AN INTRODUCTION TO TRANSVERSAL MATROIDS

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CONTENTS

1. PREFATORY REMARKS

When first introduced in the mid 1960's, transversal matroids unified many results in transversal theory, simplified their proofs, and provided a larger context for such results, thereby firmly establishing the importance of this class of matroids. Such links with other specialties fueled much interest in matroid theory within the wider research community.

This expository paper develops several ways to view transversal matroids, a selection of basic results about them, and several characterizations of these matroids, as well as some results about lattice path matroids, which form a class of transversal matroids with many very attractive properties. Thus, readers will gain a substantial but selective introduction to the theory of transversal matroids, including some recent developments, as well as some exposure to several related topics from other parts of matroid theory.

We assume that readers have already been exposed to the basic concepts of matroid theory (e.g., minors, duals, direct sums, and matrix representations over fields). All sets considered are finite. We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$.

$[n]$

2. Several Perspectives on Transversal Matroids

In this section we define transversal matroids via set systems and partial transversals, we explore their matrix and geometric representations, and we deduce some basic properties of these matroids from each of these perspectives.

set system

2.1. **Set systems, transversals, partial transversals, and Hall's theorem.** A *set system* $(S, \mathcal{A})$ is a set S along with a multiset $\mathcal{A} = (A_j : j \in J)$ of subsets of S. Thus, the same set may appear multiple times in $\mathcal{A}$, indexed by different elements of J. It is sometimes convenient to write $\mathcal{A}$ as a sequence $(A_1, A_2, \ldots, A_r)$ of subsets of S, but we must agree to not distinguish between the different possible orders in which the sets can be listed; thus, $(A_1, A_2, \ldots, A_r)$ is the same set system as $(A_{\sigma(1)}, A_{\sigma(2)}, \ldots, A_{\sigma(r)})$ for each permutation σ of $[r]$.

Set systems arise throughout mathematics. For instance, let H be a subgroup of a (finite) group G. The set of left cosets of H in G forms a set system, as does the set of right cosets, as does the multiset union of these two sets (in which H, at least, appears twice). Such set systems have features that we do not require in general set systems: for instance, the (left or right) cosets partition G; also, all cosets have the same size.

The following example of a less-structured set system will be referred to frequently.

Example: Let $S = \{a, b, c, d, e, f, g, h\}$ and $\mathcal{A} = (A, B, C, D)$ where

$$A = \{a, b, e, f, h\}, \quad B = \{b, c, g\}, \quad C = \{d, e, g, h\}, \quad \text{and} \quad D = \{d, f, h\}.$$

A set system $(S, \mathcal{A})$ can be represented by a bipartite graph: the vertex set is $\mathcal{A} \cup S$; an edge connects $A_j \in \mathcal{A}$ and $x \in S$ precisely when $x \in A_j$. Similarly, each bipartite graph can be seen as representing a set system. From the perspective of bipartite graphs, it is natural for $\mathcal{A}$ to be a multiset: in graphs, two distinct vertices can have exactly the same neighbors. The bipartite graph corresponding to the set system in the example above is shown in Figure 1.

transversal

A *transversal* of the set system $(S, \mathcal{A})$ is a subset T of S for which there is a bijection $\phi : J \to T$ with $\phi(j) \in A_j$ for all $j \in J$.

In the setting of left cosets mentioned above, a set of left coset representatives gives a transversal; a similar remark, of course, applies to right cosets and their representatives. In the set system $(S, \mathcal{A})$ given above, the sets $\{a, b, e, h\}$ and $\{e, f, g, h\}$ are among the many transversals. There may be many bijections ϕ that show that a given set T is a transversal; for instance, the underlining in the following displays shows two ways to match $\{e, f, g, h\}$ with the sets in $\mathcal{A}$.

$$A = \{a, b, \underline{e}, f, h\}, \quad B = \{b, c, \underline{g}\}, \quad C = \{d, e, g, \underline{h}\}, \quad D = \{d, \underline{f}, h\}$$

$$A = \{a, b, e, \underline{f}, h\}, \quad B = \{b, c, \underline{g}\}, \quad C = \{d, \underline{e}, g, h\}, \quad D = \{d, f, \underline{h}\}$$

Note that a transversal T is just a set; the map ϕ shows that T is indeed a transversal, but it is not part of the transversal. In graph theory terms, ϕ corresponds to a matching, while T corresponds to a set of vertices, in a fixed part of the bipartite graph, that can be matched.

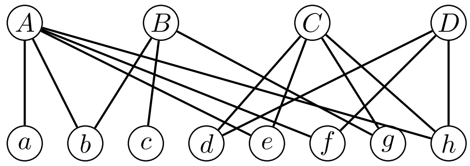


FIGURE 1. The bipartite graph representing the set system in the example.

Consistent with the terminology for cosets, some sources refer to transversals as *systems of distinct representatives*, or SDRs, but we will not use this term. systems of distinct representatives

Which set systems have transversals? Clearly the set system with $S = \{1,2,3,4\}$ and $\mathcal{A} = (\{1\},\{2\},\{1,2\},\{3,4\})$ has no transversal since three of the sets contain just two elements. In general, an obvious necessary condition for a set system to have a transversal is that each subsystem must have at least as many elements as sets. This condition turns out to be sufficient, as the following well-known theorem of P. Hall states. (To make this expository paper somewhat more self-contained, a proof is given in Section 7.)

Theorem 2.1. *A finite set system* $(S,(A_j : j \in J))$ *has a transversal if and only if, for all* $K \subseteq J$*,*

$$\Big|\bigcup_{i\in K} A_i\Big| \geq |K|.$$

We stress the hypothesis of finite in this result since the infinite counterpart is false.

EXERCISE 2.1: Give an example to show that the assertion in Theorem 2.1 can fail if the set system is not finite.

EXERCISE 2.2: Does Theorem 2.1 hold if S is infinite but the multiset is finite (i.e., $|J|$ is finite)?

EXERCISE 2.3: Use Theorem 2.1 to show that if C is an $n \times n$ matrix with entries in the nonnegative integers, and if the sum of the entries in each row of C is k and the sum of the entries in each column of C is k, then C is a sum of k permutation matrices. (Recall that a permutation matrix is a square matrix in which each row and each column contains exactly one 1, with all the other entries being 0.)

(For an important extension of Hall's theorem, due to R. Rado, in which S is the ground set of a matroid M and a transversal is required to be independent in M, see [25, 28].)

A *partial transversal* of a set system $(S,\mathcal{A})$ with $\mathcal{A} = (A_j : j \in J)$ is a transversal of some subsystem $(S,\mathcal{A}')$ where $\mathcal{A}' = (A_k : k \in K)$ with $K \subseteq J$. Thus, a partial transversal of $(S,\mathcal{A})$ is a subset X of S for which, for some $K \subseteq J$, there is a bijection $\phi : K \to X$ with $\phi(i) \in A_i$ for all $i \in K$. Equivalently, a partial transversal of $(S,\mathcal{A})$ is a subset X of S for which there is an injection $\psi : X \to J$ with $x \in A_{\psi(x)}$ for all $x \in X$. partial transversal

Of course, any transversal of $(S,\mathcal{A})$ is a partial transversal of $(S,\mathcal{A})$, as is $\emptyset$. Two other partial transversals of the set system in the example above are $\{a,b,e\}$ and $\{c\}$; two sets that are not partial transversals of that set system are $\{a,b,c\}$ and $\{a,b,e,g\}$.

2.2. **Transversal matroids via matrix encodings of set systems.** A set system $(S,\mathcal{A})$ with $\mathcal{A} = (A_j : j \in J)$ can be encoded by a zero-one matrix in which the rows are indexed by the elements of J (or by the sets themselves, if the multiplicities are all 1), the columns are indexed by the elements in S, and the entry in row j and column b is 1 if and only if $b \in A_j$. For example, the following matrix encodes the set system in our example.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

Thus, for each row, reading the places where the ones appear yields the corresponding set.

It is advantageous to replace the ones by distinct elements $x_{j,b}$ that are algebraically independent over a given field. Thus, in our example, we have the following matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{pmatrix} x_{A,a} & x_{A,b} & 0 & 0 & x_{A,e} & x_{A,f} & 0 & x_{A,h} \\ 0 & x_{B,b} & x_{B,c} & 0 & 0 & 0 & x_{B,g} & 0 \\ 0 & 0 & 0 & x_{C,d} & x_{C,e} & 0 & x_{C,g} & x_{C,h} \\ 0 & 0 & 0 & x_{D,d} & 0 & x_{D,f} & 0 & x_{D,h} \end{pmatrix} \end{array}$$

A $|J|$-element subset of S corresponds to a $|J| \times |J|$ submatrix. For instance, the set $\{b, e, g, h\}$ in the example corresponds to the 4×4 submatrix,

$$\begin{array}{c} \begin{array}{cccc} b & e & g & h \end{array} \\ \begin{pmatrix} x_{A,b} & x_{A,e} & 0 & x_{A,h} \\ x_{B,b} & 0 & x_{B,g} & 0 \\ 0 & x_{C,e} & x_{C,g} & x_{C,h} \\ 0 & 0 & 0 & x_{D,h} \end{pmatrix} \end{array},$$

which has determinant

$$-x_{A,b}x_{B,g}x_{C,e}x_{D,h} - x_{A,e}x_{B,b}x_{C,g}x_{D,h}.$$

Note that the monomials in this determinant correspond to the bijections that demonstrate that $\{b, e, g, h\}$ is a transversal. In general, the monomials in the permutation expansion of a determinant can be viewed as arising from the various ways to match row indices with column indices. In our setting, such a monomial will be nonzero precisely when the matching demonstrates that the subset indexing the columns is a transversal of the set system. Since the entries $x_{j,b}$ are algebraically independent, no cancelling occurs in the sum, so the determinant is nonzero if and only if at least one of its monomials is nonzero. Thus, linearly independent sets of $|J|$ columns arise precisely from transversals. Similar reasoning shows that the independent sets of columns in this matrix correspond to the partial transversals. Since, via linear independence, any matrix gives rise to a matroid on the columns, this proves the following fundamental theorem, which is due to J. Edmonds and D. R. Fulkerson [18].

Theorem 2.2. *The partial transversals of a set system $\mathcal{A}$ are the independent sets of a matroid.*

transversal matroid
presentation
$M[\mathcal{A}]$

We call such a matroid a *transversal matroid*; we use $M[\mathcal{A}]$ to denote it. The set system $\mathcal{A}$ is called a *presentation* of $M[\mathcal{A}]$.

