EXTENSIONS AND PRESENTATIONS OF TRANSVERSAL MATROIDS

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ABSTRACT. A transversal matroid M can be represented by a collection of sets, called a presentation of M, whose partial transversals are the independent sets of M. Minimal presentations are those for which removing any element from any set gives a presentation of a different matroid. We study the connections between (single-element) transversal extensions of M and extensions of presentations of M. We show that a presentation of M is minimal if and only if different extensions of it give different extensions of M; also, all transversal extensions of M can be obtained by extending the minimal presentations of M. We also begin to explore the partial order that the weak order gives on the transversal extensions of M.

In Memory of Michel Las Vergnas

1. INTRODUCTION

Single-element deletion is a very simple but fundamental matroid operation. Crapo [5] developed the basic theory of the reverse operation, single-element extension, which, far from simple, is the subject of a number of open problems. Given a class $\mathcal{C}$ of matroids that have a particular type of representation, it is natural to consider how representations of a matroid in $\mathcal{C}$ extend, if at all, to representations of its single-element extensions that are also in $\mathcal{C}$.

For example, consider the class $\mathcal{C}_{\mathbb{F}}$ of matroids M that are representable over a field $\mathbb{F}$. View a matrix representation of M as an embedding $\phi : E(M) \to \mathrm{PG}(r-1, \mathbb{F})$ in a projective geometry over $\mathbb{F}$. To represent a deletion $M\backslash x$ of M, restrict ϕ to $E(M) - x$. However, representing an extension is not so easy: if $|\mathbb{F}| > 3$, then ϕ may have extraneous information that could imply that none of its extensions represent a given extension of M in $\mathcal{C}_{\mathbb{F}}$. For example, for the rank-3 uniform matroid $U_{3,6}$, partition $E(U_{3,6})$ into three 2-point lines, L_1, L_2, and L_3. In one extension M' of $U_{3,6}$, add a point x to L_1, L_2, and L_3; in another extension M'' of $U_{3,6}$, add x just to L_1 and L_2, not to L_3. An embedding $\phi : E(U_{3,6}) \to \mathrm{PG}(2, \mathbb{F})$ can be extended to represent M' if and only if the intersection of the lines spanned by $\phi(L_1)$, $\phi(L_2)$, and $\phi(L_3)$ is nonempty, which is precisely when ϕ does not extend to represent M''. (The lack of what is called unique representability presents major challenges in work involving matroid representations. See Oxley [9, Section 14.6].)

In this paper, we explore questions of this nature for transversal matroids, which, like those that are representable over a field, are among the most basic types of matroids. A transversal matroid M can be represented by a presentation, which is a collection of sets whose partial transversals are the independent sets of M. There is a natural partial order on the presentations of a given transversal matroid; we focus primarily on the presentations that are minimal in this partial order. For instance, we show that a presentation of M is

Date: July 2, 2015.
1991 *Mathematics Subject Classification.* Primary: 05B35.
Key words and phrases. Transversal matroid, minimal presentation, single-element extension, weak order, extension lattice.

minimal if and only if all ways to extend the presentation yield different extensions of M; also, all transversal extensions of M are obtained by extending the minimal presentations of M. We treat such results in Section 3. In Section 4, we begin to explore the ordered set that the weak order gives on the transversal extensions of a transversal matroid.

2. Background

For basic matroid theory results, as well as the standard terminology and notation that we use, see Oxley [9]. In the first subsection we summarize the key results we need that are particular to transversal matroids; a few reminders about concepts for matroids in general are included and are signaled by referring to any matroid or to an arbitrary matroid. In the second subsection we outline essential points about single-element extensions.

2.1. **Transversal matroids.** A *set system* $\mathcal{A}$ on a set E is a multiset of subsets of E. We write $\mathcal{A}$ as $(A_1, A_2, \dots, A_r)$ or $(A_i : i \in [r])$, where $[r]$ denotes the set $\{1, 2, \dots, r\}$. We regard the set systems $(A_i : i \in [r])$ and $(A_{\sigma(i)} : i \in [r])$, for any permutation σ of $[r]$, as equal. We write $|\mathcal{A}|$ for the number of sets in $\mathcal{A}$, counting with their multiplicities, so $|\mathcal{A}| = r$ for $\mathcal{A} = (A_i : i \in [r])$.

A *partial transversal* of $\mathcal{A}$ is a subset I of E for which there is an injection $\phi : I \to [r]$ with $e \in A_{\phi(e)}$ for all $e \in I$; we call such an injection a *matching of I into $\mathcal{A}$*, or, if ϕ is onto, a *matching of I onto $\mathcal{A}$*. *Transversals* of $\mathcal{A}$ are partial transversals of size r. Edmonds and Fulkerson [6] showed that the partial transversals of $\mathcal{A}$ are the independent sets of a matroid on E; we call $\mathcal{A}$ a *presentation* of this *transversal matroid* $M[\mathcal{A}]$. Figure 1 gives an example of a transversal matroid.

A circuit in any matroid is a dependent set all of whose proper subsets are independent, so circuits have the following formulation for transversal matroids.

Lemma 2.1. *In $M[\mathcal{A}]$, a circuit is a set C for which there is no matching of C into $\mathcal{A}$, but, for each $e \in C$, there is a matching of $C - e$ into $\mathcal{A}$.*

We will use the following well-known lemmas about transversal matroids.

Lemma 2.2. *For each $A_i \in \mathcal{A}$, its complement $E - A_i$ is a flat of $M[\mathcal{A}]$.*

Indeed, to see that $r((E - A_i) \cup e) > r(E - A_i)$ whenever $e \notin E - A_i$, note that for any matching ϕ of a basis B of $E - A_i$ into $\mathcal{A}$, we have $i \notin \phi(B)$, so for any $e \in A_i$, we can extend ϕ to $B \cup e$ by mapping e to i, so $B \cup e$ is independent.

