theorem 2.2 are sharper. Note that the bounds in Theorem 2.3 are always sharper than the bound in Theorem 2.4 if the rank exceeds two. Also note that the bounds in both Theorems 2.2 and 2.5 are sharp if the girth is as large as possible; if $g = r+1$, then M is a uniform matroid with at most $r-1+d$ elements.

3. A Sharper Bound in Rank 3

Theorem 3.1 generalizes Theorem 1.3 by showing that $d + \lfloor\sqrt{d}\rfloor + 1$ is an upper bound on the number of points in a rank-3 geometry in which each cocircuit has at most d elements. Theorem 3.1 also classifies the rank-3 geometries that have exactly $d + \lfloor\sqrt{d}\rfloor + 1$ points. The three infinite families of geometries in Theorem 3.1 arise from affine and projective planes (see Section 12.1 of [9] for background on affine and projective planes). Among the geometries in Theorem 3.1, the geometry in (7) is perhaps least familiar. An elementary argument shows that there is a unique rank-3 geometry on eleven points having one 5-point line, fifteen 3-point lines, and no other lines; this is the Nwankpa plane [4]. The 5-point line of the Nwankpa plane has nonempty intersection with every other line of this geometry.

Theorem 3.1. *Let d be a positive integer and assume that G is a rank-3 geometry in which each cocircuit has at most d elements. Then G has at most $d+\lfloor\sqrt{d}\rfloor+1$ points. Furthermore, G has exactly $d+\lfloor\sqrt{d}\rfloor+1$ points in precisely the following cases:*

(1) *$d=q^2$ and G is a projective plane of order q;*
(2) *$d=q^2-q$ and G is an affine plane of order q;*
(3) *$d=q^2-q+1$ and G is a geometry of the form $M\backslash X$ where M is a projective plane of order q and X is a set of q collinear points;*
(4) *$d=1$ and G is $U_{3,3}$;*
(5) *$d=2$ and G is $U_{2,3}\oplus U_{1,1}$;*
(6) *$d=3$ and G is one of $U_{3,5}$, $U_{2,4}\oplus U_{1,1}$, or the unique five-element rank-3 geometry that has a single 3-point line;*
(7) *$d=8$ and G is the Nwankpa plane.*

Proof. Let q be $\lfloor\sqrt{d}\rfloor$; thus, cocircuits of G have fewer than $(q+1)^2$ elements.

Assume G has at least $d+\lfloor\sqrt{d}\rfloor+2$ points. Thus, each line of G contains at least $q+2$ points. Let ℓ be a line of G and let x be a point of G not in ℓ. Each line $\mathrm{cl}(\{x,y\})$ for $y\in\ell$ contains at least q points other than x in the cocircuit complementary to ℓ, so this cocircuit contains at least $(q+2)q+1$, or $(q+1)^2$, points. This contradiction to the bound on the cardinalities of cocircuits shows that G has at most $d+\lfloor\sqrt{d}\rfloor+1$ points.

It is easy to verify that the examples in (1)–(7) above have $d+\lfloor\sqrt{d}\rfloor+1$ points. We now show that these are the only examples with $d+\lfloor\sqrt{d}\rfloor+1$ points.

Assume G has $d+\lfloor\sqrt{d}\rfloor+1$ points. By the choice of q, the number of points in G is at most $((q+1)^2-1)+q+1$, or q^2+3q+1. Each line of G has at least $q+1$ points and each point not in a fixed line ℓ is in at least $|\ell|$ lines, so G has at least $|\ell|q+1$ points. Thus $|\ell|q+1\leq q^2+3q+1$, so $|\ell|\leq q+3$. Thus, all lines of G have $q+1$, $q+2$, or $q+3$ points.