Alternatively, Theorem 2.2 can be proven directly by showing that partial transversals satisfy the properties required of independent sets. Clearly subsets of partial transversals are partial transversals, as is $\emptyset$; augmentation is shown using the idea of augmenting paths in matching theory. (Readers who are unfamiliar with these ideas are encouraged to work out the proof or see [25, Theorem 1.6.2]. Similar ideas appear in the proof of Lemma 3.2 below.)

EXERCISE 2.4: Prove the following two theorems about transversal matroids. Thinking about these results from several perspectives (set systems, bipartite graphs, and matrices) is encouraged.

Theorem 2.3. *The class of transversal matroids is closed under direct sums.*

Theorem 2.4. *If M is a transversal matroid, then so is the restriction $M|X$ for any set $X \subseteq E(M)$. If the set system $(A_1, A_2, \ldots, A_r)$ is a presentation M, then its restriction $(A_1 \cap X, A_2 \cap X, \ldots, A_r \cap X)$ is a presentation of $M|X$.*

From general properties of independent sets in matroids, it is transparent that if a set system has a transversal, then each partial transversal is contained in a transversal. This simple example just begins to illustrate the clarity that matroid theory brings to transversal theory. Chapter 7 of [28] is an excellent source for further, deeper applications of matroid theory to transversal theory, including criterion for two set systems to have a common transversal.

2.3. **Properties of transversal matroids that follow easily from the matrix view.** Using algebraically independent elements (as we did above) is far stronger than needed to get a representation. Think of the entries $x_{j,b}$ in our matrix representation of $M[\mathcal{A}]$ as variables. Notice that the $|J| \times |J|$ determinants that are zero are so because all monomials in their expansion are zero; they will stay zero no matter what values in a given field replace the variables $x_{j,b}$. Observe that we need only have enough elements in a field so that, upon replacing the variables by certain field elements, the determinants that are nonzero will remain nonzero. These observations give the following result, which is due to M. Piff and D. Welsh [27]. (Their result is more general than what is stated here; see [25, Proposition 12.2.16] and [28, Theorem 9.6.1]).

Theorem 2.5. *A transversal matroid is representable over all sufficiently large fields; in particular, it is representable over all infinite fields.*

The next theorem, which is often proven from the set system or bipartite graph point of view, has a simple proof from the matrix perspective, as we demonstrate.

Theorem 2.6. *If $(A_1, A_2, \ldots, A_k)$ is a presentation of the transversal matroid M and if some basis $\{b_1, b_2, \ldots, b_r\}$ of M is a transversal of the subsystem $(A_1, A_2, \ldots, A_r)$, then $(A_1, A_2, \ldots, A_r)$ is also a presentation of M. Thus, any transversal matroid of rank r has a presentation by r sets.*

Proof. Consider a matrix representation of M using algebraically independent elements over a given field, as above; we can take the matrix to have the form

$$\left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right)$$

where X is the $r \times r$ submatrix whose rows are indexed by $A_1, A_2, \ldots, A_r$ and whose columns are indexed by $b_1, b_2, \ldots, b_r$. The hypotheses imply that the determinant of X is nonzero and that no $(r+1) \times (r+1)$ submatrix has nonzero determinant. It follows that the first r rows span the row space, so each of the last $k - r$ rows is a linear combination of them. Thus, using row operations, it follows that M is also represented by the matrix $(X \,|\, Y)$, which, interpreted as representing a set system, gives the desired conclusion. □

We will deduce that if $(A_1, \ldots, A_r)$ and $(A_1, \ldots, A_k)$, with $r(M) = r < k$, are both presentations of M, then all elements in $A_{r+1} \cup \cdots \cup A_k$ are coloops of M. Recall that an

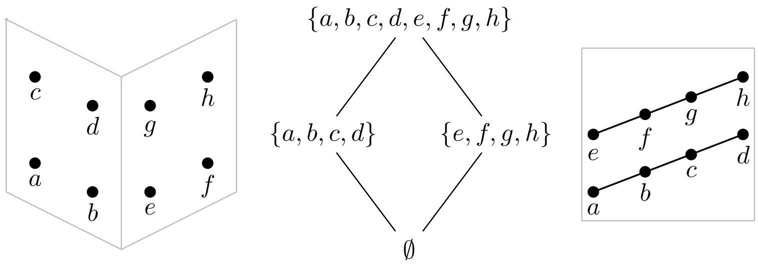


FIGURE 2. Two matroids that have the same lattice of cyclic flats.

element of $E(M)$ is a coloop of M if and only if it is in all bases of M. If a basis B of M did not contain $x \in A_{r+1} \cup \cdots \cup A_k$, then matching the elements of B to $A_1, \ldots, A_r$ and x to one of the remaining sets would yield a larger basis, which is impossible. Thus, the elements of $A_{r+1} \cup \cdots \cup A_k$ are in all bases of M and so are coloops. Thus, we have the following corollary.

Corollary 2.7. *If a transversal matroid M has no coloops, then each presentation of M has exactly $r(M)$ nonempty sets.*

2.4. **Background for the geometric perspective: cyclic flats.** A notion that arises when studying the geometry of transversal matroids (our next topic) is that of a cyclic flat. A set cyclic set X in a matroid M is *cyclic* if X is a union of circuits; equivalently, $M|X$ has no coloops. We use $\mathcal{Z}(M)$ to denote the collection of cyclic flats of M. Theorem 2.4 and Corollary 2.7 give the following useful result about cyclic flats in transversal matroids.

Corollary 2.8. *Let M be a transversal matroid with presentation $(A_1, A_2, \ldots, A_r)$. If $F \in \mathcal{Z}(M)$, then there are exactly $r(F)$ integers i with $F \cap A_i \neq \emptyset$.*

While not required for our purposes here, we make several remarks about the interesting topic of cyclic flats. First, note that, when ordered by inclusion, $\mathcal{Z}(M)$ is a lattice; the join in this lattice agrees with that in the lattice of flats; the meet of two cyclic flats is the union of the circuits in their intersection.

EXERCISE 2.5: Prove the assertions just made about joins and meets of cyclic flats. (Recall that the join, in the lattice of flats, of flats X and Y is cl$(X \cup Y)$.)

Figure 2 shows the lattice of cyclic flats of two matroids; since the lattices are the same, it follows that, unlike the lattice of flats, that of cyclic flats does not determine the matroid. However, T. Brylawski [12] made the observation in the first part of the following exercise.

EXERCISE 2.6: Show that a matroid on a given ground set is determined by its cyclic flats and their ranks. In particular, identify the independent sets and the circuits using the cyclic flats and their ranks.

For instance, the matroid on the left in Figure 2 is determined by knowing that the ranks of the 4-element cyclic flats are 3 and the rank of the ground set is 4; the matroid on the right is determined by knowing that these ranks are 2 and 3, respectively. Making these ranks 2 and 4, respectively, would give a matroid with two skew 4-point lines.

EXERCISE 2.7: Find the other valid ways that ranks can be assigned to the sets in Figure 2 to obtain matroids.

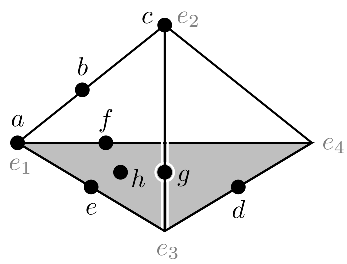


FIGURE 3. The geometric representation of a transversal matroid.

EXERCISE 2.8: Show that the rank function, expressed in terms of cyclic flats and their ranks, is given by

$$r(X) = \min\{r(A) + |X - A| \,:\, A \in \mathcal{Z}(M)\}.$$

Note that a set is a cyclic flat if and only if it is both a union of circuits (cyclic) and its complement is a union of cocircuits (recall that flats are intersections of hyperplanes; also, cocircuits are the complements of hyperplanes). Thus,

$$\mathcal{Z}(M^*) = \{E(M) - F \,:\, F \in \mathcal{Z}(M)\},$$

so the lattice of cyclic flats of M^* can be identified with the order dual of the lattice of cyclic flats of M. (See [7] for more on cyclic flats, including an axiom scheme for matroids in terms of cyclic flats and the ranks assigned to them, and the result that every finite lattice is isomorphic to the lattice of cyclic flats of some transversal matroid.)

2.5. **The geometric perspective on transversal matroids.** The argument we gave above to show that a transversal matroid M of rank r is representable over all infinite fields also shows that it can be represented over $\mathbb{R}$ by a matrix with r rows that has no negative entries. Scaling columns preserves the matroid, so we can assume that each column sum is one. Such columns are in the convex hull,

$$\{\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \cdots + \alpha_r\mathbf{e}_r \,:\, \alpha_i \geq 0 \text{ and } \alpha_1 + \alpha_2 + \cdots + \alpha_r = 1\},$$

of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r$ in $\mathbb{R}^r$, where the i-th entry of $\mathbf{e}_i$ is 1 and all others are 0. This convex hull can be viewed as a simplex Δ in $\mathbb{R}^{r-1}$ with r vertices. Thus, such a matrix representation of M gives a geometric representation in which each nonloop $x \in E(M)$ is placed in a face of Δ according to the positions of the nonzero entries in the corresponding column of the matrix. From Corollary 2.8, a key feature of such a geometric representation follows: any cyclic flat F of M is the set of elements in some $r(F)$-vertex face of Δ. Figure 3 shows the geometric representation of the transversal matroid arising from the set system in our example.

Note that from this geometric representation, we can determine the positions of the nonzero elements in the matrix and so recover the set system. We can do this for any matroid that has a geometric representation on a simplex Δ in which each cyclic flat F of M is the set of elements in some $r(F)$-vertex face of Δ, so such matroids are transversal. Thus, we have the following result of T. Brylawski [12].

Theorem 2.9. *A matroid M is transversal if and only if it has a geometric representation on a simplex Δ in which each cyclic flat F of M consists of the set of elements in some $r(F)$-vertex face of Δ.*

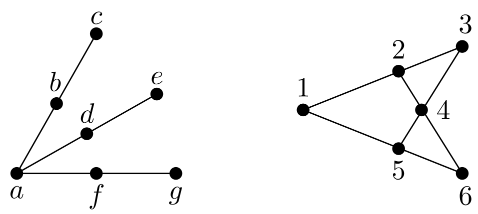


FIGURE 4. Two rank-3 matroids that are not transversal.

