A Construction of Infinite Sets of Intertwines for Pairs of Matroids

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If classes $\mathcal{C}$ and $\mathcal{C}'$ of matroids are minor-closed, then so is $\mathcal{C} \cup \mathcal{C}'$.

Any excluded minor M for $\mathcal{C} \cup \mathcal{C}'$ has a minor that is an excluded minor for $\mathcal{C}$ and another that is an excluded minor for $\mathcal{C}'$;
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also, no proper minor of M has two such minors.

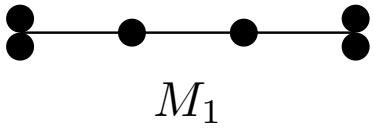
A matroid M is an intertwine of matroids M_1 and M_2 if
(i) M has minors isomorphic to M_1 and M_2, and
(ii) no proper minor of M has both M_1- and M_2-minors.

We may assume the ground sets of M_1 and M_2 are disjoint.

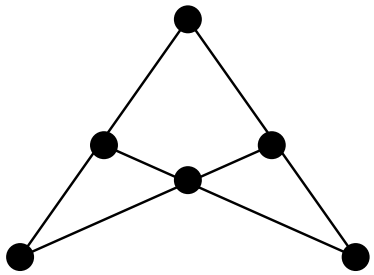
Examples

An intertwine is a minor-minimal matroid having both M_1- and M_2-minors.

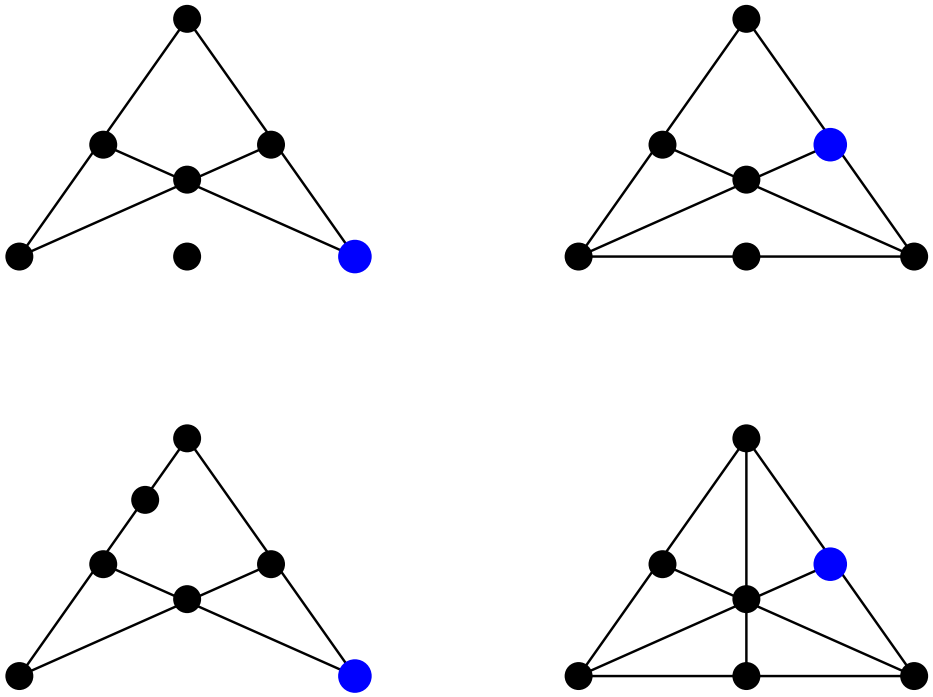
Intertwines exist: some minor of $M_1 \oplus M_2$ is an intertwine.



M_1



M_2



The rank-3 intertwines of M_1 and M_2.
(Contract the blue point to get M_1.)

Problem

Can two matroids have infinitely many intertwines?

(Brylawski, 1986; Robertson; Welsh; Oxley, Problem 14.4.6)

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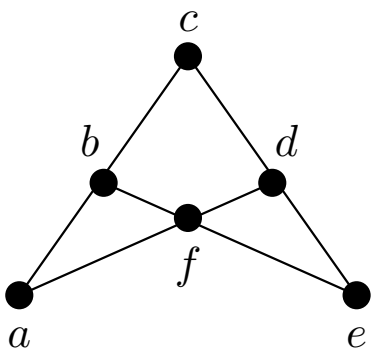
Two matroids can have infinitely many intertwines.