A proof of the following lemma can be found in Brualdi [3].

Lemma 2.3. *Let $\mathcal{A} = (A_i : i \in [r])$ be a presentation of a transversal matroid M. If $\phi : B \to [r]$ is a matching of a basis B of M into $\mathcal{A}$, then $(A_i : i \in \phi(B))$ is also a presentation of M. Thus, M has a presentation $\mathcal{A}'$ with $|\mathcal{A}'| = r(M)$. Furthermore, if M has no coloops, then all presentations of M have exactly $r(M)$ nonempty sets.*

For presentations $\mathcal{A}$ with $|\mathcal{A}| = r(M)$, the bases of M are the transversals of $\mathcal{A}$.

Throughout this paper we focus on presentations $\mathcal{A}$ of M with $|\mathcal{A}| = r(M)$, as in this lemma. (For clarity, we include this hypothesis in the statements of most results.) We will extend transversal matroids by extending their presentations by one element (we define this precisely at the start of Section 3), and Lemma 2.3 implies that the only single-element extension of a transversal matroid M that requires more than $r(M)$ sets in its presentations is the extension by a coloop. Since the extension by a coloop is a trivial type of extension, focusing on presentations with $r(M)$ sets does not significantly limit our results.

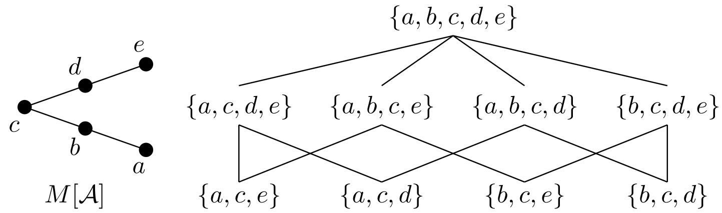


FIGURE 1. A geometric representation of the transversal matroid $M[\mathcal{A}]$ on $\{a,b,c,d,e\}$ for $\mathcal{A}=(A_1,A_2,A_3)$ where $A_1=\{a,b\}$, $A_2=\{d,e\}$, and A_3 is any of the nine sets shown on the right.

Recall that in the restriction $M|X$ of an arbitrary matroid M to a subset X of $E(M)$, its independent sets are the subsets of X that are independent in M. The next lemma follows easily from the definition of a transversal matroid.

Lemma 2.4. *If M is a transversal matroid, then so is $M|X$ for each $X \subseteq E(M)$. If $(A_i : i \in [r])$ is a presentation M, then $(A_i \cap X : i \in [r])$ is a presentation of $M|X$.*

Unions (including the empty union) of circuits of an arbitrary matroid M are called the *cyclic sets* of M. Thus, a subset X of $E(M)$ is cyclic if and only if the restriction $M|X$ has no coloops. We are most interested in flats that are cyclic. For example, in Figure 1, the cyclic flats of $M[\mathcal{A}]$ are $\emptyset$, $\{a,b,c\}$, $\{c,d,e\}$, and $\{a,b,c,d,e\}$. Lemmas 2.3 and 2.4 give the following result for transversal matroids.

Corollary 2.5. *If $(A_i : i \in [r])$ is a presentation of M and F is a cyclic flat of M, then there are exactly $r(F)$ integers i with $F \cap A_i \neq \emptyset$.*

Recall that the circuits of an arbitrary matroid M are its minimal dependent sets, and so are the minimal sets that are contained in no basis of M. The cocircuits of M are the circuits of its dual, M^*, and the cobases of M are the bases of M^*. The cocircuits of M are also the complements, $E(M)-H$, of the hyperplanes H of M, and the cobases of M are the complements of the bases of M. Thus, the cocircuits of M are the minimal sets that are contained in no cobasis of M, or, equivalently, they are the minimal sets that have nonempty intersection with all bases of M.

Now assume that M is a transversal matroid and that no set in a presentation $\mathcal{A}$ of M is empty (which must be the case when $|\mathcal{A}| = r(M)$). The bases of M are the maximal partial transversals of $\mathcal{A}$, so each set in $\mathcal{A}$ contains at least one element in each basis of M. Since, as we just noted, cocircuits are the minimal sets that have nonempty intersection with all bases, each set in $\mathcal{A}$ must contain a cocircuit of M. This is a key observation behind the next corollary, which is due to Brualdi and Dinolt [4].

Corollary 2.6. *If H is a cyclic hyperplane of a transversal matroid M, then the cocircuit $E(M)-H$ is a member of each presentation of M.*

Proof. Let $\mathcal{A} = (A_i : i \in [r])$ be a presentation of M where, by Lemma 2.3, we may take $r = r(M)$. By Corollary 2.5, exactly one set in $\mathcal{A}$, say A_r, is disjoint from H. Thus, $A_r \subseteq E(M)-H$. As noted above, A_r contains a cocircuit. Since $E(M)-H$ is a cocircuit and no cocircuit properly contains another, we have $A_r = E(M)-H$. □

In the example in Figure 1, each of $A_1 = \{a,b\}$ and $A_2 = \{d,e\}$ is the cocircuit that is the complement of a cyclic hyperplane ($\{c,d,e\}$ and $\{a,b,c\}$, respectively), so A_1 and A_2 are members of each presentation of $M[\mathcal{A}]$.

Corollary 2.7. *A transversal matroid of rank r has at most r cyclic hyperplanes. If it has r cyclic hyperplanes, then it has only one presentation.*

Minimal and maximal presentations play major roles in this paper, as do the lemmas below via which these presentations can be constructed. Minimal and maximal refer to the following partial order on the set of all presentations of a transversal matroid M that have $r = r(M)$ sets:

for two presentations $\mathcal{A} = (A_i : i \in [r])$ and $\mathcal{B} = (B_i : i \in [r])$ of M, set
$\mathcal{A} \leq \mathcal{B}$ if, up to reindexing, $A_i \subseteq B_i$ for all $i \in [r]$.