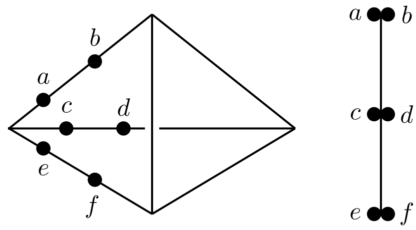


FIGURE 5. A transversal matroid (on the left) whose dual (on the right) is not transversal.

Corollary 2.10. *A rank-r transversal matroid has at most $\binom{r}{k}$ cyclic flats of rank k.*

Geometric representations make it is easy to see that some matroids are transversal and that others are not. For instance, the matroids in Figure 4 are not transversal since they clearly do not have the required geometric representation on the 3-vertex simplex (a triangle). Also, Figures 5 and 6 make it transparent that closure under duality fails for the class of transversal matroids, as does closure under contraction.

EXERCISE 2.9: Consider the set systems $([8], \mathcal{A})$ and $([8], \mathcal{A}')$ with

$$\mathcal{A} = (\{1,2,3,4,5,7,8\}, \{1,2,6,7,8\}, \{3,4,5,6,7,8\})$$

and

$$\mathcal{A}' = (\{1,2,3,4,5,6,7\}, \{1,3,4,5,6,7\}, \{1,2,6,7,8\}).$$

Are $M[\mathcal{A}]$ and $M[\mathcal{A}']$ equal? Draw their geometric representations to find out.

EXERCISE 2.10: Not counting how the vertices of the simplex are labelled and how the elements of the matroid are labelled, there are twenty-three distinct geometric representations of the uniform matroid $U_{3,6}$ on a 3-vertex simplex (a triangle). Find them. Also, pick several of these geometric representations, label the elements of $U_{3,6}$, and find the corresponding presentations. (Keep your twenty-three diagrams; they will be useful for a later exercise.)

EXERCISE 2.11: Identify the faces F of a simplex for which, if, in our geometric representation of a transversal matroid M, an element $x \in E(M)$ is placed freely in F, then M/x is transversal.

free extension

EXERCISE 2.12: The *free extension* $M+e$ of a matroid M by an element $e \notin E(M)$ is the matroid on the ground set $E(M) \cup e$ whose circuits are those of M along with the sets $B \cup e$ as B ranges over the bases of M. Geometrically, e is added to M in the freest possible way without increasing the rank. Use each of the views we have given of transversal matroids to show that if M is transversal, then so is $M+e$.

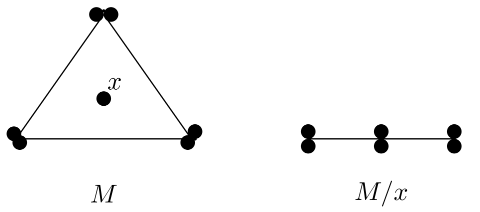


FIGURE 6. A transversal matroid M whose contraction M/x is not transversal.

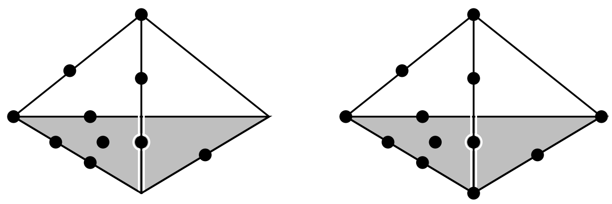


FIGURE 7. A transversal matroid and an extension to a fundamental transversal matroid.

3. CHARACTERIZATIONS OF TRANSVERSAL MATROIDS

Given a class of matroids, a natural question arises: how can members of the class be recognized? That is, what properties distinguish the members of this class from matroids outside the class? In this section, we prove two characterizations of transversal matroids, one of which involves an inequality that relates the ranks of unions and intersections of collections of cyclic flats. We do not claim that the conditions in these characterizations are easy to check for particular matroids; rather, the results are useful, for instance, to show that certain constructions, when applied to transversal matroids, yield transversal matroids (see, for example, [2]; see also [4, 7]). For the purposes of this expository paper, the role of these characterizations is, in part, to give us further insight into transversal matroids.

3.1. **Fundamental transversal matroids and the Mason-Ingleton theorem.** We start this section by using fundamental transversal matroids to motivate a characterization of transversal matroids. A *fundamental transversal matroid* (sometimes called a principal transversal matroid) is a matroid M that has a basis B (called a *fundamental basis*) such that each cyclic flat F of M is spanned by some subset of B, namely $F \cap B$. Therefore, if, in a geometric representation of M, the elements of B are at the vertices of a simplex, then each cyclic flat spans a face of the simplex. It follows that, as the name suggests, such matroids are transversal by Theorem 2.9. Indeed, a transversal matroid is fundamental if and only if some geometric representation of it on a simplex has, for each vertex of the simplex, at least one matroid element placed there. It follows that each transversal matroid can be extended to a fundamental transversal matroid of the same rank (see Figure 7).

fundamental transversal matroid

fundamental basis

EXERCISE 3.1: From the matrix perspective on transversal matroids, using algebraically independent elements over some field, show that the class of all fundamental transversal matroids is closed under duality.

EXERCISE 3.2: Prove the result in the previous exercise directly from the definition, using the following two steps. First show that a basis B is a fundamental basis of M if and only if $r(M) = r(F) + |B - F|$ for all cyclic flats F of M. Next, use the equalities

$$r^*(X) = |X| - r(M) + r(E(M) - X)$$

and

$$\mathcal{Z}(M^*) = \{E(M) - F \,:\, F \in \mathcal{Z}(M)\}$$

to show that B is a fundamental basis of M if and only if $E(M) - B$ is a fundamental basis of M^*.

EXERCISE 3.3: Complete the following statement: a transversal matroid is fundamental if and only if it has a presentation such that ... Also, describe how to a get a presentation of the dual from this perspective.

We claim that for a fundamental transversal matroid M with a fundamental basis B and for cyclic flats $X_1, X_2, \ldots, X_n$ of M,

$$r(X_1 \cup X_2 \cup \cdots \cup X_n) = \big|B \cap (X_1 \cup X_2 \cup \cdots \cup X_n)\big|$$

and

$$r(X_1 \cap X_2 \cap \cdots \cap X_n) = \big|B \cap (X_1 \cap X_2 \cap \cdots \cap X_n)\big|.$$

The first equality holds since the union of n bases, one for each of $X_1, X_2, \ldots, X_n$, spans the union of these sets, and $\bigcup_{i\in[n]}(B \cap X_i)$ is such a union that is also independent. For the second equality, the right side cannot exceed the left since subsets of B are independent, so it suffices to show that each element x in $(\bigcap_{i\in[n]} X_i) - B$ completes a circuit with elements of $\bigcap_{i\in[n]}(B \cap X_i)$. Since x is not in the basis B, the set $B \cup x$ contains a unique circuit, say C. Since $B \cap X_i$ is a basis of X_i, for $i \in [n]$, the set $(B \cap X_i) \cup x$ contains a unique circuit; the uniqueness of C gives $C - x \subseteq B \cap X_i$, so, as needed, $C - x \subseteq \bigcap_{i\in[n]}(B \cap X_i)$.

From the two displayed equalities above along with the equation

$$\big|A_1 \cap A_2 \cap \cdots \cap A_n\big| = \sum_{J \subseteq [n]} (-1)^{|J|+1} \Big|\bigcup_{j\in J} A_j\Big|$$

(which follows from the more common form of inclusion/exclusion by taking complements) applied to the sets $A_i = B \cap X_i$, for $i \in [n]$, we get the following rank equality in any fundamental transversal matroid M. For any nonempty collection $\{X_1, X_2, \ldots, X_n\}$ of cyclic flats of M,

$$r(X_1 \cap X_2 \cap \cdots \cap X_n) = \sum_{J \subseteq [n]} (-1)^{|J|+1} r\Big(\bigcup_{j\in J} X_j\Big).$$

As noted above, a transversal matroid M can be extended to a fundamental transversal matroid M' of the same rank. Each cyclic flat F of M spans a cyclic flat, $\mathrm{cl}_{M'}(F)$, of M'. The rank of a union of cyclic flats in M is the same as the rank of the union of the corresponding cyclic flats in M' since both unions span the same flat of M'; in contrast, as illustrated in Figure 7, the ranks of intersections of cyclic flats in M can be lower than the ranks of their counterparts in M'. It follows that a necessary condition for a matroid M to be transversal is that, for all nonempty collections $\{X_1, X_2, \ldots, X_n\}$ of cyclic flats of M,

$$r(X_1 \cap X_2 \cap \cdots \cap X_n) \leq \sum_{J \subseteq [n]} (-1)^{|J|+1} r\Big(\bigcup_{j\in J} X_j\Big).$$

This necessary condition turns out to be sufficient, as the next theorem states. This result was proven by J. Mason [23] for cyclic sets; the simpler formulation in Theorem 3.1 was observed by A. Ingleton [19] to be equivalent to J. Mason's result. We adopt the following abbreviations for intersections and unions of collections of set: for $\mathcal{F} \subseteq \mathcal{Z}(M)$,

$$\cap\mathcal{F} = \bigcap_{X\in\mathcal{F}} X \quad \text{and} \quad \cup\,\mathcal{F} = \bigcup_{X\in\mathcal{F}} X.$$

Theorem 3.1. *A matroid M is transversal if and only if for all nonempty subsets $\mathcal{F}$ of $\mathcal{Z}(M)$,*

$$r(\cap\mathcal{F}) \leq \sum_{\mathcal{F}'\subseteq\mathcal{F}} (-1)^{|\mathcal{F}'|+1} r(\cup\mathcal{F}'). \tag{3.1}$$

EXERCISE 3.4: We noted that the matroids in Figure 4 are not transversal. For each, find a family $\mathcal{F}$ of cyclic flats for which inequality (3.1) fails.

We have proven half of this result, namely, that inequality (3.1) holds for all nonempty subsets of $\mathcal{Z}(M)$ if M is transversal. We next develop a circle of ideas that we use to prove the remaining implication and, in the process, obtain another characterization of transversal matroids. To complete the proof of Theorem 3.1, we construct a presentation of M. Thus, we need to know what types of sets can be in a presentation and we need to know about their multiplicities.

3.2. **Sets in presentations.** Let $\mathcal{A} = (A_1, A_2, \ldots, A_r)$ be a presentation of M. We claim that for each $i \in [r]$, the complement $A_i^c = E(M) - A_i$ of A_i is a flat of M. To see this, consider a matrix representation of the type we saw in Section 2.2. The elements in A_i^c correspond to the columns that have 0 in the i-th row; obviously no other column is in their closure, so, as claimed, A_i^c is a flat.