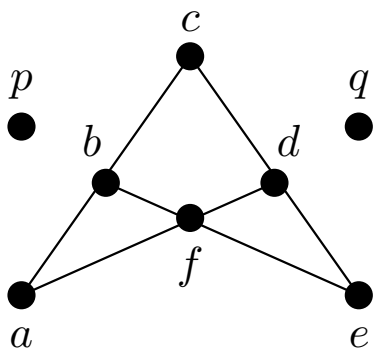
(Vertigan, mid-1990's, unpublished)

The aims of this talk are (i) to show how to construct infinite sets of intertwines for many pairs of matroids and (ii) to present some properties of and variations on this construction.

$M + X$, the free extension of M by the set X, is the freest matroid M' of rank $r(M)$ on $E(M) \cup X$ with $M' \backslash X = M$.



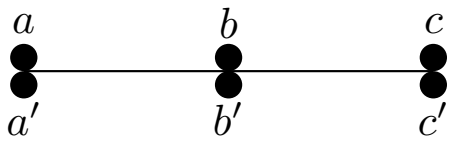
M



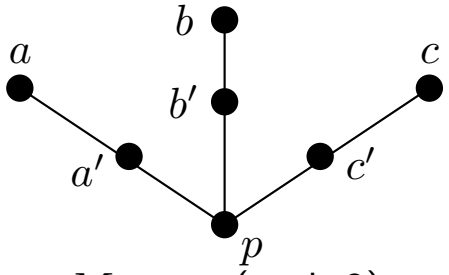
$M + \{p, q\}$

$M \times X$, the free coextension of M by the set X, is the freest matroid M' on $E(M) \cup X$ with $M'/X = M$.

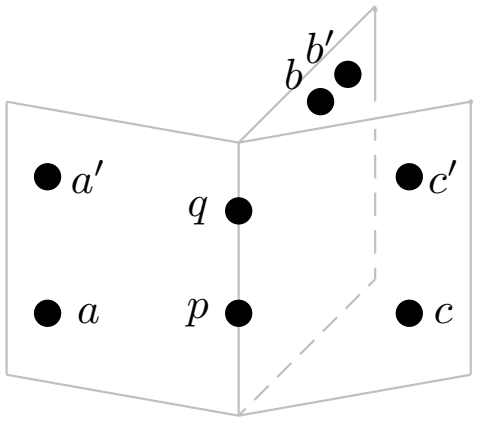
$$M \times X = (M^* + X)^*$$



M (rank 2)



$M \times p$ (rank 3)



$M \times \{p, q\}$ (rank 4)

Assume $S_1 = E(M_1)$ and $S_2 = E(M_2)$ are disjoint.

Fix $k \geq 3\max\{|S_1|, |S_2|\}$.

Pick sets T_1 and T_2 where $k = r(M_i) + |T_i|$, for $i \in \{1, 2\}$, and $S_1 \cup S_2$, T_1, and T_2 are mutually disjoint.

Theorem

If (i) all elements of M_1 and M_2 are in non-spanning circuits, and

(ii) neither M_1 nor M_2 arises, up to isomorphism, from the other by any combination of minors, truncation, and free coextension,

then the truncation of $(M_1 \times T_1) \oplus (M_2 \times T_2)$ to rank k is an intertwine of M_1 and M_2.

Theorem

If (i′) all elements of M_1 and M_2 are in dependent cocircuits, and

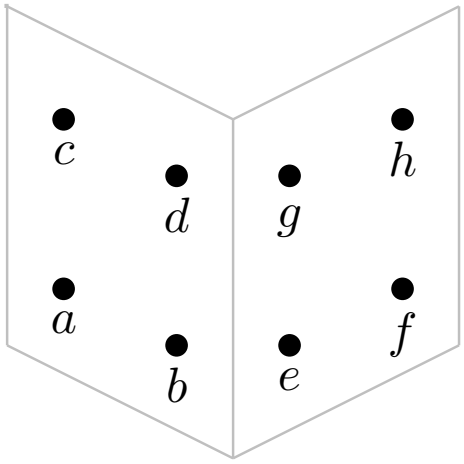
(ii′) neither M_1 nor M_2 arises, up to isomorphism, from the other by any combination of minors, free extension, and lifts,

then the lift of $(M_1 + T_1) \oplus (M_2 + T_2)$ to rank k is an intertwine.