In the example in Figure 1, we saw that $A_1 = \{a, b\}$ and $A_2 = \{d, e\}$ are members of all presentations of $M[\mathcal{A}]$, so the partial order on its presentations is determined by the inclusions among the options for A_3, which are shown on the right side of that figure.

As noted above, given a presentation $\mathcal{A}$ of M with $|\mathcal{A}| = r(M)$, each set in $\mathcal{A}$ contains a cocircuit of M. Bondy and Welsh [2] and Las Vergnas [7] showed much more: given $\mathcal{A}$, there is a presentation $\mathcal{C} = (C_i : i \in [r])$ of M where each C_i is a cocircuit of M and $\mathcal{C} \leq \mathcal{A}$. Such a presentation $\mathcal{C}$ can be constructed by repeatedly applying the following lemma of Bondy and Welsh [2] until only cocircuits remain.

Lemma 2.8. *Let $\mathcal{A} = (A_i : i \in [r])$ be a presentation of M with $r = r(M)$. Fix $j \in [r]$. If P is a transversal of $(A_i : i \in [r] - \{j\})$ for which $|P \cap A_j|$ is minimal, then*

(1) *$A_j - P$ is a cocircuit of M, and*
(2) *replacing A_j by $A_j - P$ in $\mathcal{A}$ yields a presentation of M.*

This gives the following description of minimal presentations. This result is illustrated in Figure 1 since A_1, A_2, and the four set in the bottom row are all cocircuits of $M[\mathcal{A}]$.

Corollary 2.9. *A presentation $\mathcal{A}$ of a transversal matroid M is minimal if and only if each member of $\mathcal{A}$ is a cocircuit of M.*

Note that if $\mathcal{C} = (C_i : i \in [r])$ is a minimal presentation of M, then, since any matching ϕ of a basis B of the hyperplane $E(M) - C_i$ into $\mathcal{C}$ must have $\phi(B) = [r] - \{i\}$, we get $C_i \neq C_j$ whenever $i \neq j$, so the cocircuits in any minimal presentation are all different.

We will use the next result, which appears to be new and can be seen as a refinement of Lemma 2.8. It treats the following question: in the process of deleting elements from one presentation to get a minimal presentation, can we ensure that certain elements are not removed from any of the sets they are in?

Lemma 2.10. *Let M be $M[\mathcal{A}]$ where $\mathcal{A} = (A_i : i \in [r])$ and $r = r(M)$. For $J \subseteq E(M)$, if $r(M \backslash J) = r$, then M has a minimal presentation $(C_i : i \in [r]) \leq \mathcal{A}$ with $A_i \cap J = C_i \cap J$ for all $i \in [r]$.*

Proof. The key is carefully choosing the transversal P in Lemma 2.8. First consider A_1. Set $\mathcal{A}' = (A_i : 2 \leq i \leq r)$. Since $r(M \backslash J) = r$, some transversal of $\mathcal{A}$ is disjoint from J, so there is a transversal P' of $\mathcal{A}'$ that is disjoint from J. Let P be a transversal of $\mathcal{A}'$ with $|P \cap A_1|$ minimal, and set $X = P \cap A_1 \cap J$. Thus, $X \subseteq P - P'$. By the basis-exchange property for subsets in arbitrary matroids, applied in $M[\mathcal{A}']$, there is a subset $Y \subseteq P' - P$ so that $P'' = (P - X) \cup Y$ is a basis of $M[\mathcal{A}']$, that is, P'' is a transversal of $\mathcal{A}'$. Since $X \subseteq P \cap A_1$ and $|X| = |Y|$, we have $|P'' \cap A_1| \leq |P \cap A_1|$, so the minimality condition that P satisfies forces this inequality to be equality; also, P'' is disjoint from $A_1 \cap J$; thus, when we apply Lemma 2.8, we retain all elements in $A_1 \cap J$. Likewise, each of $A_2, A_3, \ldots, A_r$ in turn can be replaced by a subset that is a cocircuit and has the same intersection with J. □

As Figure 1 illustrates, a transversal matroid can have many minimal presentations. In contrast, Bondy [1] and Mason [8] proved the following result.

Proposition 2.11. *Each transversal matroid has a unique maximal presentation.*

The following result of Bondy and Welsh [2], which plays important roles in this paper, shows how to construct the maximal presentation.

Lemma 2.12. *Let $(A_i : i \in [r])$ be a presentation of M. For $e \in E(M) - A_i$,*

$$(A_1, A_2, \dots, A_{i-1}, A_i \cup e, A_{i+1}, \dots, A_r)$$

is also a presentation of M if and only if e is a coloop of the deletion $M\backslash A_i$.

2.2. **Single-element extensions of matroids.** We briefly recall this topic; see Oxley [9] for greater depth.

Definition 2.13. *A* single-element extension *of a matroid M is a matroid N on a set $E(M) \cup x$ with $N\backslash x = M$. The extension is* rank-preserving *if $r(N) = r(M)$.*

Figure 2 gives geometric representations of three extensions, by the element x, of the matroid $M[\mathcal{A}]$ in Figure 1. Note that x and c are parallel in M_2. Also, M_3 has four cyclic hyperplanes (lines), and so it is not transversal by Corollary 2.7. One can check that both M_1 and M_2, are transversal. By Corollary 2.7, the extension M_1 has only one presentation.