EXERCISE 3.5: Give a second proof that A_i^c is a flat of M by using partial transversals (or matchings in bipartite graphs) to show $r(A_i^c \cup x) = r(A_i^c) + 1$ for all for $x \in A_i$.

As stated in the corollary below, iteratively applying the next lemma shows that the sets in any presentation of M can be enlarged to give a presentation $(A'_1, A'_2, \ldots, A'_r)$ of M so that no deletion $M\backslash A'_i$ has coloops, that is, each complement A'^c_i is a cyclic flat of M.

Lemma 3.2. *Let M be a transversal matroid with presentation $\mathcal{A} = (A_1, A_2, \ldots, A_r)$. If x is a coloop of $M\backslash A_i$, then $\mathcal{A}' = (A_1, A_2, \ldots, A_{i-1}, A_i \cup x, A_{i+1}, \ldots, A_r)$ is also a presentation of M.*

Proof. To simplify the notation slightly, we take $i = 1$, so $\mathcal{A}' = (A_1 \cup x, A_2, \ldots, A_r)$. Obviously any transversal of $\mathcal{A}$ is a transversal of $\mathcal{A}'$. The converse is clear except when x is in a transversal T of $\mathcal{A}'$ and the bijection $\phi : [r] \to T$ that is used to show that T is a transversal has $\phi(1) = x$. To treat this situation, we take the bipartite graph perspective. The argument below is illustrated in Figure 8. The bijection ϕ, when we ignore x and A_1, determines a matching of the vertices in $T - x$ with the vertices $A_2, A_3, \ldots, A_r$; color the edges in this matching red. A suitable restriction of ϕ shows that $T - (A_1 \cup x)$ is a partial transversal of $(A_2, A_3, \ldots, A_r)$. Since x is a coloop of $M\backslash A_1$, it follows that $T - A_1$ is a partial transversal of $(A_2, A_3, \ldots, A_r)$; color the edges in some matching of $T - A_1$ into $A_2, A_3, \ldots, A_r$ blue. Note that each vertex is incident with at most one edge of each color, so the components of the graph induced by the colored edges are cycles and paths. The vertex x is incident with only one colored edge (which is blue), so the component of colored edges that x is in is a path. Since each of $A_2, A_3, \ldots, A_r$ is incident with exactly one red edge, it follows that the other end-vertex of this path is an element, say y, of $A_1 \cap T$. Retaining the red edges outside of this path and swapping the blue and red edges in this

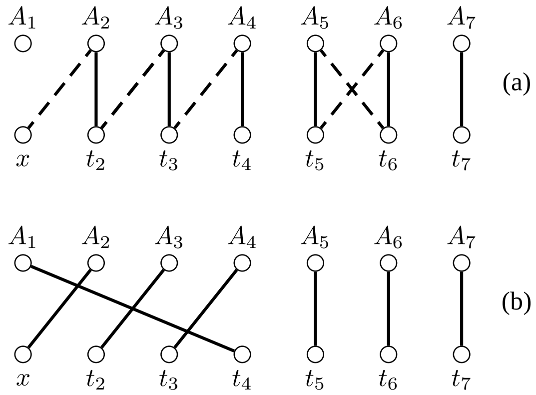


FIGURE 8. An example of the argument in the proof of Lemma 3.2. The solid edges in part (a) give a matching of $T - x$ with $(A_2, A_3, \ldots, A_r)$; the dashed edges give a matching of $T - A_1$ into $(A_2, A_3, \ldots, A_r)$. The matching of T with $(A_1, A_2, \ldots, A_r)$ in part (b) uses the the dashed edges in the component that contains x and the solid edges in the other components, and it matches t_4 (the element y in the proof) with A_1.

path gives a collection of red edges that, when augmented by matching y with A_1, shows that T is a transversal of $\mathcal{A}$, as needed. □

Corollary 3.3. *For any presentation $(A_1, A_2, \ldots, A_r)$ of a transversal matroid M, there is a presentation $(A'_1, A'_2, \ldots, A'_r)$ of M with $A_i \subseteq A'_i$ and $A'^c_i \in \mathcal{Z}(M)$ for $i \in [r]$.*

EXERCISE 3.6: Prove the converse of Lemma 3.2: if $\mathcal{A}$ and $\mathcal{A}'$ (as in the statement of the lemma) are both presentations of M, then x is a coloop of $M\backslash A_i$.

3.3. **Multiplicities of complements of cyclic flats in presentations.** As stated above, we aim to construct a presentation $\mathcal{A}$ of a matroid M if inequality (3.1) holds for all nonempty sets of cyclic flats. The presentation will consist of complements of cyclic flats. We turn to the multiplicity in $\mathcal{A}$ that we should assign to F^c for each $F \in \mathcal{Z}(M)$. Of course, the sum of the multiplicities must be $r(M)$, which we shorten to r. We will define a function β so that for $F \in \mathcal{Z}(M)$, the multiplicity of F^c in $\mathcal{A}$ will be $\beta(F)$. To motivate the definition of β, we note that, by Corollary 2.8, for each $F \in \mathcal{Z}(M)$ we must have

$$\sum_{Y \in \mathcal{Z}(M)\,:\,F \cap Y^c \neq \emptyset} \beta(Y) = r(F), \tag{3.2}$$

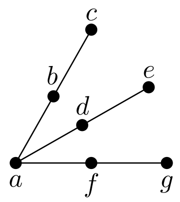
or, equivalently,

$$\sum_{Y \in \mathcal{Z}(M)\,:\,F \subseteq Y} \beta(Y) = r - r(F). \tag{3.3}$$

definition of $\beta(X)$

With this motivation, we define β recursively on all subsets X of $E(M)$ by

$$\beta(X) = r - r(X) - \sum_{Y \in \mathcal{Z}(M)\,:\,X \subset Y} \beta(Y). \tag{3.4}$$



$$\beta(a,b,c) = 1 - 0 = 1$$
$$\beta(a) = 2 - 1 - 1 - 1 = -1$$
$$\beta(\emptyset) = 3 - 1 - 1 - 1 = 0$$

FIGURE 9. Several values of β that arise from a rank-3 matroid.

Note that equation (3.3) holds for $F \in \mathcal{Z}(M)$. Applying equation (3.3) to the least cyclic flat, cl($\emptyset$), gives

$$\sum_{Y \in \mathcal{Z}(M)} \beta(Y) = r. \tag{3.5}$$

Equation (3.2) now follows.

Figure 9 gives an example of computing some values of β on a non-transversal matroid; as we will see below, it is no mere coincidence that β takes on negative values. As an example of computing β on a transversal matroid, take the uniform matroid $U_{r,n}$ of rank r on $[n]$ with $r < n$. We have $\beta(X) = r - r(X)$ for each $X \subseteq [n]$; the only cyclic flat F with $\beta(F) \neq 0$ is $F = \emptyset$, so the resulting presentation of $U_{r,n}$ has just one set, $[n]$, and this set has multiplicity $\beta(\emptyset)$, which is r. For another example, let M be a transversal matroid whose geometric representation on the r-vertex simplex has at least two parallel elements at each vertex (and possibly more elements). Each cyclic hyperplane H (which is spanned by $r-1$ parallel pairs) has $\beta(H) = 1$; it follows that β is zero on all other cyclic flats. Thus, the resulting presentation of M consists of the complements of the r cyclic hyperplanes.

EXERCISE 3.7: For the second matroid in Figure 4, find a set X with $\beta(X) < 0$.

EXERCISE 3.8: For the transversal matroid shown in Figure 3, find the presentation that β gives. Also, draw the geometric representation that corresponds to this presentation. (The geometric representation will differ from the one in Figure 3.)

The matroid in Figure 9 is not transversal; also, $\beta(a) < 0$. This is consistent with the following theorem, which, when combined with the half of Theorem 3.1 that we have already proven, shows that β is never negative on transversal matroids. In the proof, we use the following basic algebraic result: if K is a nonempty set, then

$$\sum_{J\,:\,\emptyset \subset J \subseteq K} (-1)^{|J|+1} = 1. \tag{3.6}$$

This equality holds since, upon setting $|K| = n$, the sum can be written as

$$\binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \cdots + (-1)^{n+1}\binom{n}{n};$$

the alternating sum of all binomial coefficients $\binom{n}{i}$ is zero, so the displayed sum is $\binom{n}{0} = 1$.

Theorem 3.4. *If inequality (3.1) holds for all nonempty collections of cyclic flats of M, then $\beta(X) \geq 0$ for all $X \subseteq E(M)$.*

Proof. For $X \subseteq E(M)$, set $\mathcal{F}(X) = \{Y \in \mathcal{Z}(M) : X \subset Y\}$, so $\mathcal{F}(X)$ contains the sets Y in the range of summation in the definition of $\beta(X)$. By the definition of β, we have

$\beta(X) \geq 0$ if and only if

$$\sum_{Y \in \mathcal{F}(X)} \beta(Y) \leq r - r(X).$$

The displayed inequality trivially holds if $\mathcal{F}(X) = \emptyset$. If $\mathcal{F}(X) \neq \emptyset$, then we have

$$\sum_{Y \in \mathcal{F}(X)} \beta(Y) = \sum_{Y \in \mathcal{F}(X)} \beta(Y) \sum_{\emptyset \subset \mathcal{F}' \subseteq \{Z \in \mathcal{F}(X)\,:\, Z \subseteq Y\}} (-1)^{|\mathcal{F}'|+1} \tag{3.7}$$

$$= \sum_{\emptyset \subset \mathcal{F}' \subseteq \mathcal{F}(X)} (-1)^{|\mathcal{F}'|+1} \sum_{Y \in \mathcal{F}(X)\,:\, \cup \mathcal{F}' \subseteq Y} \beta(Y) \tag{3.8}$$

$$= \sum_{\emptyset \subset \mathcal{F}' \subseteq \mathcal{F}(X)} (-1)^{|\mathcal{F}'|+1} \big(r - r(\cup \mathcal{F}')\big) \tag{3.9}$$

$$= r - \sum_{\emptyset \subset \mathcal{F}' \subseteq \mathcal{F}(X)} (-1)^{|\mathcal{F}'|+1} r(\cup \mathcal{F}') \tag{3.10}$$

$$\leq r - r(\cap \mathcal{F}(X)) \tag{3.11}$$

$$\leq r - r(X).$$

(Line (3.7) and the simplification from line (3.9) to (3.10) use equation (3.6). Line (3.8) changes the order of summation. Line (3.9) uses equation (3.3) and the observation that the union spans a cyclic flat. Inequality (3.1) connects lines (3.10) and (3.11). Clearly $X \subseteq \cap \mathcal{F}(X)$, so $r(X) \leq r(\cap \mathcal{F}(X))$, from which the last line follows.) □

3.4. **Completion of the proof of the Mason-Ingleton theorem.** Theorems 3.4 and 3.5 complete the proof of Theorem 3.1.