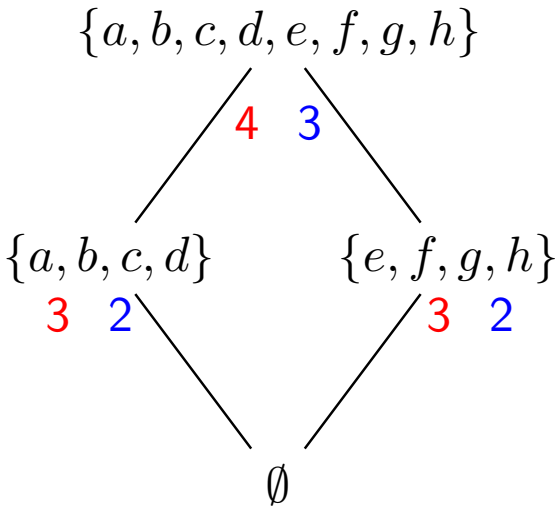
The more general construction we will see applies under a unified form of conditions (ii) and (ii′),
it avoids the limitations imposed by conditions (i) and (i′),
and it gives many intertwines of each rank.

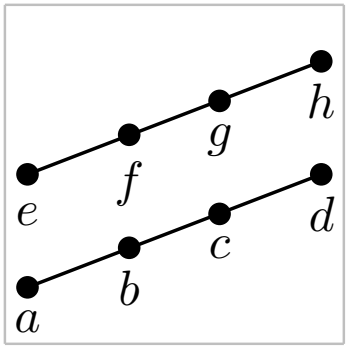
A set in M is cyclic if it is a (possibly empty) union of circuits.

Let $\mathcal{Z}(M)$ be the set (in fact, lattice) of cyclic flats of M.



M





N

A matroid is determined by its cyclic flats along with their ranks.

(Brylawski, 1975)

Theorem

For $\mathcal{Z} \subseteq 2^S$ and $r : \mathcal{Z} \to \mathbb{Z}$, there is a matroid M on S with $\mathcal{Z} = \mathcal{Z}(M)$ and $r = r_M|_{\mathcal{Z}}$ iff

(Z0) *$\mathcal{Z}$ is a lattice under inclusion,*

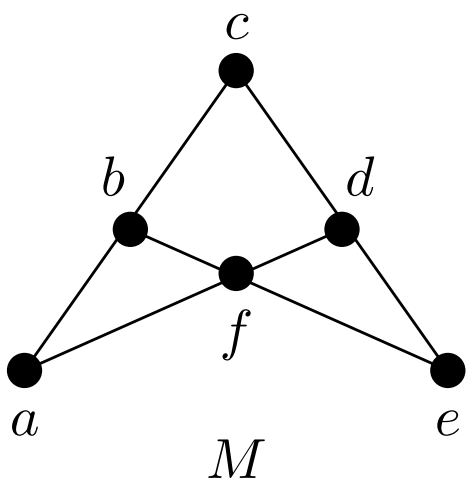
(Z1) *$r(0_{\mathcal{Z}}) = 0$, where $0_{\mathcal{Z}}$ is the least element of $\mathcal{Z}$,*