If $M = N\backslash x$, then the set $\mathcal{F}(M)$ of flats of M is $\{F - x : F \in \mathcal{F}(N)\}$, so $\mathcal{F}(M)$ is partitioned into the following three sets (some may be empty) that determine N:

$$\begin{aligned}
\mathcal{M} &= \{A \in \mathcal{F}(M) \,:\, A \cup x \in \mathcal{F}(N) \text{ and } A \notin \mathcal{F}(N)\} \\
&= \{A \in \mathcal{F}(M) \,:\, \mathrm{cl}_N(A) = A \cup x\}, \\
\\
\mathcal{C} &= \{A \in \mathcal{F}(M) \,:\, A \in \mathcal{F}(N) \text{ and } A \cup x \notin \mathcal{F}(N)\}, \\
\\
\mathcal{I} &= \{A \in \mathcal{F}(M) \,:\, A, A \cup x \in \mathcal{F}(N)\}.
\end{aligned}$$

For example, for the extension M_1 in Figure 2, the members of $\mathcal{M}$ are $\{a, e\}$ and $\{a, b, c, d, e\}$; for M_2, the members of $\mathcal{M}$ are all flats that contain c; and for M_3, the members of $\mathcal{M}$ are $\{a, d\}$, $\{b, e\}$, and $\{a, b, c, d, e\}$.

The collection $\mathcal{M}$ has the properties in the next definition and so is a modular cut.

Definition 2.14. *A* modular cut *of a matroid M is a set $\mathcal{M}$ of flats of M that has the following properties:*

(1) *for $A \in \mathcal{M}$, if $B \in \mathcal{F}(M)$ with $A \subseteq B$, then $B \in \mathcal{M}$, and*
(2) *if A and B are in $\mathcal{M}$ and (A, B) is a modular pair, that is,*

$$r(A) + r(B) = r(A \cup B) + r(A \cap B),$$

then $A \cap B \in \mathcal{M}$.

In the theory of single-element extensions, we focus on the set $\mathcal{M}$ because $\mathcal{C}$ and $\mathcal{I}$ can be derived from it. Specifically, one can show that a flat A is in $\mathcal{C}$ if and only if $A \notin \mathcal{M}$ but there is a superset A' of A with $A' \in \mathcal{M}$ and $r(A') = r(A) + 1$; also the collection $\mathcal{I}$ is the order ideal of flats not in $\mathcal{M} \cup \mathcal{C}$. Moreover, single-element extensions are equivalent to modular cuts by the following result of Crapo [5].

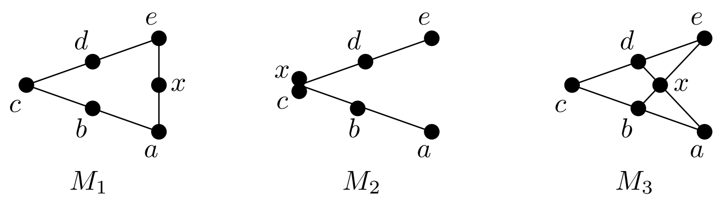


FIGURE 2. Three extensions, by the element x, of the matroid $M[\mathcal{A}]$ in Figure 1.

Proposition 2.15. *The map that takes each single-element extension of M to its modular cut is bijective; the image is the set of all modular cuts of M. The extension by a coloop corresponds to the empty modular cut, so the rank-preserving extensions correspond to the nonempty modular cuts.*

Note that the set $\mathcal{M} = \{E(M)\}$ is a modular cut of M. The corresponding extension is the *free extension* of M, denoted $M + x$. Geometrically, to get $M + x$, put x in the most general position in M without increasing the rank. The following lemma is well known.

Lemma 2.16. *The set of circuits of the free extension $M + x$ is*

$$\{C : C \text{ is a circuit of } M\} \cup \{B \cup x : B \text{ is a basis of } M\}.$$

3. PRESENTATIONS AND SINGLE-ELEMENT EXTENSIONS

Let $\mathcal{A} = (A_i : i \in [r])$ be a presentation of a rank-r transversal matroid M. For an element x not in $E(M)$ and a subset I of $[r]$, let $\mathcal{A}^I$ be $(A^I_i : i \in [r])$ where

$$A^I_i = \begin{cases} A_i \cup x, & \text{if } i \in I, \\ A_i, & \text{otherwise.} \end{cases}$$

Such a set system $\mathcal{A}^I$ is a *single-element extension of* $\mathcal{A}$. By Lemma 2.4, the matroid $M[\mathcal{A}^I]$ on $E(M) \cup x$ is a rank-preserving single-element extension of M.

For example, for the matroid $M[\mathcal{A}]$ in Figure 1 with $\mathcal{A} = (A_1, A_2, A_3)$, $A_1 = \{a, b\}$, $A_2 = \{d, e\}$, and $A_3 = \{b, c, d\}$, the extension $M[\mathcal{A}^{\{1,2\}}]$ is the matroid M_1 of Figure 2, and $M[\mathcal{A}^{\{3\}}]$ is the matroid M_2. If $\mathcal{A}$ is the only presentation of M under discussion, then we shorten $M[\mathcal{A}^I]$ to M^I. It follows from Lemmas 2.1 and 2.16 that $M^{[r]}$ is the free extension of M. Also, $M^{\emptyset}$ extends M by a loop.

All extensions of transversal matroids that we consider are by a single element, which we always call x. The only extension that does not preserve the rank is the extension by a coloop, which is a trivial type of extension, so we consider only extensions that preserve the rank. Therefore we typically omit the adjectives "rank-preserving" and "single-element" unless we are discussing the general theory of single-element extensions.

The 2^r extensions of $\mathcal{A}$ need not give distinct extensions of M; for example, if M is a rank-r uniform matroid, then its maximal presentation $\mathcal{A}$ consists of r copies of $E(M)$, and each extension M^I other than $M^{\emptyset}$ is the free extension of M, which is the rank-r uniform matroid on $E(M) \cup x$. Our first result answers the following question: for which presentations $\mathcal{A}$ do its 2^r extensions give 2^r different transversal extensions of M?