Assume all lines of G have exactly $q+1$ points. Thus, there is an integer k such that each point of G is in k lines. Since G has at least q^2+q+1 points and at most q^2+3q+1 points, k is $q+1$, $q+2$, or $q+3$. If k is $q+1$, then G has q^2+q+1 points and each line of G has $q+1$ points; it is an easy exercise to show that G is $U_{3,3}$ or a projective plane of order q. If k is $q+2$, then G has $(q+1)^2$ points and each line of G has $q+1$ points; it is easy to show that G is an affine plane of order $q+1$. If k is $q+3$, then G has q^2+3q+1 points. It follows that the number of lines in G is $\binom{q^2+3q+1}{2}/\binom{q+1}{2}$. For this quotient to be an integer, q must be 1, so G is $U_{3,5}$.

Assume that all lines of G have exactly $q+1$ or $q+2$ points and that at least one line has $q+2$ points. Therefore G has at least $(q+2)q+1$, or q^2+2q+1, points. Since G has at most q^2+3q+1 points, some line of G has only $q+1$ points. First assume G has at most q^2+3q points, say q^2+2q+k points where $1\leq k\leq q$. Note that each point of G is in at most $q+2$ lines of G. Therefore each line of G has nonempty intersection with each $(q+2)$-point line of G. Let ℓ and ℓ' be lines of G with $q+2$ and $q+1$ points, respectively, and let X be the set of point of G not in $\ell\cup\ell'$. Thus $|X|=(q^2+2q+k)-(2q+2)=q^2+k-2$. Since $|\ell|=q+2$, each point in X is in exactly $q+2$ lines of G. Thus each point of X is in exactly one line that has empty intersection with ℓ'; necessarily this is a $(q+1)$-point line of G, and one point on this line is in ℓ. Thus, such lines partition X into blocks of size q. Since $|X|=q^2+k-2$, it follows that q divides $k-2$. Since $1\leq k\leq q$, either $q=1$ or $k=2$. If $q=1$, then $k=1$ and G has four points; it follows that G is $U_{2,3}\oplus U_{1,1}$. Assume $k=2$, so G has q^2+2q+2, or $(q+1)^2+1$, points. Since each point not on ℓ is on $q+2$ lines of G, it follows that each such point is on $q+1$ lines that each have $q+1$ points and on exactly one line that has $q+2$ points. From this, it follows that all $(q+2)$-point lines of G intersect ℓ in the same point, say x, and all lines of G containing x are $(q+2)$-point lines. It follows that the deletion $G\backslash x$ is an affine plane of order $q+1$, so G is given by (3) above. Now assume that G has q^2+3q+1 points. Note that each point of G is either on $q+3$ lines of G, each of which has $q+1$ points, or on two $(q+1)$-point lines and q lines with $q+2$ points; assume there are t points of the first type and so $q^2+3q+1-t$ of the second type. By counting the number of pairs (x,ℓ) where x is a point of G and ℓ is a $(q+1)$-point line of G containing x, we get $s(q+1) = t(q+3) + (q^2+3q+1-t)2$ where s is the number of $(q+1)$-point lines of G. Thus $s(q+1) = t(q+1) + ((q+1)^2+q)2$, so $q+1$ divides $2q$. Thus $q=1$, so G has five points and d is 3. All such geometries are included in (3) and (6).

Finally, assume G has a line ℓ that contains $q+3$ points. It follows that G has exactly q^2+3q+1 points; also, all lines of G other than ℓ have exactly $q+1$ points and have nonempty intersection with ℓ. Let y be a point of ℓ and assume that y is contained in t lines of G other than ℓ. By counting the points on lines through y, we get $q^2+3q+1 = q+3+qt$, or $q(q+2-t) = 2$. It follows that q divides 2 and so is either 1 or 2. It follows that if $q=2$, then G is the Nwankpa plane, while if $q=1$, then G is $U_{2,4} \oplus U_{1,1}$. □

We close this section by noting the following properties of the Nwankpa plane. These observations follow directly from the techniques in Section 6.4 of [7].