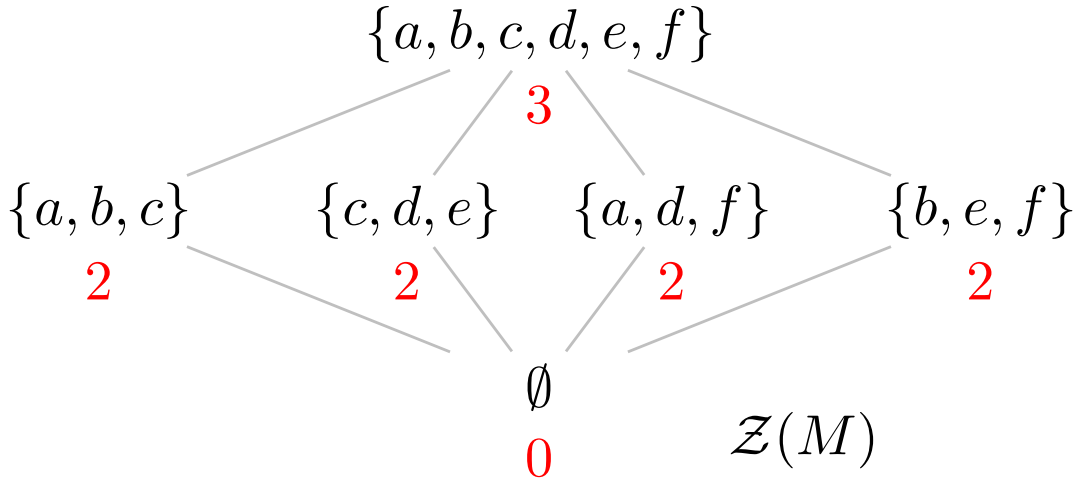
(Z2) *$0 < r(Y) - r(X) < |Y - X|$ for all $X, Y \in \mathcal{Z}$ with $X \subset Y$,*

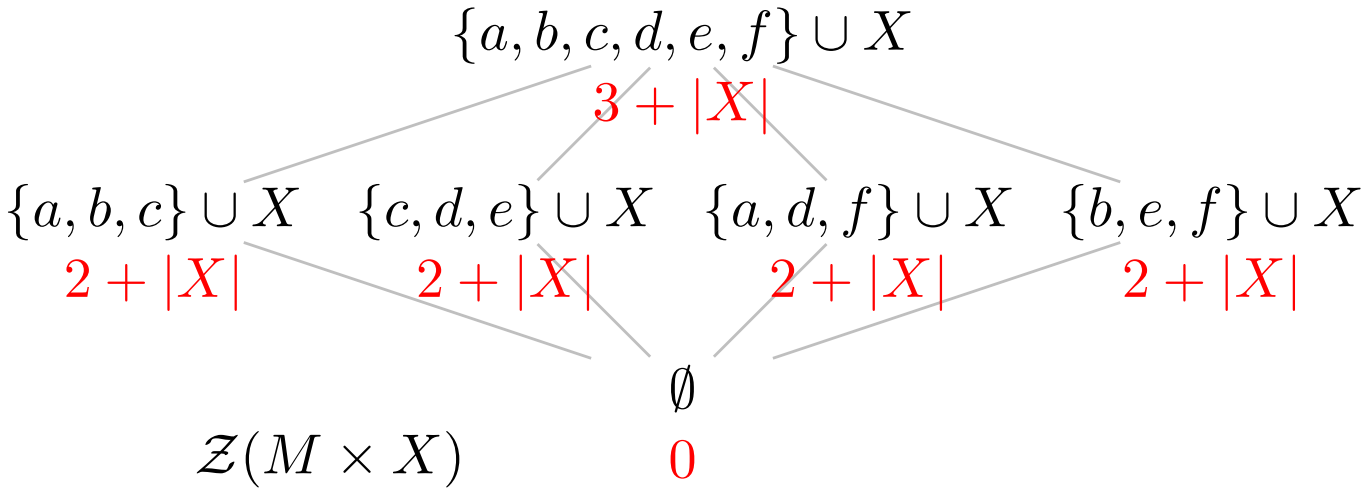
(Z3) *for all pairs of incomparable sets $X, Y \in \mathcal{Z}$,*
$r(X) + r(Y) \geq r(X \vee Y) + r(X \wedge Y) + |(X \cap Y) - (X \wedge Y)|$.

(Sims, 1980; Bonin and de Mier, 2008)

Free Coextension from the Perspective of Cyclic Flats







Pick $S_1' \subseteq S_1$ and $S_2' \subseteq S_2$ and sets T_1 and T_2 with T_1, T_2, S_1, and S_2 disjoint and so that

$$M_1 \times (T_1 \cup S_2') \qquad \text{and} \qquad M_2 \times (T_2 \cup S_1') \qquad (*)$$

have the same rank, say k, with $k \geq 4 \max\{|S_1|, |S_2|\}$.