Theorem 3.1. *Let M be $M[\mathcal{A}]$ where $\mathcal{A} = (A_i : i \in [r])$ and $r = r(M)$. The following statements are equivalent:*

(1) *the presentation $\mathcal{A}$ of M is minimal, and*

(2) *for subsets I and J of $[r]$, if $M^I = M^J$, then $I = J$.*

Proof. First assume statement (1) fails, so, up to relabeling, we have $C \subsetneq A_r$ for some cocircuit C of M. Thus, the flat $E(M)-A_r$ is a proper subset of the hyperplane $E(M)-C$, so $r(M\backslash A_r) < r-1$. It follows that x is a coloop of $M[\mathcal{A}^{[r-1]}]\backslash A_r$, so Lemma 2.12 gives $M[\mathcal{A}^{[r-1]}] = M[\mathcal{A}^{[r]}]$, so statement (2) fails.

Now assume statement (1) holds. For $i \in [r]$, let H_i be the hyperplane $E(M) - A_i$. Thus, either H_i or $H_i \cup x$ is a hyperplane of the extension M^I of M. Any matching ϕ of a basis B of H_i into $\mathcal{A}$ has $\phi(B) = [r] - \{i\}$, so $B \cup x$ is a basis of M^I if and only if $x \in A_i^I$, that is, if and only if $i \in I$. Thus, we can reconstruct I from $\mathcal{A}$ and M^I, so statement (2) holds. □

The second paragraph of the proof shows that $H_i \cup x$ is a hyperplane of M^I if and only if $A_i^I = A_i$. Thus, A_i^I is the cocircuit complementary to the hyperplane spanned by H_i in M^I, which proves the next lemma.

Lemma 3.2. *Any extension $\mathcal{C}^I$ of a minimal presentation $\mathcal{C} = (C_i : i \in [r])$ of rank-r transversal matroid is a minimal presentation of $M[\mathcal{C}^I]$.*

By Theorem 3.1, any minimal presentation of M extends to presentations of $2^{r(M)}$ different transversal extensions of M, whereas any non-minimal presentation extends to presentations of fewer than $2^{r(M)}$ transversal extensions of M. Thus, if M has more than one presentation, then not all transversal extensions of M are obtained by extending the maximal presentation of M. (Note the example of uniform matroids before Theorem 3.1.) We next show that for each transversal extension N of M, some minimal presentation of M can be extended to a presentation of N (necessarily minimal, by Lemma 3.2).

Theorem 3.3. *If N is a transversal matroid and $x \in E(N)$ is not a coloop, then N has a presentation that is an extension of some minimal presentation of $N\backslash x$.*

Proof. Let $r = r(N)$ and let x be in exactly k sets in the maximal presentation $\mathcal{A}$ of N. By Lemma 2.10, there is a minimal presentation of N of the form

$$\mathcal{D} = (D_1 \cup x, \dots, D_k \cup x, D_{k+1}, \dots, D_r).$$

Let M be $N\backslash x$. Now $(D_i : i \in [r])$ is a presentation of M, so it suffices to show that each set D_j is a cocircuit of M. This holds if $j \in [k]$ since $E(N) - (D_j \cup x)$, that is, $E(M) - D_j$, is a hyperplane of N and so of M. Assume $k+1 \le j \le r$. Since x is in as many sets in $\mathcal{D}$ as in $\mathcal{A}$ (the maximal presentation), Lemma 2.12 implies that x is not a coloop of $N\backslash D_j$, so $N\backslash(D_j \cup x)$ and $N\backslash D_j$ have the same rank, which is $r-1$ since $E(N) - D_j$ is a hyperplane of N. The set $E(M) - D_j$ is therefore a flat of M of rank $r-1$, and so is a hyperplane of M, so D_j is a cocircuit of M, as needed. □

While this result represents a gain in efficiency since only minimal presentations need be extended, M may have many minimal presentations; for example, if M is the rank-r uniform matroid on $[2r]$, its $\binom{2r}{r}$ presentations of the form $(X \cup y : y \in [2r] - X)$, where X is an r-subset of $[2r]$, account for only some of its minimal presentations.

We introduce some notation to facilitate discussing several topics that are motivated by the previous result. For a transversal matroid M, let $\mathcal{T}(M)$ be the set of rank-preserving transversal extensions of M to $E(M) \cup x$. Also, let $\mathcal{P}_0$ be the set of minimal presentations of M. For any set $\mathcal{P}$ of presentations of M with $r = r(M)$ sets, define $\mathcal{T}_{\mathcal{P}}$ by

$$\mathcal{T}_{\mathcal{P}} = \{M[\mathcal{A}^I] : \mathcal{A} \in \mathcal{P},\, I \subseteq [r]\}.$$

Thus, $\mathcal{T}_\mathcal{P} \subseteq \mathcal{T}(M)$, and, by Theorem 3.3, if $\mathcal{P}_0 \subseteq \mathcal{P}$, then $\mathcal{T}_\mathcal{P} = \mathcal{T}(M)$. The next result identifies $\mathcal{T}_\mathcal{P}$ if $\mathcal{P}$ contains all but the minimal presentations of M.

Corollary 3.4. *For $N \in \mathcal{T}(M)$, the following statements are equivalent:*

(1) *N has more than one presentation, and*
(2) *$N = M[\mathcal{A}^I]$ for some set I and some presentation $\mathcal{A}$ of M that is not minimal.*

Proof. If statement (1) holds, then the maximal presentation of N, say $(A_i : i \in [r])$, is not minimal, so the presentation $(A_i - x : i \in [r])$ of M is not minimal by Lemma 3.2, so statement (2) holds. Theorem 3.3 gives the converse. □

While the inclusion $\mathcal{P}_0 \subseteq \mathcal{P}$ guarantees the equality $\mathcal{T}_\mathcal{P} = \mathcal{T}(M)$, we next show that it is possible for the equality to hold even when $\mathcal{P} \cap \mathcal{P}_0 = \emptyset$. Specifically, we show that $\mathcal{T}_\mathcal{P} = \mathcal{T}(M)$ whenever M is a rank-r uniform matroid, with $r > 1$, and $\mathcal{P}$ is the set of its non-minimal presentations. By Corollary 3.4, it suffices to show that each $N \in \mathcal{T}(M)$ has multiple presentations. That clearly holds if x is a loop of N. If x is not a loop, then N has a minimal presentation $(A_1 \cup x, \dots, A_k \cup x, A_{k+1}, \dots, A_r)$ with $k \geq 1$. Since $r(N) > 1$, the cocircuit $A_i \cup x$, for $i \in [k]$, is not all of $E(N)$; also, $N \backslash (A_i \cup x)$ is the restriction of the uniform matroid M to a hyperplane, so it contains only coloops. Thus, Lemma 2.12 implies that $(E(N), \dots, E(N), A_{k+1}, \dots, A_r)$ is another presentation of N.