Theorem 3.2. *The Nwankpa plane is representable over a field F if and only if F contains an element α such that $2\alpha = 0$ and $\alpha^2+\alpha+1 = 0$. Thus, the Nwankpa plane is representable only over fields of characteristic two; the finite fields over which it is representable are precisely the fields $\mathrm{GF}(2^{2k})$ for positive integers k.*

4. Results for Rank-4 Geometries

In this section, we prove that if each cocircuit in a connected rank-4 geometry G has at most d elements, then G has at most $d + d^{2/3} + d^{1/3} + 1$ elements; furthermore, this upper bound is strict unless d is the cube of a prime power q, in which case the only such geometry with q^3+q^2+q+1 points is the projective geometry $\mathrm{PG}(3,q)$. We treat these results in two cases according to whether or not d is a perfect cube (Theorems 4.3 and 4.4, respectively). Note that $U_{2,d+1} \oplus U_{2,d+1}$ shows the need for the hypothesis of connectivity in Theorems 4.3 and 4.4. The proof of Theorem 4.3 uses the following result [5, Theorem 4.3].

Lemma 4.1. *Let $q > 1$ be an integer. Any rank-r geometry with no minor isomorphic to the $(q+2)$-point line has at most $(q^r-1)/(q-1)$ points. This upper bound is attained only by projective geometries of rank r and order q.*

Note that if G has a minor isomorphic to a $(q+2)$-point line, then, by the Scum Theorem [7, Theorem 3.3.7], some coline of G is contained in at least $q+2$ hyperplanes of G.

We also use the following lemma.

Lemma 4.2. *Let $q > 1$ be an integer. Let G be a rank-r geometry in which each cocircuit has at most q^{r-1} elements and in which all lines have at least $q+1$ points. Then the number of points in G is exactly $(q^r-1)/(q-1)$ and G is a projective geometry of rank r and order q.*

Proof. Since all lines of G have at least $q+1$ points, it follows inductively that each rank-i flat of G has at least $(q^i-1)/(q-1)$ points. Since there are at most q^{r-1} points in any cocircuit and at least $q+1$ points in each line of G, it follows that each hyperplane has at most $(q^{r-1}-1)/(q-1)$ points. Therefore each hyperplane has exactly $(q^{r-1}-1)/(q-1)$ points, and, more generally, each rank-i flat of G has exactly $(q^i-1)/(q-1)$ points. Therefore the number of hyperplanes over each coline is exactly

$$\Big(\frac{q^r-1}{q-1}-\frac{q^{r-2}-1}{q-1}\Big)\Big/\Big(\frac{q^{r-1}-1}{q-1}-\frac{q^{r-2}-1}{q-1}\Big),$$

or $q+1$. Therefore Lemma 4.1 applies, from which we conclude that G is a rank-r projective geometry of order q. □

We are now ready to address the case in which the upper bound on the cardinalities of cocircuits is a perfect cube.

Theorem 4.3. *Let $q > 1$ be an integer. Let G be a connected rank-4 geometry in which each cocircuit has at most q^3 elements. Then the number of points in G is at most $q^3 + q^2 + q + 1$. Furthermore, G has $q^3 + q^2 + q + 1$ points if and only if q is a prime power and G is the rank-4 projective geometry $\mathrm{PG}(3, q)$.*

Proof. Assume that G satisfies the hypotheses of the theorem and has at least $q^3 + q^2 + q + 1$ points. From the upper bound on the cardinalities of cocircuits, this implies that each plane of G has at least $q^2 + q + 1$ points. We will show that G is a rank-4 projective geometry of order q, thereby proving the upper bound and analyzing the case of equality simultaneously.

The following observation will be used.

(4.3.1) *Any line contained in at least $q + 2$ planes contains at least $2q + 1$ points.*

To prove (4.3.1), let ℓ be a line that is contained in at least $q+2$ planes. The number of points outside any plane containing ℓ is at least $(q + 1)(q^2 + q + 1 - |\ell|)$. Since cocircuits have at most q^3 points, we have $(q + 1)(q^2 + q + 1 - |\ell|) \le q^3$. Thus, $|\ell| \ge 2q + 1$, as claimed.