Let $\mathcal{Z}$ consist of $S_1 \cup S_2 \cup T_1 \cup T_2$ and the proper cyclic flats of the matroids in (*).

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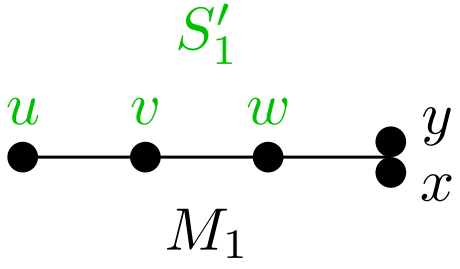
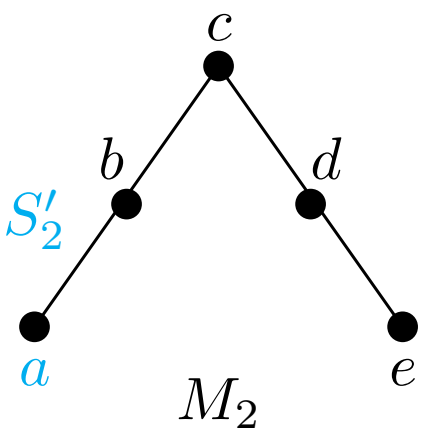
Define $r : \mathcal{Z} \to \mathbb{Z}$ by

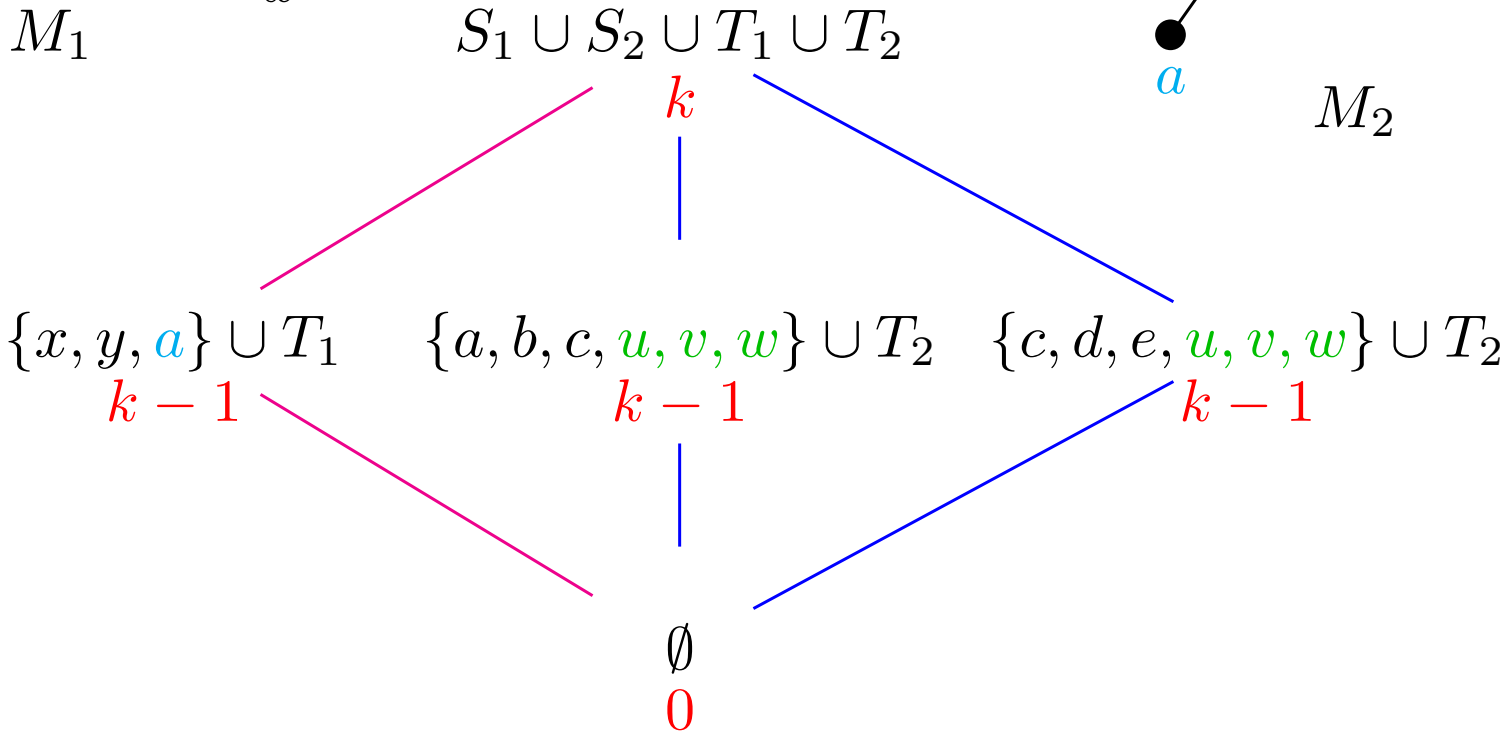
1. $r(F \cup T_1 \cup S_2') = r_1(F) + |T_1 \cup S_2'|$ for $F \in \mathcal{Z}'(M_1)$,
2. $r(F \cup T_2 \cup S_1') = r_2(F) + |T_2 \cup S_1'|$ for $F \in \mathcal{Z}'(M_2)$,
3. $r(\emptyset) = 0$, and
4. $r(S_1 \cup S_2 \cup T_1 \cup T_2) = k$,