In Corollary 3.6, we give sufficient conditions under which, for all proper subsets $\mathcal{P}$ of $\mathcal{P}_0$, the difference $\mathcal{T}(M) - \mathcal{T}_\mathcal{P}$ is nonempty, that is, $\mathcal{P}_0$ is an inclusion-minimal set of presentations of M whose extensions yield all transversal extensions of M. In its proof and that of Theorem 3.7, we use the following lemma.

Lemma 3.5. *Let $\mathcal{A} = (A_i : i \in [r])$ be a presentation of M with $r = r(M)$. For $i \in [r]$, let $X_i = E(M) - A_i$ and $I(i) = [r] - \{i\}$. If X_i is a hyperplane of M, then $X_i \cup x$ is a cyclic hyperplane of $M^{I(i)}$.*

Proof. Let B be a basis of the hyperplane X_i of M. Observe that there is no matching of $B \cup x$ into $\mathcal{A}^{I(i)}$, but there are matchings of each of its proper subsets into $\mathcal{A}^{I(i)}$. Thus, by Lemma 2.1, the set $B \cup x$ is a circuit of $M^{I(i)}$. In $M^{I(i)}$, the set $X_i \cup x$ is the closure of this circuit and so is a cyclic hyperplane of $M^{I(i)}$. □

(If X_i is a hyperplane of M, then $M^{I(i)}$ is what is called the principal extension of M by X_i, but we will not need this fact.)

Part (2) of the next corollary applies to the example in Figure 1.

Corollary 3.6. *Let M be a transversal matroid of rank r.*

(1) *For $\mathcal{A} \in \mathcal{P}_0$, if there is an extension $\mathcal{A}^I$ of $\mathcal{A}$ for which $M[\mathcal{A}^I]$ has only one presentation, then $\mathcal{T}_{\mathcal{P}_0 - \{\mathcal{A}\}} \subsetneq \mathcal{T}(M)$.*
(2) *If M has at least $r - 1$ cyclic hyperplanes and $\mathcal{P} \subsetneq \mathcal{P}_0$, then $\mathcal{T}_\mathcal{P} \subsetneq \mathcal{T}(M)$.*

Proof. With the hypothesis of part (1), the only way to get the only presentation of $M[\mathcal{A}^I]$ is to extend $\mathcal{A}$, so the conclusion follows. By Corollary 2.6 and the hypothesis of part (2), in any minimal presentation $\mathcal{A} = (A_i : i \in [r])$ of M, at most one set, say A_r, is not the complement of a cyclic hyperplane of M. By Lemma 3.5, $(E(M) - A_r) \cup x$ is a cyclic hyperplane of $M[\mathcal{A}^{[r-1]}]$, which therefore has r cyclic hyperplanes. Part (2) now follows from part (1) and Corollary 2.7. □

Behind the proof of that corollary is part of the proof of the next result.

Theorem 3.7. *For a transversal matroid M of rank r, the following two statements are equivalent:*

(1) *M has exactly 2^r transversal extensions of rank r, and*
(2) *M has a unique minimal presentation.*

If, in addition, M has no coloops, then the statements below are equivalent to those above:

(3) *M has r cyclic hyperplanes, and*
(4) *M has only one presentation.*

Proof. Statement (2) implies statement (1) by Theorems 3.1 and 3.3. For the converse, let $\mathcal{C} = (C_i : i \in [r])$ be a minimal presentation of M. For $i \in [r]$, let $I(i) = [r] - \{i\}$. By Lemma 3.5, $(E(M) - C_i) \cup x$ is a cyclic hyperplane of $M[\mathcal{C}^{I(i)}]$, so by Corollary 2.6, its complement, C_i, is in each presentation of $M[\mathcal{C}^{I(i)}]$. Theorem 3.1 and statement (1) imply that for any minimal presentation $\mathcal{D}$ of M, some extension $\mathcal{D}^J$ of $\mathcal{D}$ is a presentation of $M[\mathcal{C}^{I(i)}]$. Thus, $C_i \in \mathcal{D}^J$, so $C_i \in \mathcal{D}$. Thus, $\mathcal{C} = \mathcal{D}$, so statement (2) holds.

Now assume M has no coloops. By Corollary 2.7, statement (3) implies statement (4), which implies statement (2), so it suffices to show that if statement (3) fails, so does statement (2). Let $\mathcal{C} = (C_i : i \in [r])$ be a minimal presentation of M. Assuming that statement (3) fails, some hyperplane $E(M) - C_i$, say $E(M) - C_1$, is not cyclic, so $M\backslash C_1$ has a coloop, say e, so $\mathcal{C}' = (C_1 \cup e, C_2, \dots, C_r)$ is a presentation of M by Lemma 2.12. If e is in exactly k sets of $\mathcal{C}$, then, since e is in $k+1$ sets of $\mathcal{C}'$, it is in more than k sets in the maximal presentation of M, so, by Lemma 2.10, it is in more than k sets in some minimal presentation $\mathcal{D}$ of M. Thus, $\mathcal{C} \neq \mathcal{D}$, so statement (2) fails. □

The free matroid, $U_{r,r}$, which satisfies statements (1) and (2) only, shows the need for the assumption on coloops in order for statements (1) and (2) to imply statement (4).

The maximal presentation of a transversal matroid M need not extend to the maximal presentation of an extension of M, but their maximal presentations are related, as the next result shows.