By Lemma 4.2, it suffices to prove that all lines of G have at least $q + 1$ points. Assume, to the contrary, that ℓ is a line of G having at most q points. Note that ℓ is in at most q planes, for otherwise, by counting points in the planes through ℓ, excluding points on ℓ, we deduce that the cocircuit complementary to any one of these planes contains at least $q(q^2 + 1)$ points, contrary to our assumption about the cardinalities of cocircuits. It follows that any line that is not coplanar with ℓ has at most q points.

Let π be a plane through ℓ. We claim that the only points in π on at least $q + 2$ lines of π are in ℓ. Indeed, assume x is such a point. Let ℓ^* be a line of G that intersects π in x. Note that ℓ^* is in at least $q + 2$ planes, namely $\mathrm{cl}(\ell^* \cup \ell')$ as ℓ' ranges over the $q + 2$ or more lines in π that contain x. By (4.3.1), we conclude that ℓ^* contains at least $2q+1$ points. Since lines not coplanar with ℓ have at most q points, we deduce that x is in ℓ, as claimed.

We claim that in each plane π through ℓ, the points of $\pi - \ell$ are collinear. Indeed, assume this is not so. From the conclusion of the last paragraph, it follows that all lines in π have at most $q + 1$ points. Let z be a point of $\pi - \ell$. Since z is on at most $q + 1$ lines, each of which has at most $q + 1$ points, π has at most $q^2 + q + 1$ points. Therefore π has exactly $q^2 + q + 1$ points, so each point of $\pi - \ell$ is on exactly $q+1$ lines and each of these contains exactly $q+1$ points. Let y be a point of ℓ. Note that lines through y other than ℓ have exactly $q + 1$ points. Since ℓ has q or fewer points, π cannot have exactly $q^2 + q + 1$ points. This contradiction shows that the points of $\pi - \ell$ are collinear.

Thus G consists of ℓ together with at most q lines, $\ell_1, \ell_2, \dots, \ell_t$, all in different planes with ℓ. Since ℓ has at most q points and all planes of G have at least $q^2 + q + 1$ points, each set $\ell_i - \ell$ has at least $q^2 + 1$ points.

We claim that each point in ℓ is in at most one of the lines $\ell_1, \ell_2, \dots, \ell_t$. Assume instead that $x \in \ell$ is in ℓ_i and ℓ_j. Fix $u \in \ell - x$ and $v \in \ell_i - x$. As w ranges over the $q^2 + 1$ or more points in $\ell_j - x$, we get at least $q^2 + 1$ distinct planes $\mathrm{cl}(\{u, v, w\})$, all containing the line $\mathrm{cl}(\{u, v\})$. By (4.3.1), the line $\mathrm{cl}(\{u, v\})$ has at least $2q + 1$ points. However, this contradicts the fact that the points of $\mathrm{cl}(\ell_i \cup \ell) - \ell$ are collinear. Thus each point of ℓ is in at most one of the lines $\ell_1, \ell_2, \dots, \ell_t$.

We claim that t is 2 and that ℓ contains just two points x_1 and x_2 where $\{x_i\}$ is $\ell \cap \ell_i$. If this were not so, then there are at least two lines ℓ_i and ℓ_j and a point $u \in \ell$ with u on neither ℓ_i nor ℓ_j. The same type of argument as above, using u, a fixed point $v \in \ell_i - \ell$, and the q^2+1 or more points w of $\ell_j - \ell$, yields a contradiction.

Thus G consists precisely of the two complementary lines ℓ_1 and ℓ_2. This contradiction to the assumption that G is connected completes the proof that all lines have at least $q+1$ points, and hence completes the proof of the theorem. □

Theorem 4.3 applies when d, the upper bound on the cardinalities of cocircuits, is a perfect cube. We now provide a strict upper bound on the number of points for a rank-4 geometry when d is not a perfect cube.