where $\mathcal{Z}'(M)$ is the set of proper, nonempty cyclic flats of M.

An Example







$$|T_1| = k - 2 - 1 = k - 3, \quad |T_2| = k - 3 - 3 = k - 6$$

Theorem

The pair $(\mathcal{Z}, r)$ satisfies axioms (Z0)–(Z3).

The Main Theorem

Assume neither M_1 nor M_2 yields an isomorphic copy of the other upon taking minors, free extensions, and free coextensions.

Assume S_i', for $i \in \{1, 2\}$, contains all free elements and isthmuses of M_i, but no cofree elements and no loops.

The matroid M defined above is an intertwine of M_1 and M_2.

Knowing More About M_1 and M_2 Lets Us Say More

If M_1 and M_2 have no circuit-hyperplanes, then augmenting $\mathcal{Z}$ by any collection of k-subsets H_i of $T_1 \cup T_2$ with $|H_i \triangle H_j| \neq 2$ for all i, j and setting $r(H_i) = k - 1$ yields another intertwine.

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Define a, b, and c by $|T_1| = k - a$, $\quad |T_2| = k - b$, $|T_1 \cup T_2| = 2k - c$.

Fix $C \subset T_1 \cup T_2$ with $|C| = c$.

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Focus on intertwines in which all circuit-hyperplanes contain C.

Assume $k - c$ is even; set $h = (k - c)/2$.

Pair off the $4h$ elements in $|(T_1 \cup T_2) - C|$ into $2h$ pairs.

Forming each circuit-hyperplane from C and any h of the $2h$ pairs guarantees $|H_i \triangle H_j| \neq 2$ and so yields $2^{\binom{2h}{h}}$ intertwines.

Knowing More About M_1 and M_2 Lets Us Say More

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Now $\binom{2h}{h} > 2^{2h}/(2h+1)$ and $n! \leq \sqrt{n+1}\; n^n\, e^{-n+1}$.

The resulting lower bound on the number of non-isomorphic rank-k intertwines is 2^α where α is $4^h/(2h+1)$ minus several terms, each smaller than $k \log_2(k/e)$.

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Open Problem

What can be said about $i(k+1; M_1, M_2) - i(k; M_1, M_2)$ where $i(k; M_1, M_2)$ is the number of rank-k intertwines of M_1 and M_2?

Theorem

The matroid M is transversal iff M_1 and M_2 are.

The key is the Mason/Ingleton characterization of transversal matroids:

Theorem

A matroid M is transversal iff for all nonempty $\mathcal{A} \subseteq \mathcal{Z}(M)$,

$$r(\cap \mathcal{A}) \leq \sum_{\mathcal{F} \subseteq \mathcal{A}} (-1)^{|\mathcal{F}|+1} r(\cup \mathcal{F}).$$

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Corollary

If M_1 and M_2 satisfy the conditions in the main theorem and are transversal, then infinitely many of their intertwines are transversal.