Theorem 3.5. *If $\beta(X) \geq 0$ for all $X \subseteq E(M)$, then M is transversal. One presentation $\mathcal{A}$ of M consists of the sets F^c, for $F \in \mathcal{Z}(M)$, with F^c having multiplicity $\beta(F)$ in $\mathcal{A}$.*

Proof. Let $r = r(M)$. By equation (3.5), the number of sets in $\mathcal{A}$, counting multiplicity, is r. Our goal is to show $M = M[\mathcal{A}]$.

To show that each dependent set X of M is dependent in $M[\mathcal{A}]$, it suffices to show this when X is a circuit of M. In this case, cl(X) is a cyclic flat of M, so, by equation (3.2) and the definition of $\mathcal{A}$, it has nonempty intersection with exactly $r(X)$ sets of $\mathcal{A}$, counting multiplicity. However, this conclusion and the inequality $r(X) < |X|$ imply that X cannot be a partial transversal of $\mathcal{A}$, so it is dependent in $M[\mathcal{A}]$.

Write $\mathcal{A}$ as $(F_1^c, F_2^c, \dots, F_r^c)$. To show that each independent set of M is independent in $M[\mathcal{A}]$, it suffices to show this for an arbitrary basis B of M. For this, it suffices to show that the set system $\mathcal{A}_B = (F_1^c \cap B, F_2^c \cap B, \dots, F_r^c \cap B)$ has a transversal (which must be B). We apply Theorem 2.1: $\mathcal{A}_B$ has a transversal if and only if, for each $j \leq r$, each union, say X, of any j sets in $\mathcal{A}_B$ contains at least j elements. Such a set X has the form $X = \bigcup_{i \in J}(F_i^c \cap B)$ with $J \subseteq [r]$ and $|J| = j$. It follows that $B - X \subseteq \bigcap_{i \in J} F_i$, so there are at least j sets F_k^c in $\mathcal{A}$ with $B - X \subseteq F_k$; thus, as needed,

$$\begin{aligned} j &\leq \sum_{Y \in \mathcal{Z}(M)\,:\, B - X \subseteq Y} \beta(Y) \\ &\leq r - r(B - X) \\ &= |B| - |B - X| \\ &\leq |X|. \end{aligned}$$

(The assumption that β is nonnegative plays a role in the first line and it is used to get the second line from that; recall the first steps in the proof of Theorem 3.4.) □

Observe that we have proven the cycle of implications in the following theorem. (The equivalence of statements (1) and (3) is the dual of a result in [24], which was also shown in [20, 22].)

Theorem 3.6. *For any matroid M, the following statements are equivalent:*

(1) *M is transversal,*
(2) *for every nonempty subset $\mathcal{F}$ of $\mathcal{Z}(M)$,*

$$r(\cap\mathcal{F}) \leq \sum_{\mathcal{F}'\subseteq\mathcal{F}} (-1)^{|\mathcal{F}'|+1} r(\cup\mathcal{F}'), \tag{3.12}$$

(3) *$\beta(X) \geq 0$ for all $X \subseteq E(M)$.*

Note that for $X, Y \in \mathcal{F}$ with $X \subset Y$, using $\mathcal{F} - \{Y\}$ in place of $\mathcal{F}$ does not change either side of inequality (3.12); with $\mathcal{F}$, the terms on the right side that include Y cancel via the involution that maps $\mathcal{F}'$ to the symmetric difference $\mathcal{F}' \triangle \{X\}$. Thus, statements (1)–(3) are equivalent to the statement that inequality (3.12) holds for all nonempty sets $\mathcal{F}$ of incomparable flats in $\mathcal{Z}(M)$. Note also that two cases of this inequality automatically hold: equality holds when $|\mathcal{A}| = 1$ and the case of $|\mathcal{A}| = 2$ is the semimodular inequality.

EXERCISE 3.9: Use Theorem 3.1 to prove the theorem below. (As with Theorem 3.1, this result was first formulated by J. Mason using cyclic sets; A. Ingleton observed that the refinement to cyclic flats follows easily from that.)

Theorem 3.7. *A matroid M of rank r is transversal if and only if there is an injection ϕ from $\mathcal{Z}(M)$ into the set of subsets of $[r]$ with*

(1) *$|\phi(F)| = r(F)$ for all $F \in \mathcal{Z}(M)$,*
(2) *$\phi(\mathrm{cl}(F \cup G)) = \phi(F) \cup \phi(G)$ for all $F, G \in \mathcal{Z}(M)$, and*
(3) *$r(\cap\mathcal{F}) \leq |\cap\{\phi(F) : F \in \mathcal{F}\}|$ for every subset $\mathcal{F}$ of $\mathcal{Z}(M)$.*

3.5. **Several further results.** Let $\mathcal{A} = (A_1, A_2, \ldots, A_r)$ be a presentation of M. We say that $\mathcal{A}$ is *maximal* if, whenever $(A'_1, A'_2, \ldots, A'_r)$ is a presentation of M with $A_i \subseteq A'_i$ for $i \in [r]$, then $A_i = A'_i$ for $i \in [r]$. In other words, $\mathcal{A}$ is maximal if $M[\mathcal{A}'] \neq M$ for every set system $\mathcal{A}'$ that is obtained by adding elements of $E(M)$ to some sets in $\mathcal{A}$. Similarly, $\mathcal{A}$ is *minimal* if, whenever $(A'_1, A'_2, \ldots, A'_r)$ is a presentation of M with $A'_i \subseteq A_i$ for $i \in [r]$, then $A_i = A'_i$ for $i \in [r]$. Thus, $\mathcal{A}$ is minimal if $M[\mathcal{A}'] \neq M$ for every set system $\mathcal{A}'$ that is obtained by removing elements from some sets in $\mathcal{A}$.

maximal presentation

minimal presentation

EXERCISE 3.10: Think about what it means geometrically for both $(A_1, A_2, \ldots, A_r)$ and $(A'_1, A'_2, \ldots, A'_r)$, where $A'_i \subseteq A_i$ for $i \in [r]$, to be presentations of the same transversal matroid. With this insight, follow up your work on Exercise 2.10 by determining which of the twenty-three distinct affine representations of $U_{3,6}$ are minimal.

In contrast to what you just saw in the exercise above, we have the following result of J. Mason on maximal presentations.

Theorem 3.8. *Each transversal matroid has exactly one maximal presentation.*

Proof. By Corollary 3.3, in any maximal presentation $(A_1, A_2, \ldots, A_r)$ of a transversal matroid M, each complement A_i^c is a cyclic flat of M. By Corollary 2.8, for each cyclic flat F of M, there are exactly $r(F)$ integers i with $F \cap A_i \neq \emptyset$, i.e., $F \not\subseteq A_i^c$. Arguing recursively from the largest cyclic flat to the smallest, it follows that only one assignment of multiplicities (the one that β gives) meets these conditions. □

Beyond being inherently interesting, this result says that the maximal presentation is a canonical form that can be used to determine whether two presentations yield the same transversal matroid. From each presentation, one can use Lemma 3.2 to enlarge the sets in the presentation until the maximal presentation is obtained; then one checks whether the maximal presentations are the same. (In contrast, it is a frequent source of difficulty in many parts of matroid theory that many matroids have many different representations without having a way of getting from one to another, or to a preferred representation. This is a major issue when studying matroids that are representable over a given field, except in the cases of a few very small fields.)

In Section 3.1 we showed that equality holds in inequality (3.12) for any fundamental transversal matroid. The following result from [6] includes the converse.

Theorem 3.9. *For any matroid M, the following statements are equivalent:*

(1) *M is a fundamental transversal matroid,*
(2) *for every nonempty subset $\mathcal{F}$ of $\mathcal{Z}(M)$, we have*

$$r(\cap\mathcal{F}) = \sum_{\mathcal{F}'\subseteq\mathcal{F}} (-1)^{|\mathcal{F}'|+1} r(\cup\mathcal{F}'),$$

(3) *for every filter $\mathcal{F}$ in $\mathcal{Z}(M)$, we have*

$$\sum_{Y\in\mathcal{F}} \beta(Y) = r(M) - r(\cap\mathcal{F}).$$

filter

A *filter* in $\mathcal{Z}(M)$ is a subset $\mathcal{F}$ of $\mathcal{Z}(M)$ such that if $A, B \in \mathcal{Z}(M)$ with $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$. As in Theorem 3.6, the collections $\mathcal{F}$ in statement (2) can be limited to collections of incomparable sets; they can also be limited to filters (this can be done in Theorems 3.6 and 3.7 too).

To close this section, we note that [10, 19, 28] contain several other characterizations of transversal matroids as well as a variety of results about presentations that have special properties.

4. Lattice Path Matroids

To illustrate a variety of basic concepts in matroid theory and to expose you to some recent developments in the theory of transversal matroids, we turn to transversal matroids that arise from lattice paths. This class of matroids (introduced in [9] and studied further in [3, 8]), while fairly large, is small relative to the class of transversal matroids: in [12] it is shown that the number of fundamental transversal matroids on n elements is on the order of c^{n^2} for some constant c; in comparison, the number of lattice path matroids on n elements is on the order of d^n for some d (see the fourth exercise in Section 4.2). However, this smaller class of matroids has many attractive properties that are not shared by transversal matroids in general.

lattice path

4.1. Sets of lattice paths and related set systems.

A *lattice path* is a sequence of east and north steps of unit length, starting at $(0, 0)$. We sometimes write paths as strings of Es and Ns. For instance, the dotted path in Figure 10 is $NENEEEENENE$, or, slightly more compactly, $(NE)^2E^2(NE)^2$.