Theorem 4.4. *Let $d \geq 2$ be an integer that is not a perfect cube. Let G be a connected rank-4 geometry in which each cocircuit has at most d elements. Then G has fewer than $d + d^{2/3} + d^{1/3} + 1$ elements.*

Proof. We treat the three values of 2, 3, and 4 for d separately. In these cases, $d + d^{2/3} + d^{1/3} + 1$, rounded down, is 5, 7, and 9, respectively. The first and last agree with the bound on the number of elements given in Theorem 2.3; the middle value is not as tight as the bound of 6 given in Theorem 2.3. (One can give a geometric argument to show that the best bound for $d = 4$ is eight. Note that $U_{4,5}$, $U_{4,6}$, and $\mathrm{AG}(3,2)$ show that the upper bounds 5, 6, and 8 are sharp for these values of d.) For the rest of the proof, we assume $d \geq 5$.

Assume that G has at least $d + d^{2/3} + d^{1/3} + 1$ points. Therefore, each plane of G has at least $d^{2/3} + d^{1/3} + 1$ points. We will derive the same contradiction as in the proof of Theorem 4.3, namely, that G is disconnected.

Since d is not a perfect cube, either a line is contained in more than $d^{1/3} + 1$ planes or a line is contained in fewer than $d^{1/3} + 1$ planes. We will use such observations freely. The proof of the following observation is similar to that of assertion (4.3.1).

> **(4.4.1)** *Any line contained in more than $d^{1/3} + 1$ planes contains more than $d^{1/3} + 1$ points.*

Assume first that all lines of G have more than $d^{1/3} + 1$ points. Let π be a plane of G and let x be a point of G not in π. The lines cl$(\{x, y\})$ for $y \in \pi$ contain more than $(d^{1/3} - 1)(d^{2/3} + d^{1/3} + 1) + 1$ points in the cocircuit complementary to π; however this exceeds d. This contradiction shows that some line of G has fewer than $d^{1/3} + 1$ points. Let ℓ be such a line.

From (4.4.1), we get that ℓ is in fewer than $d^{1/3} + 1$ planes. Thus any line that is not coplanar with ℓ has fewer than $d^{1/3} + 1$ points. As in the proof of Theorem 4.3, it follows that any point that is in more than $d^{1/3} + 1$ lines of a plane that contains ℓ is in ℓ.

Let π be a plane containing ℓ and let y be a point in $\pi - \ell$. Since π has at least $d^{2/3} + d^{1/3} + 1$ points and y is in fewer than $d^{1/3} + 1$ lines, it follows that at least one line, say ℓ', of π containing y has more than $d^{1/3} + 1$ points. From this and the last paragraph, it follows that π is $\ell \cup \ell'$.

Thus G consists of ℓ together with fewer than $d^{1/3} + 1$ lines, $\ell_1, \ell_2, \ldots, \ell_t$, all in different planes with ℓ. Since ℓ has fewer than $d^{1/3} + 1$ points and all planes of G have at least $d^{2/3} + d^{1/3} + 1$ points, each set $\ell_i - \ell$ has more than $d^{2/3}$ points. Assume that x is in ℓ, ℓ_i, and ℓ_j. Fix $u \in \ell - x$ and $v \in \ell_i - x$. As w ranges over the more than $d^{2/3}$ points in $\ell_j - \ell$, we get more than $d^{2/3}$ distinct planes cl$(\{u, v, w\})$, all containing the line cl$(\{u, v\})$. Since $d \geq 5$, we have $d^{2/3} > d^{1/3} + 1$, so by (4.4.1) the line cl$(\{u, v\})$ has more than $d^{1/3} + 1$ points. Thus cl$(\{u, v\})$ has more than two points. However, this contradicts the fact that the points of cl$(\ell_i \cup \ell) - \ell$ are collinear. It follows that each point of ℓ is on at most one of the lines $\ell_1, \ell_2, \ldots, \ell_t$.

