A Construction of Infinite Sets of Intertwines for Pairs of Matroids

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If classes $C$ and $C'$ of matroids are minor-closed, then so is $C \cup C'$.

Any excluded minor $M$ for $C \cup C'$ has a minor that is an excluded minor for $C$ and another that is an excluded minor for $C'$; also, no proper minor of $M$ has two such minors.
If classes $C$ and $C'$ of matroids are minor-closed, then so is $C \cup C'$.

Any excluded minor $M$ for $C \cup C'$ has a minor that is an excluded minor for $C$ and another that is an excluded minor for $C'$; also, no proper minor of $M$ has two such minors.

A matroid $M$ is an intertwine of matroids $M_1$ and $M_2$ if (i) $M$ has minors isomorphic to $M_1$ and $M_2$, and (ii) no proper minor of $M$ has both $M_1$- and $M_2$-minors.

We may assume the ground sets of $M_1$ and $M_2$ are disjoint.
Examples

An intertwine is a minor-minimal matroid having both $M_1$- and $M_2$-minors.

Intertwines exist: some minor of $M_1 \oplus M_2$ is an intertwine.

The rank-3 intertwines of $M_1$ and $M_2$.
(Contract the blue point to get $M_1$.)

$M_1$

$M_2$
Problem

*Can two matroids have infinitely many intertwines?*

*(Brylawski, 1986; Robertson; Welsh; Oxley, Problem 14.4.6)*

The analogous question for graphs has a negative answer by the graph minors theorem.
The Basic Problem of Intertwines

Problem

Can two matroids have infinitely many intertwines?

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The analogous question for graphs has a negative answer by the graph minors theorem.

Two matroids can have infinitely many intertwines.

(Vertigan, mid-1990’s, unpublished)

The aims of this talk are (i) to show how to construct infinite sets of intertwines for many pairs of matroids and (ii) to present some properties of and variations on this construction.
$M + X$, the free extension of $M$ by the set $X$, is the freest matroid $M'$ of rank $r(M)$ on $E(M) \cup X$ with $M' \setminus X = M$. 
\( M \times X \), the free coextension of \( M \) by the set \( X \), is the freest matroid \( M' \) on \( E(M) \cup X \) with \( M'/X = M \).

\[
M \times X = (M^* + X)^*
\]
A Special Instance of the Construction of Intertwines

Assume $S_1 = E(M_1)$ and $S_2 = E(M_2)$ are disjoint.

Fix $k \geq 3 \max\{|S_1|, |S_2|\}$.

Pick sets $T_1$ and $T_2$ where $k = r(M_i) + |T_i|$, for $i \in \{1, 2\}$, and $S_1 \cup S_2, T_1,$ and $T_2$ are mutually disjoint.

**Theorem**

If (i) all elements of $M_1$ and $M_2$ are in non-spanning circuits, and (ii) neither $M_1$ nor $M_2$ arises, up to isomorphism, from the other by any combination of minors, truncation, and free coextension, then the truncation of $(M_1 \times T_1) \oplus (M_2 \times T_2)$ to rank $k$ is an intertwine of $M_1$ and $M_2$. 
The Dual Construction

**Theorem**

If (i′) all elements of $M_1$ and $M_2$ are in dependent cocircuits, and (ii′) neither $M_1$ nor $M_2$ arises, up to isomorphism, from the other by any combination of minors, free extension, and lifts, then the lift of $(M_1 + T_1) \oplus (M_2 + T_2)$ to rank $k$ is an intertwine.

The more general construction we will see applies under a unified form of conditions (ii) and (ii′), it avoids the limitations imposed by conditions (i) and (i′), and it gives many intertwines of each rank.
A set in $M$ is **cyclic** if it is a (possibly empty) union of circuits.

Let $\mathcal{Z}(M)$ be the set (in fact, lattice) of cyclic flats of $M$.

A matroid is determined by its cyclic flats along with their ranks.

(Brylawski, 1975)
Theorem

For $\mathcal{Z} \subseteq 2^S$ and $r : \mathcal{Z} \to \mathbb{Z}$, there is a matroid $M$ on $S$ with $\mathcal{Z} = \mathcal{Z}(M)$ and $r = r_M\big|_{\mathcal{Z}}$ iff

(Z0) $\mathcal{Z}$ is a lattice under inclusion,

(Z1) $r(0_{\mathcal{Z}}) = 0$, where $0_{\mathcal{Z}}$ is the least element of $\mathcal{Z}$,

(Z2) $0 < r(Y) - r(X) < |Y - X|$ for all $X, Y \in \mathcal{Z}$ with $X \subset Y$,

(Z3) for all pairs of incomparable sets $X, Y \in \mathcal{Z}$,$$
 r(X) + r(Y) \geq r(X \vee Y) + r(X \wedge Y) + |(X \cap Y) - (X \wedge Y)|.
$$