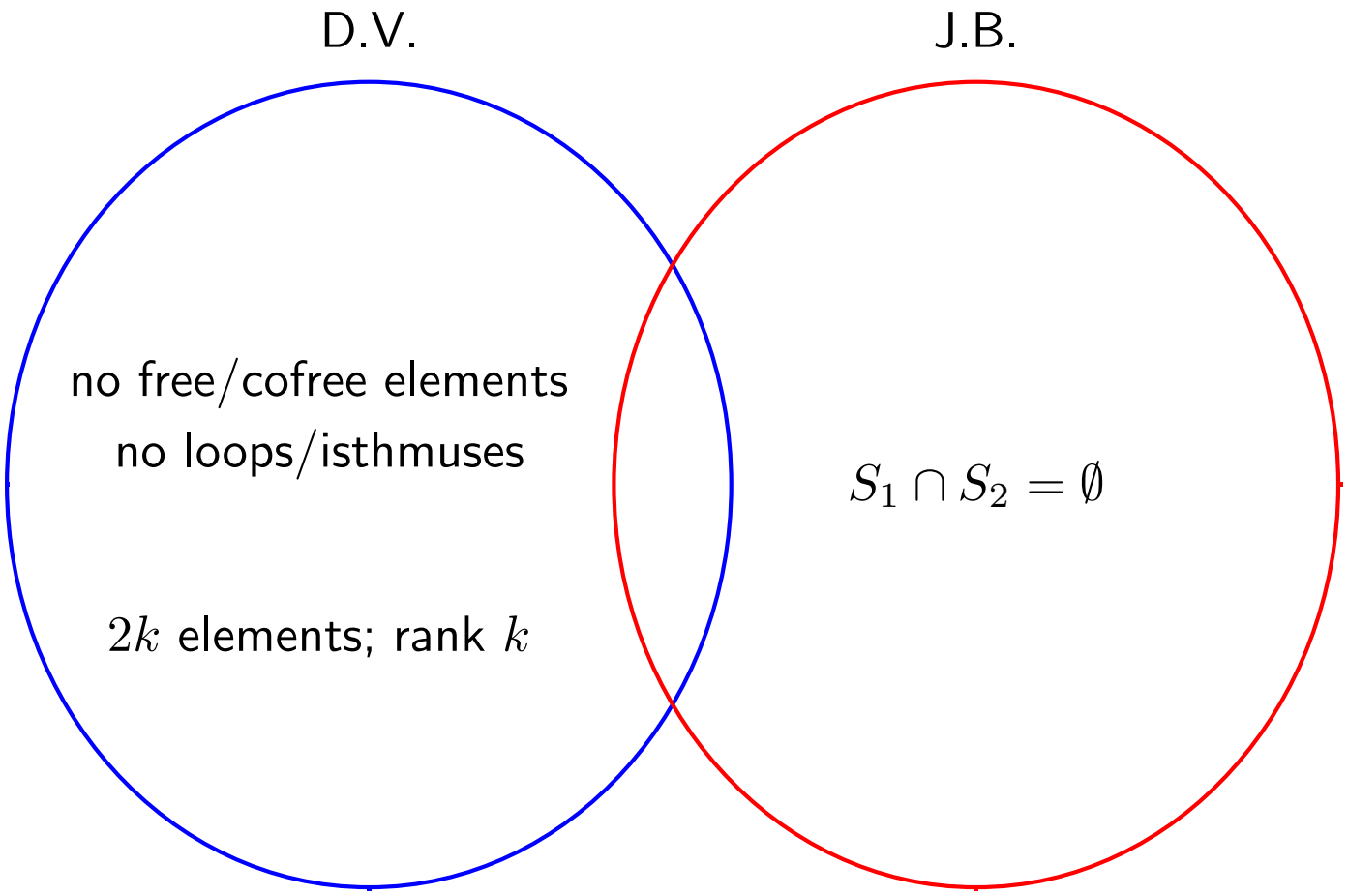
The same holds for cotransversal matroids, bitransversal matroids, and gammoids.