A similar adaptation of the final argument in the proof of Theorem 4.3 shows that $t = 2$ and G is the direct sum of the restrictions to the two lines ℓ_1 and ℓ_2. This contradiction to the assumption that G is connected completes the proof of the theorem. □

5. Results for Rank-5 Geometries

If we assume only 2-connectivity, Theorem 4.3 does not have a counterpart for rank 5. To see this, consider the direct sum of three $\lfloor q^4/2+1 \rfloor$-point lines. Truncate this to get a rank-5 geometry. This geometry is connected but not 3-connected. It has $3\lfloor q^4/2+1 \rfloor$ points, which exceeds $q^4+q^3+q^2+q+1$ for sufficiently large q. Hyperplanes consist of either two of the original lines or one original line with one point from each of the other two; in both cases, the complementary cocircuits have at most q^4 elements.

In higher ranks, there are such examples that are also 3-connected. Consider, for example, rank 10. Choose a prime power t with $3t^2 \leq q^9$. Consider the direct sum of four projective planes of order t and truncate twice to get a rank 10 geometry G. It is easy to check that G is 3-connected. Hyperplanes of G are of three types:

(a) three of the projective planes,
(b) two of the projective planes, a line from another, and a point from the fourth,
(c) one of the projective planes and a line from each of the three others.

The complementary cocircuits have a) t^2+t+1, b) $2t^2+t$, and c) $3t^2$ points each. By the choice of t, none of these exceed q^9. The total number of points is $4(t^2+t+1)$, which, for suitable choices of q and t, exceeds $(q^{10}-1)/(q-1)$.

Note that requiring connectivity higher than three is not of interest for problems of this type since the connectivity of projective geometries is exactly three. However, one may require that the vertical connectivity $\kappa(M)$ of M be higher. (Vertical connectivity, also known as Whitney connectivity, is discussed in [2] and Section 8.2 of [7]; our terminology follows [7].) The condition we impose in Theorem 5.2 is that $\kappa(M)$ is $r(M)$; this holds if M is a projective geometry. By Theorem 5 of [2], $\kappa(M) = r(M)$ (or $\kappa(M) = \infty$ in the notation of [2]) if and only if each cocircuit of M spans M. Equivalently, $\kappa(M) = r(M)$ if and only if the ground set of M is not the union of two proper flats of M. In [3], such matroids are called nonsplitting; this is the terminology we adopt here. We will use the following elementary lemma about nonsplitting matroids.

Lemma 5.1. *Assume M is a nonsplitting matroid and the restriction $M|\{x,y,z,w\}$ is the uniform matroid $U_{2,4}$. Then the deletion $M\backslash x$ is nonsplitting.*

Proof. Assume the ground set of $M\backslash x$ is the union of two proper flats, X and Y, of $M\backslash x$. At least one of $X \cap \{y,z,w\}$ and $Y \cap \{y,z,w\}$ has at least two elements. It follows that the closures of X and Y in M are proper flats in M whose union is the ground set of M, contradicting the assumption that M is nonsplitting. □

We now turn to the main result of this section.

Theorem 5.2. *Let $q > 1$ be an integer. Let G be a nonsplitting rank-5 geometry in which each cocircuit has at most q^4 elements. Then the number of points in G is at most $q^4+q^3+q^2+q+1$. Furthermore, G has $q^4+q^3+q^2+q+1$ points if and only if q is a prime power and G is the rank-5 projective geometry $\mathrm{PG}(4,q)$.*

Proof. It suffices to show that if G has at least $q^4+q^3+q^2+q+1$ points, then G is isomorphic to $\mathrm{PG}(4,q)$. We will assume that G has at least $q^4+q^3+q^2+q+1$ points and that G is not isomorphic to $\mathrm{PG}(4,q)$. From this assumption we will derive a contradiction.