P, Q

$\mathcal{P}$

Our interest is not in individual lattice paths per se, but in collections of paths. To obtain a collection of interest, fix two lattice paths P and Q from $(0, 0)$ to (m, r) with P never going above Q. (The paths may meet at points in addition to the ends.) Let $\mathcal{P}$ be the set of lattice paths from $(0, 0)$ to (m, r) that stay in the region that P and Q bound. Such bounding paths and a typical path in $\mathcal{P}$ are illustrated in Figure 10.

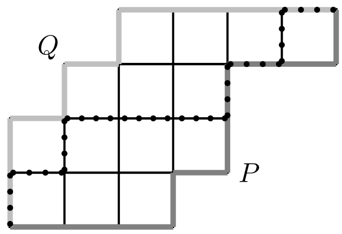


FIGURE 10. A region bounded by two lattice paths (in two shades of gray) and a path (dotted) in the region.

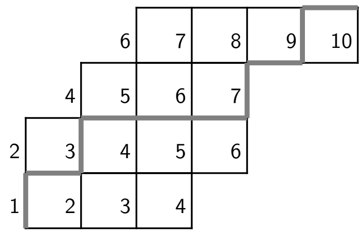


FIGURE 11. The labelling of the north steps for the diagram in Figure 10 along with the path whose first, third, seventh, and ninth steps are north.

A simple but important observation is that each path in $\mathcal{P}$ is determined by its north steps. For example, knowing that first, third, seventh, and ninth steps are north (and that paths in the given region have ten steps) specifies the dotted path in Figure 10.

Our aim is to interpret the paths in $\mathcal{P}$ as the bases of a transversal matroid. To do this, label each north step in the region by the position it has in each path it is in. Note that many north steps can receive a given label, but that all north steps with a given label have a line of slope -1 through their centers. Such a labelling is shown in Figure 11.

For $1 \leq i \leq r$, let N_i be the set of all labels on the north steps in row i of the diagram, $(N_1, N_2, \ldots, N_r)$
indexed from the bottom up. Thus, N_i is the set of positions (in the sequence of steps) where the i-th north step of a path in $\mathcal{P}$ could be. In the example in Figure 11, these sets are $N_1 = \{1,2,3,4\}$, $N_2 = \{2,3,4,5,6\}$, $N_3 = \{4,5,6,7\}$, and $N_4 = \{6,7,8,9,10\}$.

In general, each set N_i is an interval $[a_i, b_i]$ of integers; also, $1 \leq a_1 < a_2 < \cdots < a_r$ $N_i = [a_i, b_i]$
and $b_1 < b_2 < \cdots < b_r \leq m+r$. Note that none of these intervals contains another. From this collection of intervals, the bounding paths P and Q can be recovered.

4.2. **Lattice path matroids; interpreting paths as bases.** Using the notation established above, the matroid $M[P,Q]$ is the transversal matroid $M[(N_1, N_2, \ldots, N_r)]$ on $[m+r]$. $M[P,Q]$
Thus, the set system $(N_1, N_2, \ldots, N_r)$ is a presentation of $M[P,Q]$. A *lattice path matroid* is a matroid that is isomorphic to $M[P,Q]$ for some such P and Q. lattice path matroid

From the remarks at the end of Section 4.1, it follows that an equivalent definition of a lattice path matroid is that it is a transversal matroid that has a presentation by a collection of intervals in the ground set, relative to some fixed linear order, such that no interval contains another. For our purposes, we favor the lattice path view since it makes many properties transparent.

In the next result, we achieve the goal stated above: we interpret the paths in $\mathcal{P}$ as the bases of the transversal matroid $M[P,Q]$.

Theorem 4.1. *The map $R \mapsto \{i : \text{the } i\text{-th step of } R \text{ is north}\}$ is a bijection between $\mathcal{P}$ and the set of bases of $M[P,Q]$.*

Proof. Clearly each path maps to a transversal of the set system. Also, the map is injective since a path is determined by where its north steps occur, so we need only show that each transversal $T = \{t_1, t_2, \dots, t_r\}$ of $(N_1, N_2, \dots, N_r)$ arises from a path. We may assume that $t_1 < t_2 < \cdots < t_r$. Note that if we show that t_i is in N_i, then it is clear that T arises from a path. If, to the contrary, $t_i \notin N_i = [a_i, b_i]$, then we have the following two possibilities, both of which contradict T being a transversal: if $t_i < a_i$, then the i elements $t_1, t_2 \dots, t_i$ are in at most $i-1$ sets, namely, $N_1, N_2, \dots, N_{i-1}$; if $t_i > b_i$, then the $r-i+1$ elements $t_i, t_{i+1}, \dots, t_r$ are in at most $r-i$ sets, namely $N_{i+1}, N_{i+2}, \dots, N_r$. □

Corollary 4.2. *The number of bases of $M[P,Q]$ is $|\mathcal{P}|$.*

EXERCISE 4.1: Show that the element i is a coloop of $M[P,Q]$ if and only if the i-th steps of P and Q coincide and are north.

EXERCISE 4.2: Show that $(N_1, N_2, \dots, N_r)$ is a minimal presentation of $M[P,Q]$.

EXERCISE 4.3: Consider lattice path matroids $M[P,Q]$ where P has the form $E^m N^r$. Show that if the path presentation $(N_1, N_2, \dots, N_r)$ of $M[P,Q]$ is

$$([a_1, m+1], [a_2, m+2], \dots, [a_r, m+r]),$$

then another presentation of $M[P,Q]$ is given by the system of nested intervals

$$([a_1, m+r], [a_2, m+r], \dots, [a_r, m+r]).$$

(Lemma 3.2 may be useful. For more assistance, look ahead to Lemma 4.7 and its proof.) Use this result to get a geometric representation of $M[P,Q]$ on the r-vertex simplex and to identify the cyclic flats of $M[P,Q]$ along with their ranks. What simple structure does the lattice of cyclic flats have? Conclude that $M[P,Q]$ is isomorphic to $M[P',Q]$ if and only if $P = P'$.

EXERCISE 4.4: Show that the number of lattice path matroids $M[P,Q]$ on $[n]$ is between 2^n and 4^n.

4.3. **Duality, direct sums, connectivity, spanning circuits, and minors.** A number of matroid operations have appealing interpretations for lattice path matroids. One of the simplest and most attractive interpretations is for duality. Recall that the dual M^* of M has $\{E(M) - B : B \text{ is a basis of } M\}$ as its set of bases. The next result, which stands in contrast to what we saw for transversal matroids, is transparent once we observe that reflecting one of our diagrams about the line $y = x$ interchanges north steps and east steps: what are the east steps of paths (the complements of bases) in the original diagram are the north steps of paths (the bases) in the reflected diagram. This is illustrated in Figure 12.

Theorem 4.3. *The class of lattice path matroids is closed under duality.*

EXERCISE 4.5: With duality and the first exercise in Section 4.2, characterize the loops of $M[P,Q]$.

Since the bases of the direct sum $M_1 \oplus M_2$ are the sets $B_1 \cup B_2$, where B_i is a basis of M_i, thinking about Figure 13 makes the following result obvious.

Theorem 4.4. *The class of lattice path matroids is closed under direct sums.*

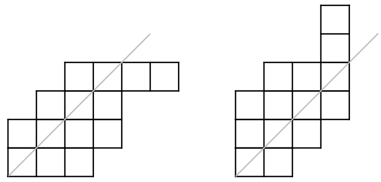


FIGURE 12. The path representations of a lattice path matroid and its dual.

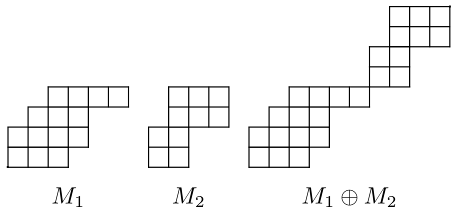


FIGURE 13. The path representations of two lattice path matroids and their direct sum.

(Recall that $M_1 \oplus M_2$ is defined only when $E(M_1)$ and $E(M_2)$ are disjoint, yet all sets $E(M[P,Q])$ include 1. This is part of why we defined lattice path matroids to be matroids that are isomorphic to matroids of the form $M[P,Q]$. Thus, in Figure 13, we have in mind that the elements have been relabelled so that the ground sets are disjoint.)

It follows that if $M[P,Q]$ is connected, then P and Q meet only at the ends. The converse is also true, as we will show next. We will use the following exercise.

EXERCISE 4.6: Show that any matroid M that has no loops and that has a spanning circuit (that is, a circuit of rank $r(M)$) is connected.

Assume P and Q meet only at the ends. Matroids with just one element are connected, so assume $M[P,Q]$ has at least two elements. From the result you obtained in the first exercise in this section, it follows that $M[P,Q]$ has no loops. Thus, if we can show that $M[P,Q]$ has a spanning circuit, it will follow that $M[P,Q]$ is connected. Let x be any element of $M[P,Q]$ that is in at least two sets among $N_1, N_2, \ldots, N_r$, say $x \in N_i \cap N_{i+1}$. We will construct a spanning circuit of $M[P,Q]$ that contains x. The assumption gives $a_h \in N_{h-1} \cap N_h$ if $h > 1$ and $b_k \in N_k \cap N_{k+1}$ if $k < r$. It follows easily that each r-subset of $C = \{a_1, a_2, \ldots, a_i, x, b_{i+1}, \ldots, b_r\}$ is a transversal of $(N_1, N_2, \ldots, N_r)$, and hence a basis of $M[P,Q]$, so C is a spanning circuit. Note also that a_1 could be replaced by any element that is in only N_1; similarly, b_r could be replaced by any element that is in only N_r. Thus, we have the following result.

Theorem 4.5. *The matroid $M[P,Q]$ is connected if and only if P and Q intersect only at the ends. Also, an element of $M[P,Q]$ is in a spanning circuit of $M[P,Q]$ if and only if it is in at least two sets among $N_1, N_2, \ldots, N_r$, or it is in either N_1 or N_r.*

Now that we understand the connected components of lattice path matroids, we can address another basic question. We know that the bases of $M[P,Q]$ correspond to the paths in $\mathcal{P}$. Does the isomorphism type of $M[P,Q]$ capture the paths? It does provided we take certain symmetries into account. The following operations on regions, which are illustrated in Figure 14, yield isomorphic matroids:

(1) permuting the connected components and

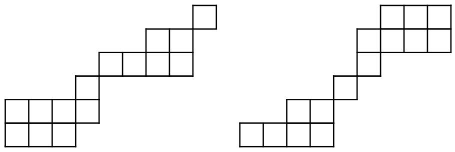


FIGURE 14. The path representations of two isomorphic lattice path matroids.