(Sims, 1980; Bonin and de Mier, 2008)
Free Coextension from the Perspective of Cyclic Flats

\[ \mathbb{Z}(M \times X) \]

\[ \{a, b, c, d, e, f\} \cup X \]

\[ \{a, b, c\} \cup X \quad \{c, d, e\} \cup X \quad \{a, d, f\} \cup X \quad \{b, e, f\} \cup X \]

\[ 2 + |X| \quad 2 + |X| \quad 2 + |X| \quad 2 + |X| \]

\[ 0 \]

\[ 3 + |X| \]

\[ \mathbb{Z}(M) \]

\[ \emptyset \]

\[ \{a, b, c\} \quad \{c, d, e\} \quad \{a, d, f\} \quad \{b, e, f\} \]

\[ 2 \quad 2 \quad 2 \quad 2 \]

\[ 0 \]
The Construction of Intertwines

Pick $S_1' \subseteq S_1$ and $S_2' \subseteq S_2$ and sets $T_1$ and $T_2$ with $T_1$, $T_2$, $S_1$, and $S_2$ disjoint and so that

$$M_1 \times (T_1 \cup S_2') \quad \text{and} \quad M_2 \times (T_2 \cup S_1') \quad (*)$$

have the same rank, say $k$, with $k \geq 4 \max\{|S_1|, |S_2|\}$.

Let $\mathcal{Z}$ consist of $S_1 \cup S_2 \cup T_1 \cup T_2$ and the proper cyclic flats of the matroids in $(*)$. 
The Construction of Intertwines

Pick $S'_1 \subseteq S_1$ and $S'_2 \subseteq S_2$ and sets $T_1$ and $T_2$ with $T_1$, $T_2$, $S_1$, and $S_2$ disjoint and so that

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Let $\mathcal{Z}$ consist of $S_1 \cup S_2 \cup T_1 \cup T_2$ and the proper cyclic flats of the matroids in $(*)$.

Define $r : \mathcal{Z} \to \mathbb{Z}$ by

1. $r(F \cup T_1 \cup S'_2) = r_1(F) + |T_1 \cup S'_2|$ for $F \in \mathcal{Z}'(M_1)$,
2. $r(F \cup T_2 \cup S'_1) = r_2(F) + |T_2 \cup S'_1|$ for $F \in \mathcal{Z}'(M_2)$,
3. $r(\emptyset) = 0$, and
4. $r(S_1 \cup S_2 \cup T_1 \cup T_2) = k$,

where $\mathcal{Z}'(M)$ is the set of proper, nonempty cyclic flats of $M$. 
An Example

\[ S_1' \cup S_2' \cup T_1 \cup T_2 \]

\[ \{x, y, a \} \cup T_1 \]
\[ k - 1 \]

\[ \{a, b, c, u, v, w\} \cup T_2 \]
\[ k - 1 \]

\[ \{c, d, e, u, v, w\} \cup T_2 \]
\[ k - 1 \]

\[ \emptyset \]
\[ 0 \]

\[ |T_1| = k - 2 - 1 = k - 3, \quad |T_2| = k - 3 - 3 = k - 6 \]
The Basic Results

Theorem
The pair \((\mathcal{Z}, r)\) satisfies axioms (Z0)–(Z3).

The Main Theorem
Assume neither \(M_1\) nor \(M_2\) yields an isomorphic copy of the other upon taking minors, free extensions, and free coextensions.

Assume \(S'_i\), for \(i \in \{1, 2\}\), contains all free elements and isthmuses of \(M_i\), but no cofree elements and no loops.

The matroid \(M\) defined above is an intertwine of \(M_1\) and \(M_2\).
Knowing More About $M_1$ and $M_2$ Lets Us Say More

If $M_1$ and $M_2$ have no circuit-hyperplanes, then augmenting $Z$ by any collection of $k$-subsets $H_i$ of $T_1 \cup T_2$ with $|H_i \triangle H_j| \neq 2$ for all $i,j$ and setting $r(H_i) = k - 1$ yields another intertwine.

Let’s apply this to get a doubly exponential lower bound (as a function of $k$) on the number of rank-$k$ intertwines in this setting.
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Define $a$, $b$, and $c$ by $|T_1| = k - a$, $|T_2| = k - b$, $|T_1 \cup T_2| = 2k - c$.

Fix $C \subset T_1 \cup T_2$ with $|C| = c$.

Focus on intertwines in which all circuit-hyperplanes contain $C$. 
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Fix $C \subset T_1 \cup T_2$ with $|C| = c$.

Focus on intertwines in which all circuit-hyperplanes contain $C$.

Assume $k - c$ is even; set $h = (k - c)/2$.