By the upper bound on the cardinalities of cocircuits, each hyperplane of G has at least q^3+q^2+q+1 points. By the assumption that G is not isomorphic to $\mathrm{PG}(4,q)$ together with Lemma 4.1 and the Scum Theorem, it follows that G has a plane that is in at least $q+2$ hyperplanes. Let π be such a plane. The number of points outside any one of the hyperplanes containing π is at least $(q+1)(q^3+q^2+q+1-|\pi|)$. Since cocircuits have at most q^4 points, we have $(q+1)(q^3+q^2+q+1-|\pi|) \leq q^4$. Thus, $|\pi| \geq 2q^2+2$. This proves the following assertion.

(5.2.1) *Any plane contained in more than $q+1$ hyperplanes contains at least $2q^2+2$ points.*

We next claim that some lines of G have many points.

(5.2.2) *At least one line of G contains at least $q+2$ points.*

To prove (5.2.2), let π be a plane of G in at least $q+2$ hyperplanes. By (5.2.1), π has at least $2q^2+2$ points. We may assume no line of π has $q+2$ or more points, otherwise there is nothing to show. It follows that each point of π is in at least $q+2$ lines of π. Thus for a line ℓ complementary to π and any point x in π, the plane $\mathrm{cl}(\ell \cup \{x\})$ is in at least $q+2$ hyperplanes, namely those spanned by this plane and one of the $q+2$ or more lines of π through x. Therefore by (5.2.1), for each point x of π, the plane $\mathrm{cl}(\ell \cup \{x\})$ has at least $2q^2+2$ points. Fix a line ℓ' of π. Since ℓ' has at most $q+1$ points, there are at least $2q^2-q+1$ points x in $\pi-\ell'$, and for each such point x, the points of $\mathrm{cl}(\ell \cup \{x\})-\ell$ are in the cocircuit complementary to the hyperplane $\mathrm{cl}(\ell \cup \ell')$. Therefore there are at least $(2q^2-q+1)(2q^2+2-|\ell|)$ points in this cocircuit. Since this is at most q^4, it follows that ℓ has at least $q+2$ points. This proves (5.2.2).

From (5.2.2) and the assumption $q \geq 2$, we have a line with at least four points. From Lemma 5.1, it follows that if G has more than $q^4+q^3+q^2+q+1$ points, we can delete at least one point of G and still satisfy all hypotheses. We can apply this argument repeatedly. Since all cocircuits of $\mathrm{PG}(4,q)$ have q^4 elements, $\mathrm{PG}(4,q)$ has no extension satisfying the hypotheses. Therefore we may assume the following.

(5.2.3) *The geometry G has exactly $q^4+q^3+q^2+q+1$ points and is not $\mathrm{PG}(4,q)$.*

We claim that planes with many points cannot contain more than one line with many points. This is the content of (5.2.5), which follows immediately from (5.2.4).

(5.2.4) *In a plane of G with at least $2q^2+2$ points, there must be more than one point on fewer than $q+2$ lines in the plane.*

(5.2.5) *Each plane of G with at least $2q^2+2$ points contains exactly one line with more than $q+1$ points.*

To prove (5.2.4), let π be a plane of G with at least $2q^2+2$ points and assume all but at most one point of π is on $q+2$ or more lines of π. Let ℓ be a line complementary to π. With one possible exception, each plane $\mathrm{cl}(\ell \cup \{x\})$, for a point x of π, has at least $2q^2+2$ points; this follows from (5.2.1) since the $q+2$ or more lines through x in π give rise to at least $q+2$ hyperplanes containing $\mathrm{cl}(\ell \cup \{x\})$. Since there are at least $2q^2+1$ such points x and $q^4+q^3+q^2+q+1$ points in G, we get

$$(2q^2+1)(2q^2+2-|\ell|)+|\ell| \leq q^4+q^3+q^2+q+1.$$

Therefore ℓ has at least q^2+2 points. Thus, each line complementary to π has at least q^2+2 points. Let H be a hyperplane containing π and at most one point in a complementary line ℓ. Since G is nonsplitting, there is a point a not in $H \cup \ell$. Since the plane $\mathrm{cl}(\ell \cup \{a\})$ together with π spans G, it follows that for all points b on ℓ with at most one exception, the line $\mathrm{cl}(\{a,b\})$ is complementary to π. Therefore each such line has at least q^2+2 points, and so at least q^2 points not in $H \cup a$. Therefore there are at least $(q^2+1)q^2+1$ points in the complement of H, contrary to the upper bound on the cardinalities of cocircuits. This proves (5.2.4).