(2) rotating, by 180°, the region corresponding to any connected component.

The next theorem follows from results in [8].

Theorem 4.6. *The paths P and Q are determined by any matroid isomorphic to $M[P, Q]$ up to operations (1) and (2) above.*

To show that the class of lattice path matroids (unlike that of transversal matroids) is closed under minors, with Theorem 4.3 and the fact that deletion and contraction are dual operations, it suffices to show that the class is closed under deletion. One can prove this from the lattice path perspective (see [8]); to illustrate another approach, we instead use the interval perspective. As noted before Theorem 4.1, a lattice path matroid can be seen as a transversal matroid that has a presentation by a collection of incomparable intervals relative to a fixed linear order on the ground set. The next lemma shows that some comparabilities among the intervals can be allowed.

Lemma 4.7. *Fix a linear order on a set S. Let $\mathcal{A}$ be $(N_1, N_2, \dots, N_r)$ where N_i is an interval $[a_i, b_i]$ in the linear order. If $a_1 \leq a_2 \leq \cdots \leq a_r$ and $b_1 \leq b_2 \leq \cdots \leq b_r$, then $M[\mathcal{A}]$ is a lattice path matroid.*

Proof. We claim that if $a_i = a_{i+1} < a_{i+2}$, then $M[\mathcal{A}] = M[\mathcal{A}']$ where $\mathcal{A}'$ is obtained from $\mathcal{A}$ by replacing N_{i+1} by $[a_i', b_{i+1}]$, where a_i' is the successor of a_i in the linear order. Since $N_i - [a_i', b_{i+1}]$ is the singleton $\{a_i\}$, it follows that a_i becomes a coloop when we delete $[a_i', b_{i+1}]$ from $M[\mathcal{A}']$, so, as desired, we get $M[\mathcal{A}] = M[\mathcal{A}']$ by Lemma 3.2. The result in the lemma follows since this process and its counterpart for upper endpoints can be iterated until we obtain a presentation by incomparable intervals. □

The next theorem follows from this lemma and Theorem 2.4, using the induced linear order on any subset of a linear order.

Theorem 4.8. *The class of lattice path matroids is closed under deletions and so under minors.*

Given that the class of lattice path matroids is closed under minors, one can ask for the excluded-minor characterization of this class, that is, the characterization of this class that results by finding the minor-minimal obstructions to being in the class. (More precisely, excluded minor an *excluded minor* for a minor-closed class $\mathcal{C}$ of matroids is a matroid M with $M \notin \mathcal{C}$ such that for all $x \in E(M)$, both $M\backslash x$ and M/x are in $\mathcal{C}$.) A result in this spirit that readers may be familiar with is Kuratowski's characterization of planar graphs: a graph is planar if and only if it has neither K_5 nor $K_{3,3}$ as a minor. Thus, K_5 and $K_{3,3}$ are the two excluded minors for the minor-closed class of planar graphs. The class of lattice path matroids has such a characterization [3]; however, in contrast to Kuratowski's theorem, there are infinitely many excluded minors. Fortunately, all but four of the excluded minors naturally fall into five infinite families that have common geometric features. (As we saw in Section 2.5, the class of transversal matroids is not closed under minors; therefore, it has no characterization by excluded minors. Attempts to characterize transversal matroids by the more restrictive notion of excluded series-minors (which are related to topological minors of graphs) have proven unwieldy.)

5. TUTTE POLYNOMIALS OF LATTICE PATH MATROIDS

In this section, we present one of the most striking enumerative properties of lattice path matroid: a very important polynomial that is extremely difficult to compute for most matroids can be computed easily for lattice path matroid.

5.1. **A brief introduction to Tutte polynomials.** By far the most important polynomial associated with a matroid is its Tutte polynomial. Special evaluations of Tutte polynomials, for certain matroids, give many other much-studied polynomial and numerical invariants in combinatorics and other subjects, such as flow and chromatic polynomials of graphs, weight enumerators of linear codes, invariants of arrangements of hyperplanes (e.g., the number of regions and the number of bounded regions), Jones polynomials of alternating knots, and the partition function of the Ising model in statistical physics. Despite its many important specializations, this polynomial has a rather simple definition. The *Tutte polynomial* of a matroid M is Tutte polynomial

(5.1) $$t(M;x,y) = \sum_{A\subseteq E(M)} (x-1)^{r(M)-r(A)}(y-1)^{|A|-r(A)}.$$ $t(M;x,y)$

The Tutte polynomial is related to the generating function for the subsets of $E(M)$ indexed according to their rank and size, that is,

$$f(M;x,y) = \sum_{A\subseteq E(M)} x^{r(A)}y^{|A|},$$

by a change of variables, but formulating $t(M;x,y)$ as we did leads to many attractive properties, such as that in Corollary 5.3. For information on some of the many application of the Tutte polynomial, see [13, 29].

EXERCISE 5.1: Find the formulas that express each of $t(M;x,y)$ and $f(M;x,y)$ in terms of the other.

To illustrate computing the Tutte polynomial from the definition, we give an example that also shows that nonisomorphic matroids can have the same Tutte polynomial. The matroids M_1 and M_2 are given in Figure 15. For these matroids, we have

$$\begin{aligned}
t(M_1;x,y) = t(M_2;x,y) &= (x-1)^3 && \text{(the empty set)}\\
&+ 6(x-1)^2 && \text{(the six singleton sets)}\\
&+ 15(x-1) && \text{(the fifteen pairs)}\\
&+ 18 && \text{(eighteen of the 3-subsets have rank 3)}\\
&+ 2(x-1)(y-1) && \text{(the other two 3-subsets have rank 2)}\\
&+ 15(y-1) && \text{(the 4-element subsets)}\\
&+ 6(y-1)^2 && \text{(the 5-element subsets)}\\
&+ (y-1)^3 && \text{(the entire set).}
\end{aligned}$$

Expanding this polynomial gives $x^3+3x^2+4x+2xy+4y+3y^2+y^3$, which is symmetric in x and y. Symmetry is not typical for Tutte polynomials. We will see below what accounts for the symmetry in these examples.

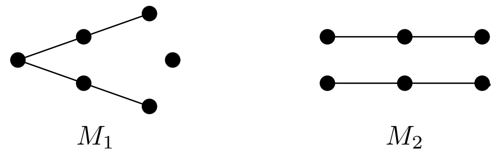


FIGURE 15. Two transversal matroids that have the same Tutte polynomial.

Computing Tutte polynomials in the manner we just did can be done only for matroids on small ground sets. With its numerous important applications, many of which are known to be computationally very hard, it is not surprising that computing the Tutte polynomial is hard (see [21]). Even for transversal matroids, computing evaluations of their Tutte polynomials at most points is hard (see [14]). (Computational complexity allows one to make the notion of hardness precise.) In sharp contrast, as we will see, computing the Tutte polynomial of a lattice path matroid is easy. The key is a different perspective on this polynomial, which we treat in the next section.

EXERCISE 5.2: Can one determine the rank of a matroid from its Tutte polynomial? If so, how?

EXERCISE 5.3: Give counting interpretations of the following evaluations of the Tutte polynomial: (a) $t(M;2,2)$, (b) $t(M;2,1)$, (c) $t(M;1,2)$, and (d) $t(M;1,1)$.

5.2. **Tutte polynomials as generating functions for basis activities.** (See Björner [1] for an excellent account of the following approach to the Tutte polynomial and for the proof of Theorem 5.1.) Fix an arbitrary linear order on the ground set of M. An element b of a basis B of M is *internally active for* B if

internally active

$$b = \min\{x : (B - b) \cup x \text{ is a basis of } M\}.$$

Equivalently, $b \in B$ is internally active if it is the least element of the cocircuit that is the complement of the hyperplane $\mathrm{cl}(B - b)$. Let $i(B)$ be the number of internally active elements in B. Dually, $a \in E(M) - B$ is *externally active for* B if

externally active

$$a = \min\{x : (B \cup a) - x \text{ is a basis of } M\};$$

equivalently, a is the least element of the unique circuit that is contained in $B \cup a$. Let $e(B)$ be the number of externally active elements in $E(M) - B$. The following result of Crapo [15], which generalizes a result of Tutte for graphs, shows that the Tutte polynomial is the generating function for the activities of bases.

Theorem 5.1. *For any matroid M and any linear order of $E(M)$,*

$$t(M;x,y) = \sum_{\text{bases } B} x^{i(B)} y^{e(B)}.$$

Note that which elements are active in a basis depends on the linear order, as do the numbers $i(B)$ and $e(B)$, so it is striking that the sum in this theorem does not depend on the order.

EXERCISE 5.4: Prove the lemma below using the definition of activities.

Lemma 5.2. *Relative to a fixed linear order of $E(M)$, an element $b \in B$ is internally active for B in M if and only if b is externally active for $E(M) - B$ in M^*.*

This lemma has the following immediate corollary.

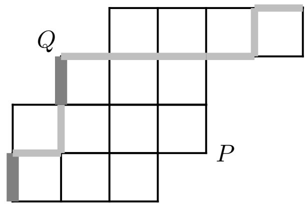


FIGURE 16. The path (in gray) corresponding to a basis and the (darker) steps that give internally active basis elements.

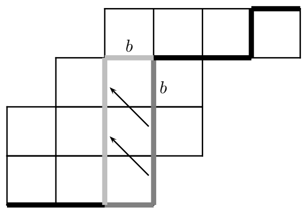


FIGURE 17. To show that the step labelled b in the black and dark gray path (basis) is not internally active, replace the dark gray part of the path with the light gray steps.

Corollary 5.3. *For any matroid, $t(M^*; x, y) = t(M; y, x)$. Thus, if M is self-dual (i.e., isomorphic to its dual), then $t(M; x, y)$ is symmetric in x and y.*

EXERCISE 5.5: Give another proof of Corollary 5.3 directly from equation (5.1).

EXERCISE 5.6: Explain why the Tutte polynomial we computed in the last section is symmetric.

5.3. **Basis activities in lattice path matroids; computing Tutte polynomials.** The key to understanding the Tutte polynomial of a lattice path matroid is interpreting basis activities in terms of lattice paths, using the natural order $1 < 2 < \cdots < m + r$. This is done in the following result, which Figure 16 illustrates; Figure 17 illustrates the proof.