Corollary

If M_1 and M_2 satisfy the conditions in the theorem, then, for any integer n, infinitely many of their intertwines

1. *are n-connected and*
2. *have $U_{n,2n}$-minors.*

Relation to Dirk Vertigan's Construction



Both constructions can be extended to cover, and agree on, the union of what they individually cover.

Relation to Dirk Vertigan's Construction

Dirk defined his intertwine using the set of bases, which is the intersection of the sets of bases for two other matroids.

To phrase his construction in our terms, set

$$M_1' = \big(M_1 \times (T_1 \cup S_2')\big) + \big(T_2 \cup (S_2 - S_2')\big)$$

and

$$M_2' = \big(M_2 \times (T_2 \cup S_1')\big) + \big(T_1 \cup (S_1 - S_1')\big)$$

where $S_i' \subseteq S_i - S_j$ for $\{i, j\} = \{1, 2\}$.

Both M_1' and M_2' have rank k; also, $\mathcal{Z}(M) = \mathcal{Z}(M_1') \cup \mathcal{Z}(M_2')$.

It follows that a subset of $S_1 \cup S_2 \cup T_1 \cup T_2$ is a basis of M iff it is a basis of both M_1' and M_2'.

Open Problem

Under precisely what conditions do M_1 and M_2 have infinitely many intertwines?

Open Problem

Can a matroid and a uniform matroid have infinitely many intertwines? *(Geelen, Some Open Problems on Excluding a Uniform Matroid)*

Open Problem

Is the intertwine we constructed algebraic over a given field iff M_1 and M_2 are?

Open Problem

What can be said about $i(k+1; M_1, M_2) - i(k; M_1, M_2)$ where $i(k; M_1, M_2)$ is the number of rank-k intertwines of M_1 and M_2?

Theorem

With $k' = |S_1| + |S_2| + |T_1| + |T_2| - k$*, we have*

$$\big(M_k(M_1, S_1', T_1; M_2, S_2', T_2)\big)^* = M_{k'}(M_1^*, S_1 - S_1', T_2; M_2^*, S_2 - S_2', T_1).$$

It suffices to show that no single-element deletion $M \backslash a$ has both an M_1- and an M_2-minor.

The inequality on k implies that if $M\backslash a\backslash X/Y \simeq M_i$, then either $|X \cap T_1| \geq \eta(M_1)$ or $|Y \cap T_1| \geq r(M_2)$.

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(1) $M\backslash(X \cap T_1) = \big(M_2 \times (T_2 \cup S_1')\big) + \big((S_1 - S_1') \cup (T_1 - X)\big)$,

(2) $M\backslash(X \cap T_1)$ has no M_1-minor, and

(3) $\eta_2(F) = \eta_{M\backslash(X\cap T_1)}(F \cup T_2 \cup S_1')$ for $F \in \mathcal{Z}'(M_2)$.

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The second implies

(1′) $M/(Y \cap T_1) = \big(M_1 \times ((T_1 - Y) \cup S_2')\big) + \big((S_2 - S_2') \cup T_2\big)$,

(2′) $M/(Y \cap T_1)$ has no M_2-minor, and

(3′) $\eta_1(F) = \eta_{M/(Y\cap T_1)}(F \cup (T_1 - Y) \cup S_2')$ for $F \in \mathcal{Z}'(M_1)$.

For $a \in (S_1 - S_1') \cup T_1$, assume $M \backslash a$ has a minor $M \backslash a \backslash X / Y \simeq M_1$.

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(3′) $\eta_1(F) = \eta_{M/(Y \cap T_1)}(F \cup (T_1 - Y) \cup S_2')$ for $F \in \mathcal{Z}'(M_1)$.

By (3′), since a is in proper cyclic flats of $M/(Y \cap T_1)$, deleting a makes the sum of the nullities of the proper cyclic flats less than that sum in M_1, so $M \backslash a \backslash X / Y \not\simeq M_1$.

A similar argument covers $a \in S_1'$; symmetry covers $a \in S_2 \cup T_2$.