Pair off the $4h$ elements in $|(T_1 \cup T_2) - C|$ into $2h$ pairs.

Forming each circuit-hyperplane from $C$ and any $h$ of the $2h$ pairs guarantees $|H_i \triangle H_j| \neq 2$ and so yields $2^{(2h\choose h)}$ intertwines.
Knowing More About $M_1$ and $M_2$ Lets Us Say More

Dividing to get rid of isomorphic matroids leaves at least

$$\frac{2\binom{2h}{h}}{(k - a)!(k - b)!}$$

non-isomorphic intertwines.
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\frac{2^{2^h}}{(k - a)!(k - b)!}
\]

non-isomorphic intertwines.

Now \( \binom{2^h}{h} > \frac{2^h}{2h + 1} \) and \( n! \leq \sqrt{n + 1} \ n^n \ e^{-n+1} \).

The resulting lower bound on the number of non-isomorphic rank-\( k \) intertwines is \( 2^\alpha \) where \( \alpha \) is \( \frac{4^h}{2h + 1} \) minus several terms, each smaller than \( k \log_2(k/e) \).
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Now \( \left(\begin{array}{c} 2h \\ h \end{array}\right) > \frac{2^h}{2h + 1} \) and \( n! \leq \sqrt{n + 1} \ n^n e^{-n+1} \).

The resulting lower bound on the number of non-isomorphic rank-\( k \) intertwines is \( 2^\alpha \) where \( \alpha \) is \( \frac{4^h}{2h + 1} \) minus several terms, each smaller than \( k \log_2(k/e) \).

**Open Problem**

What can be said about \( i(k + 1; M_1, M_2) - i(k; M_1, M_2) \) where \( i(k; M_1, M_2) \) is the number of rank-\( k \) intertwines of \( M_1 \) and \( M_2 \)?
Corollaries of the Construction

Theorem

*The matroid $M$ is transversal iff $M_1$ and $M_2$ are.*

The key is the Mason/Ingleton characterization of transversal matroids:

**Theorem**

*A matroid $M$ is transversal iff for all nonempty $\mathcal{A} \subseteq \mathcal{Z}(M)$,

$$r(\cap \mathcal{A}) \leq \sum_{\mathcal{F} \subseteq \mathcal{A}} (-1)^{|\mathcal{F}|+1} r(\cup \mathcal{F}).$$

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$$r(\cap A) \leq \sum_{\mathcal{F} \subseteq A} (-1)^{|\mathcal{F}|+1} r(\cup \mathcal{F}).$$

Corollary
If $M_1$ and $M_2$ satisfy the conditions in the main theorem and are transversal, then infinitely many of their intertwines are transversal.

The same holds for cotransversal matroids, bitransversal matroids, and gammoids.
Corollaries of the Construction

Corollary

If $M_1$ and $M_2$ satisfy the conditions in the theorem, then, for any integer $n$, infinitely many of their intertwines

1. are $n$-connected and
2. have $U_{n,2n}$-minors.
Both constructions can be extended to cover, and agree on, the union of what they individually cover.
Relation to Dirk Vertigan’s Construction

Dirk defined his intertwine using the set of bases, which is the intersection of the sets of bases for two other matroids.

To phrase his construction in our terms, set

\[ M'_1 = (M_1 \times (T_1 \cup S'_2)) + (T_2 \cup (S_2 - S'_2)) \]

and

\[ M'_2 = (M_2 \times (T_2 \cup S'_1)) + (T_1 \cup (S_1 - S'_1)) \]

where \( S'_i \subseteq S_i - S_j \) for \( \{i, j\} = \{1, 2\} \).

Both \( M'_1 \) and \( M'_2 \) have rank \( k \); also, \( \mathcal{Z}(M) = \mathcal{Z}(M'_1) \cup \mathcal{Z}(M'_2) \).

It follows that a subset of \( S_1 \cup S_2 \cup T_1 \cup T_2 \) is a basis of \( M \) iff it is a basis of both \( M'_1 \) and \( M'_2 \).
Open Problem

Under precisely what conditions do $M_1$ and $M_2$ have infinitely many intertwines?

Open Problem

Can a matroid and a uniform matroid have infinitely many intertwines? (Geelen, Some Open Problems on Excluding a Uniform Matroid)

Open Problem

Is the intertwine we constructed algebraic over a given field iff $M_1$ and $M_2$ are?