Theorem 5.4. *The internally active elements in a basis B of $M[P, Q]$ correspond to the north steps in the associated path P_B that lie on the upper bounding path Q.*

Proof. Assume first that the north step labelled b in P_B is not in Q. We must show that b is not internally active in B, that is, some path in the region of interest arises by replacing b by some smaller element. Let I be the collection of north steps strictly below the b-th step and with the same x-coordinate. Since the b-th step is not in Q, it follows that each step in I can be moved to the step with the same label one unit up and to the left, the b-th step can be converted into an east step, and the east step before the first step in I can become north, and the resulting path stays in the region. Thus, as claimed, b can be replaced by a smaller element. When the north step labelled b is in Q, any replacement of it by a smaller element would result in a path that goes outside of the region of interest, so such elements are internally active, and the converse holds. □

Combining this theorem with Lemma 5.2 and our lattice path interpretation of duality gives the following corollary.

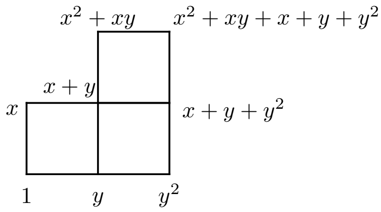


FIGURE 18. A small example of computing the Tutte polynomial of a lattice path matroid recursively.

Corollary 5.5. *The externally active elements in a basis of $M[P, Q]$ correspond to the east steps in the associated path that lie in P.*

Together, these results give the following interpretation of the coefficients of the Tutte polynomial of a lattice path matroid.

Theorem 5.6. *The coefficient of $x^i y^j$ in the Tutte polynomial of $M[P, Q]$ is the number of paths in $\mathcal{P}$ that share i north steps with Q and j east steps with P.*

This theorem yields a recurrence relation with which the Tutte polynomial of a lattice path matroid can be computed quickly. For a lattice point (integer point) (i, j) in the region of interest, let $f_{(i,j)}(x, y)$ be the polynomial in which the coefficient of $x^s y^t$ is the number of the lattice paths from $(0, 0)$ to (i, j) that share s north steps with Q and t east steps with P. We have the following initial condition and recurrence:

(1) $f_{(0,0)}(x, y) = 1$,
(2) if the step from $(i - 1, j)$ to (i, j) is on P, then $f_{(i,j)}(x, y) = y\, f_{(i-1,j)}(x, y)$,
(3) if the step from $(i, j - 1)$ to (i, j) is on Q, then $f_{(i,j)}(x, y) = x\, f_{(i,j-1)}(x, y)$, and
(4) if (i, j), $(i - 1, j)$, and $(i, j - 1)$ are all in the region of interest, then we have $f_{(i,j)}(x, y) = f_{(i-1,j)}(x, y) + f_{(i,j-1)}(x, y)$.

If P and Q end at (m, r), then $t(M[P, Q]; x, y) = f_{(m,r)}(x, y)$. Figure 18 shows a small example of applying these rules.

Since the number of lattice points in the region is at most quadratic in the number of elements in $M[P, Q]$, we get the following result.

Corollary 5.7. *The Tutte polynomial of a lattice path matroid can be computed in polynomial time relative to the number of elements in the ground set.*

Items (1)–(4) above are special cases of the following well-known deletion/contraction rules for the Tutte polynomial.

(1′) For the empty matroid M, we have $t(M; x, y) = 1$.
(2′) If e is a loop of M, then $t(M; x, y) = y\, t(M/e; x, y)$.
(3′) If e is a coloop of M, then $t(M; x, y) = x\, t(M\backslash e; x, y)$.
(4′) If $e \in E(M)$ is neither a loop nor a coloop of M, then

$$t(M; x, y) = t(M\backslash e; x, y) + t(M/e; x, y).$$

In essence, the lattice path diagram allows us to do something that rarely can be done, namely, use the deletion/contraction rules to compute the Tutte polynomial efficiently. The problem that one runs into when applying properties (1′)–(4′) in general is that $2^{|E(M)|}$ minors arise; however, for lattice path matroids, the path diagrams effectively collect these minors into relatively few isomorphism types.

EXERCISE 5.7: From equation (5.1), prove properties (1′)–(4′) above.

EXERCISE 5.8: Let P and Q be the paths $EENENN$ and $NNENEE$, respectively. Use the algorithm above to compute the Tutte polynomial of the lattice path matroid $M[P,Q]$. You should get the same polynomial as we got for the examples in Section 5.1. Draw the geometric representation of $M[P,Q]$ to see why.

Our brief treatment of the Tutte polynomial would have a glaring gap if we did not at least touch on why the Tutte polynomial arises in so many applications. We just noted that the Tutte polynomial satisfies a deletion/contraction rule. In each of the applications cited in Section 5.1 (e.g., chromatic polynomials of graphs), there is a similar rule. Much of the power of the Tutte polynomial stems from its being universal with respect to such rules in the following sense: every invariant of matroids that satisfies a deletion/contraction rule is an evaluation of the Tutte polynomial. The following theorem makes this precise.

Theorem 5.8. *Let R be a commutative ring that has unity. For each choice of elements u, v, σ, τ of R, there is a unique function T from the class of all matroids into R that has the following properties.*

(1) *If M is the empty matroid, then $T(M) = 1$.*
(2) *If e is a loop of M, then $T(M) = v\,T(M/e)$.*
(3) *If e is a coloop of M, then $T(M) = u\,T(M\backslash e)$.*
(4) *If $e \in E(M)$ is neither a loop nor a coloop, then $T(M) = \sigma\,T(M\backslash e) + \tau\,T(M/e)$.*

Furthermore, T is the following evaluation of the Tutte polynomial $t(M; x, y)$:

$$T(M) = \sigma^{|E(M)|-r(M)}\tau^{r(M)}t(M; u/\tau, v/\sigma).$$

This result is often called the "recipe theorem" since it gives a recipe for writing any invariant that satisfies a deletion/contraction rule as an evaluation of the Tutte polynomial. Brylawski [11] proved this theorem in the case of $\sigma = \tau = 1$; Oxley and Welsh [26] observed that the same argument yields the general result. This result is very useful, but its proof is just a simple induction based on the deletion/contraction rule.

EXERCISE 5.9: Prove Theorem 5.8.

6. FURTHER COMMENTS AND OPEN PROBLEMS

Given the limited scope of these notes, we have omitted much interesting material, such as the result that a matroid is transversal if and only if it is a union of rank-one matroids (the theory of submodular functions sheds much light on the operation of matroid union; see [25, Chapter 12]). We recommend [10, 19, 25, 28] for topics we have omitted.

A much-studied class of transversal matroids that we have omitted is that of bicircular matroids. The *bicircular matroid* $B(G)$ of a graph G is the matroid on the set of edges of G in which a set X of edges is independent if and only if each connected component of the subgraph induced by X has at most one cycle (simple closed path). bicircular matroid

EXERCISE 6.1: Show that bicircular matroids are transversal and that they are precisely the transversal matroids that have geometric representations on the simplex in which each element is placed either at a vertex or on an edge of the simplex. How are such geometric representations reflected in set systems?

We have also not addressed the duals of transversal matroids, which are cotransversal matroids or strict gammoids. For an attractive graph-theoretic description of such matroids, see [10, 25].

We close with three open problems related to transversal matroids, the first two of which appear in [25]. The first is a problem that D. Welsh posed in 1971.

Open Problem 6.1. *Characterize the matroids M for which both M and its dual M^* are transversal.*

In some sense, Theorem 3.1 and its dual provide such a characterization, but we would like an explicit description of all such matroids. Such matroids include, but are not limited to, fundamental transversal matroids, lattice path matroids, and multi-path matroids (a generalization of lattice path matroids; see [5]). The theory in [4] identifies many more such matroids, and the class of matroids M for which both M and M^* are transversal is preserved under both the direct sum and the free product of matroids (see [16, 17, 6]).

The second problem concerns gammoids, which are minors of transversal matroids. (The class of gammoids is closed under duality. For more on gammoids, see [25, 28].)

Open Problem 6.2. *Find an effective criterion to determine which matroids are gammoids.*

The third problem is motivated by Theorem 2.9 and concerns a class of matroids that has not yet been explored. One could formulate an analogous problem for any sequence of polytopes.

Open Problem 6.3. *Characterize the matroids that have geometric representations on cubes in which each cyclic flat of rank k in such a matroid consists of the elements in some $(k-1)$-dimensional face of the cube.*

7. Supplement: Hall's Theorem

Since it is a fundamental topic behind some of the material in these notes, and since it may be new to some readers, we provide a proof of Hall's theorem (Theorem 2.1), which we recast as follows.

Theorem 7.1. *The set system $\mathcal{A} = (A_1, A_2, \dots, A_r)$ has a transversal if and only if*

> **(H)** *for each i with $1 \le i \le r$, each subsystem of i sets (counting multiplicities) contains, in its union, at least i elements.*

Proof. It is clear that if $\mathcal{A}$ has a transversal, then condition (H) holds. We prove the converse by strong induction on r. The base case $r = 1$ is obvious.

We divide the inductive step into two cases. A subsystem $(A_{j_1}, A_{j_2}, \dots, A_{j_h})$ of $\mathcal{A}$ is *critical* if (i) $h < r$ and (ii) $|A_{j_1} \cup A_{j_2} \cup \cdots \cup A_{j_h}| = h$. First assume $\mathcal{A}$ has no critical subsystem. Fix $x \in A_r$ and consider $\mathcal{A}_x = (A_1 - x, A_2 - x, \dots, A_{r-1} - x)$. Since $\mathcal{A}$ has no critical subsystem, it follows that $\mathcal{A}_x$ satisfies condition (H), so, by induction, $\mathcal{A}_x$ has a transversal; this transversal of $\mathcal{A}_x$ along with x clearly gives a transversal of $\mathcal{A}$.

Assume $\mathcal{A}$ has a critical subsystem, which we may take to be $\mathcal{A}' = (A_1, A_2, \dots, A_h)$. By induction and the assumption that $\mathcal{A}'$ is critical, $T' = A_1 \cup A_2 \cup \cdots \cup A_h$ is a transversal of $\mathcal{A}'$. Note that property (H) holds for $\mathcal{A}'' = (A_{h+1} - T', A_{h+2} - T', \dots, A_r - T')$ since, for any subsystem of $(A_{h+1}, A_{h+2}, \dots, A_r)$, property (H) holds when we adjoin the critical subsystem $\mathcal{A}'$ to this subsystem. Thus, by induction, $\mathcal{A}''$ has a transversal, say T; it follows that $T \cup T'$ is a transversal of $\mathcal{A}$. □

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