(5.2.6) *In a plane of G with at least $2q^2+2$ points, there is a (unique) line with at least q^2+2 points.*

To prove (5.2.6), note that from (5.2.5) such a plane π contains exactly one line, say ℓ, with $q+2$ or more points. From (5.2.4), since each point of $\pi-\ell$ is on at least $q+2$ lines, there is a point x of ℓ on at most $q+1$ lines. Since the q or fewer lines through x other than ℓ each have at most q points besides x, it follows that there are at most q^2 points in $\pi-\ell$. Therefore ℓ has at least q^2+2 points, proving (5.2.6).

Thus we have at least one line ℓ with at least $q^2 + 2$ points. Planes complementary to ℓ are in at least $q^2 + 2$ hyperplanes. Assume that the plane π' is in at least $q^2 + 2$ hyperplanes. By considering the points in the cocircuit complementary to any of the hyperplanes containing π', we get the inequality

$$(q^2 + 1)(q^3 + q^2 + q + 1 - |\pi'|) \leq q^4.$$

Thus, $|\pi'| \geq q^3 + q + 2$. This proves the following assertion.

(5.2.7) *Any plane contained in more than q^2+1 hyperplanes contains at least q^3+q+2 points.*

There is at least one line with at least $q^2 + 2$ points. By (5.2.7) and (5.2.6), any complementary plane also has a line with at least $q^2 + 2$ points. Thus there is a pair ℓ and ℓ' of noncoplanar lines, each with at least $q^2 + 2$ points. Let x be a point so that cl$(\{x\} \cup \ell')$ is a plane complementary to ℓ. Fix a point w on ℓ and consider the hyperplane cl$(\ell' \cup \{x, w\})$. Since G is nonsplitting, the cocircuit complementary to cl$(\ell' \cup \{x, w\})$ spans G, so there are lines ℓ_1 and ℓ_2 through x that are not in cl$(\ell' \cup \{x, w\})$ and are such that cl$(\ell \cup \ell_1 \cup \ell_2)$ has rank 5. In particular, cl$(\ell_1 \cup \ell_2)$ is a plane complementary to ℓ. For each point u of ℓ', we have a plane cl$(\ell_1 \cup \{u\})$. Note that at most one of these is not complementary to ℓ. Thus at least $q^2 + 1$ of these planes cl$(\ell_1 \cup \{u\})$ are complementary to ℓ and so contain at least $q^3 + q + 2$ points. Therefore counting the points in each such plane and the points on ℓ, we have at least $(q^2 + 1)(q^3 + q + 2 - |\ell_1|) + |\ell_1| + q^2 + 2$ points in G. Therefore ℓ_1 has at least $q^3 - q^2 + q + 3$ points. Similarly ℓ_2 has at least $q^3 - q^2 + q + 3$ points. Since cl$(\ell_1 \cup \ell_2)$ is complementary to the line ℓ that has at least $q^2 + 2$ points, cl$(\ell_1 \cup \ell_2)$ has at least $2q^2 + 2$ points. However the plane cl$(\ell_1 \cup \ell_2)$ also has two lines, ℓ_1 and ℓ_2, each having more than $q + 1$ points. This contradiction to (5.2.5) completes the proof. □

Acknowledgement

The authors thank the anonymous referee for noting the properties of the Nwankpa plane given in Theorem 3.2. The first author thanks Tom Dowling and Tom Zaslavsky for interesting discussions on the topics of this paper.

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