Theorem 3.8. *Let M be $M[\mathcal{A}]$ where $\mathcal{A} = (A_i : i \in [r])$ and $r = r(M)$. Fix $I \subseteq [r]$. If $(B_i : i \in [r])$ and $(C_i : i \in [r])$ are the maximal presentations of M and $M[\mathcal{A}^I]$, respectively, then $C_i = B_i \cup x$ for all $i \in I$.*

Proof. For $i \in I$, the matroids $M\backslash A_i$ and $(M[\mathcal{A}^I])\backslash(A_i \cup x)$ are equal and so have the same coloops, so the result follows from Lemma 2.12. □

3.1. **A digression: a view of Theorem 3.1 in terms of linear subclasses.** Those who know the theory of linear subclasses (another way to describe single-element extensions, which we briefly recall below; see Oxley [9, Exercise 7.2.6]) may see a presentation-free formulation of part of Theorem 3.1 that might suggest possible generalizations. Below we give examples to show that, while analogous conclusions follow from a different type of hypothesis, for what it treats Theorem 3.1 is optimal. We start with a definition.

Definition 3.9. *A* linear subclass *of a matroid M is a set $\mathcal{H}$ of hyperplanes M that has the following property: if $H_1, H_2 \in \mathcal{H}$ and $r(H_1 \cap H_2) = r(M) - 2$, then $\mathcal{H}$ contains all hyperplanes of M that contain $H_1 \cap H_2$.*

The motivation for this definition is the observation that condition (2) in Definition 2.14 implies that, given a modular cut $\mathcal{M}$ of M, the set

$$\mathcal{H}_{\mathcal{M}} = \{H \in \mathcal{M} : r(H) = r(M) - 1\}$$

of hyperplanes in $\mathcal{M}$ has the property above. Note that $\mathcal{H}_{\emptyset} = \emptyset = \mathcal{H}_{\{E(M)\}}$, that is, the extension by a coloop and the free extension yield the same linear subclass. However, if

$\mathcal{M}$ is nonempty (so the extension is rank-preserving), then a flat is in $\mathcal{M}$ if and only if all hyperplanes that contain it are in $\mathcal{H}_{\mathcal{M}}$. Moreover, if $\mathcal{H}$ is a linear subclass, then setting

$$\mathcal{M}_{\mathcal{H}} = \{A \in \mathcal{F}(M) \ : \ \text{all hyperplanes } H \text{ with } A \subseteq H \text{ are in } \mathcal{H}\}$$

gives a modular cut of M. Thus, linear subclasses give another way to encode the rank-preserving single-element extensions of M. (Storing linear subclasses is more efficient than storing modular cuts, so software (such as Sage) that, for instance, finds single-element extensions of matroids typically works with linear subclasses.)

The fact that statement (1) of Theorem 3.1 implies statement (2) can be recast in terms of linear subclasses this way: if M is a transversal matroid and $\mathcal{S}$ is the set of hyperplanes that are the complements of the cocircuits in a given minimal presentation of M, then, for each subset $\mathcal{S}'$ of $\mathcal{S}$, there is a linear subclass $\mathcal{H}$ of M, corresponding to some transversal extension of M, with $\mathcal{H} \cap \mathcal{S} = \mathcal{S}'$. Note that the intersection of any i hyperplanes in $\mathcal{S}$ has rank at most $r(M) - i$. This suggests the following question.

> *Assume a matroid M has rank r and $\mathcal{S}$ is a set of r hyperplanes of M, the intersection of any i of which has rank at most $r - i$. For each subset $\mathcal{S}'$ of $\mathcal{S}$, is there a linear subclass $\mathcal{H}$ of M with $\mathcal{H} \cap \mathcal{S} = \mathcal{S}'$?*

First observe that the answer is positive if we change the assumption about $\mathcal{S}$: for any matroid M (whether or not it is transversal), the answer is positive if $r(H \cap H') < r(M) - 2$ for all $H, H' \in \mathcal{S}$ since each subset of $\mathcal{S}$ is a linear subclass. We now show that the answer can be negative even if M is transversal but $\mathcal{S}$ is not the set of complements of the cocircuits in a minimal presentation. For example, let M be $M[\mathcal{A}]$ on $\{a, b, b', c, c', d, d'\}$ where

$$\mathcal{A} = (\{a, b, b'\}, \{a, c, c'\}, \{a, d, d'\}, \{a, b', c', d'\}).$$

Let $\mathcal{S}$ be consist of the following four hyperplanes (planes) of M:

$$P_1 = \{a, b, b'\}, \quad P_2 = \{a, c, c'\}, \quad P_3 = \{b, b', d, d'\}, \quad P_4 = \{c, c', d, d'\}.$$

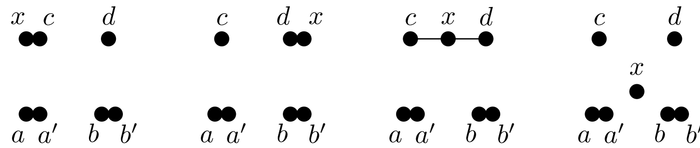
Let $\mathcal{H}$ be any linear subclass that contains P_1, P_2, and P_3. Since $P_1 \cap P_3$ is the line $\{b, b'\}$, by the definition of a linear subclass, the plane $\{b, b', c, c'\}$ is in $\mathcal{H}$; this plane intersects P_2 in $\{c, c'\}$, so P_4 is in $\mathcal{H}$, thus giving a negative answer for M and $\mathcal{S}$.