Open Problem

What can be said about $i(k + 1; M_1, M_2) - i(k; M_1, M_2)$ where $i(k; M_1, M_2)$ is the number of rank-$k$ intertwines of $M_1$ and $M_2$?
Theorem

With $k' = |S_1| + |S_2| + |T_1| + |T_2| - k$, we have

$$\left( M_k(M_1, S'_1, T_1; M_2, S'_2, T_2) \right)^* = M_{k'}(M^*_1, S_1 - S'_1, T_2; M^*_2, S_2 - S'_2, T_1).$$

It suffices to show that no single-element deletion $M \setminus a$ has both an $M_1$- and an $M_2$-minor.
The inequality on $k$ implies that if $M \setminus a \setminus X / Y \simeq M_i$, then either $|X \cap T_1| \geq \eta(M_1)$ or $|Y \cap T_1| \geq r(M_2)$. 
The inequality on \( k \) implies that if \( M \setminus a \setminus X / Y \simeq M_i \), then either \(|X \cap T_1| \geq \eta(M_1)\) or \(|Y \cap T_1| \geq r(M_2)\).

The first implies

1. \( M \setminus (X \cap T_1) = (M_2 \times (T_2 \cup S_1')) + ((S_1 - S_1') \cup (T_1 - X)) \),
2. \( M \setminus (X \cap T_1) \) has no \( M_1 \)-minor, and
3. \( \eta_2(F) = \eta_{M \setminus (X \cap T_1)}(F \cup T_2 \cup S_1') \) for \( F \in \mathcal{Z}'(M_2) \).
The inequality on $k$ implies that if $M \setminus a \setminus X / Y \simeq M_i$, then either $|X \cap T_1| \geq \eta(M_1)$ or $|Y \cap T_1| \geq r(M_2)$.

The first implies

(1) $M \setminus (X \cap T_1) = (M_2 \times (T_2 \cup S'_1)) + ((S_1 - S'_1) \cup (T_1 - X))$,

(2) $M \setminus (X \cap T_1)$ has no $M_1$-minor, and

(3) $\eta_2(F) = \eta_{M \setminus (X \cap T_1)}(F \cup T_2 \cup S'_1)$ for $F \in \mathcal{Z}'(M_2)$.

The second implies

(1') $M / (Y \cap T_1) = (M_1 \times ((T_1 - Y) \cup S'_2)) + ((S_2 - S'_2) \cup T_2)$,

(2') $M / (Y \cap T_1)$ has no $M_2$-minor, and

(3') $\eta_1(F) = \eta_{M / (Y \cap T_1)}(F \cup (T_1 - Y) \cup S'_2)$ for $F \in \mathcal{Z}'(M_1)$. 
For $a \in (S_1 - S'_1) \cup T_1$, assume $M \backslash a$ has a minor $M \backslash a \backslash X/Y \simeq M_1$.

Either

1. $M \backslash (X \cap T_1) = (M_2 \times (T_2 \cup S'_1)) + ((S_1 - S'_1) \cup (T_1 - X))$,
2. $M \backslash (X \cap T_1)$ has no $M_1$-minor, and
3. $\eta_2(F) = \eta_{M \backslash (X \cap T_1)}(F \cup T_2 \cup S'_1)$ for $F \in \mathcal{Z}'(M_2)$

or

1'. $M / (Y \cap T_1) = (M_1 \times ((T_1 - Y) \cup S'_2)) + ((S_2 - S'_2) \cup T_2)$,
2'. $M / (Y \cap T_1)$ has no $M_2$-minor, and
3'. $\eta_1(F) = \eta_{M / (Y \cap T_1)}(F \cup (T_1 - Y) \cup S'_2)$ for $F \in \mathcal{Z}'(M_1)$. 
For $a \in (S_1 - S'_1) \cup T_1$, assume $M \setminus a$ has a minor $M \setminus a \setminus X/Y \simeq M_1$.

Either

(1) $M \setminus (X \cap T_1) = (M_2 \times (T_2 \cup S'_1)) + ((S_1 - S'_1) \cup (T_1 - X))$,
(2) $M \setminus (X \cap T_1)$ has no $M_1$-minor, and
(3) $\eta_2(F) = \eta_{M \setminus (X \cap T_1)}(F \cup T_2 \cup S'_1)$ for $F \in \mathcal{Z}'(M_2)$

or

(1') $M/(Y \cap T_1) = (M_1 \times ((T_1 - Y) \cup S'_2)) + ((S_2 - S'_2) \cup T_2)$,
(2') $M/(Y \cap T_1)$ has no $M_2$-minor, and
(3') $\eta_1(F) = \eta_{M/(Y \cap T_1)}(F \cup (T_1 - Y) \cup S'_2)$ for $F \in \mathcal{Z}'(M_1)$.

By (3'), since $a$ is in proper cyclic flats of $M/(Y \cap T_1)$, deleting $a$ makes the sum of the nullities of the proper cyclic flats less than that sum in $M_1$, so $M \setminus a \setminus X/Y \not\simeq M_1$.

A similar argument covers $a \in S'_1$; symmetry covers $a \in S_2 \cup T_2$. 