4. Transversal extensions and the weak order

In this section, we pose and begin to investigate a problem about the structure of the set of all rank-preserving transversal extensions of M. First, recall that the collection of all matroids on a set E is ordered by the *weak order*, denoted by $\leq_w$, where $M \leq_w N$ if and only if $r_M(X) \leq r_N(X)$ for all $X \subseteq E$; equivalently, each independent set in M is also independent in N. This order is not a lattice. Let M_1 and M_2 be rank-preserving single-element extensions of M, and, for $i \in \{1, 2\}$, let $\mathcal{M}_i$ be the modular cut of M that gives M_i. It is well known and easy to check that $M_1 \leq_w M_2$ if and only if $\mathcal{M}_2 \subseteq \mathcal{M}_1$. Also, the set of all rank-preserving single-element extensions of M is a lattice, the *extension lattice* $\mathcal{E}(M)$ of M, under the weak order; this holds since $\mathcal{M}_1 \cap \mathcal{M}_2$ is a modular cut and so gives the join, $M_1 \vee M_2$. Much about extension lattices remains unknown.

Problem 4.1. *Let M be a transversal matroid. Is the set $\mathcal{T}(M)$ of all rank-preserving transversal extensions of M to $E(M) \cup x$, ordered by the weak order, a lattice?*

As in $\mathcal{E}(M)$, the least extension in $\mathcal{T}(M)$ adds x as a loop, and the greatest is the free extension of M. The following examples show that certain pairs of extension have meets or joins in $\mathcal{T}(M)$ that differ from those in $\mathcal{E}(M)$.



M_c M_d $M_c \vee M_d$ $M + x$

FIGURE 3. The extensions in Example 1, their join in $\mathcal{E}(M)$, and their transversal join.

EXAMPLE 1. Consider the matroid M obtained from the uniform matroid $U_{3,4}$ on the set $\{a, b, c, d\}$ by adding a' parallel to a, and b' parallel to b. The following extensions of M are shown in Figure 3: let M_c be the transversal extension of M obtained by adding x parallel to c; for M_d, extend M by adding x parallel to d. The join, $M_c \vee M_d$, in $\mathcal{E}(M)$ has x placed freely on the line $\{c, d\}$, but this is not transversal; their join in $\mathcal{T}(M)$ exists and is the free extension, $M + x$. (Corollary 2.6 makes it relatively easy to show that $M_c \vee M_d$ is not transversal since that result gives two of the three sets in any candidate presentation.)

EXAMPLE 2. Let M be $U_{3,6}$ on $E = \{a, b, c, d, e, f\}$. One minimal presentation of M is $\mathcal{A} = (A_1, A_2, A_3)$ where

$$A_1 = \{a, b, c, d\}, \qquad A_2 = \{a, b, e, f\}, \quad \text{and} \quad A_3 = \{c, d, e, f\}.$$

The transversal extension $M[\mathcal{A}^{\{1\}}]$ arises from the modular cut $\mathcal{M}_1 = \{\{a, b\}, \{c, d\}, E\}$; in this extension, $\{x, a, b\}$ and $\{x, c, d\}$ are lines. The transversal extension $M[\mathcal{A}^{\{2\}}]$ arises from the modular cut $\mathcal{M}_2 = \{\{a, b\}, \{e, f\}, E\}$; in this extension, $\{x, a, b\}$ and $\{x, e, f\}$ are lines. Their meet, $M[\mathcal{A}^{\{1\}}] \wedge M[\mathcal{A}^{\{2\}}]$, in $\mathcal{E}(M)$ arises from the modular cut $\mathcal{M}_1 \cup \mathcal{M}_2$; in this extension, $\{x, a, b\}$, $\{x, c, d\}$, and $\{x, e, f\}$ are lines, and this extension is not transversal (again use Corollary 2.6; there is only one candidate presentation to check); their meet in $\mathcal{T}(M)$ exists and is $M^{\emptyset}$, in which x is a loop.

Although this example has the meet of $M[\mathcal{A}^I]$ and $M[\mathcal{A}^J]$ in $\mathcal{T}(M)$ being $M[\mathcal{A}^{I\cap J}]$, the next example shows that this does not hold in general.

EXAMPLE 3. For $i \in [4]$, let A_i be $[5] - \{i\}$. Then $\mathcal{A} = (A_i : i \in [4])$ is a presentation of the rank-4 uniform matroid M on $[5]$ (i.e., a circuit). One can easily check that both $\mathcal{A}^{\{1,2\}}$ and $\mathcal{A}^{\{3,4\}}$ are presentations of the free extension of M, so this is also their meet, yet $\mathcal{A}^{\{1,2\}\cap\{3,4\}}$, that is, $\mathcal{A}^{\emptyset}$, is the extension by a loop.

In contrast to these examples, we show in Theorem 4.4 that the join of $M[\mathcal{A}^I]$ and $M[\mathcal{A}^J]$ in $\mathcal{E}(M)$ is $M[\mathcal{A}^{I\cup J}]$, so their join in $\mathcal{T}(M)$ exists and also is $M[\mathcal{A}^{I\cup J}]$. The heart of the proof is Lemma 4.3, in which we identify the modular cut for an extension $M[\mathcal{A}^I]$ of $M[\mathcal{A}]$.

We need the following two definitions. A transversal matroid is *fundamental* if it has a presentation $(A_i : i \in [r])$ for which no difference $A_i - \bigcup_{j\in[r]-\{i\}} A_j$ is empty. These matroids were introduced by Las Vergnas [7], who showed that the class of fundamental transversal matroids, unlike that of all transversal matroids, is closed under duality. For a subset X of $E(M)$, the $\mathcal{A}$-*support* $s_{\mathcal{A}}(X)$ *of* X is given by

$$s_{\mathcal{A}}(X) = \{i : X \cap A_i \neq \emptyset\}.$$

We will abbreviate this to $s(X)$ if no ambiguity results. By Hall's famous theorem on matchings in bipartite graphs, a subset Y of $E(M)$ is independent in M if and only if $|s(Z)| \geq |Z|$ for all subsets Z of Y.

Lemma 4.2. *Let M be $M[\mathcal{A}]$ where $\mathcal{A} = (A_i : i \in [r])$ and $r = r(M)$. For subsets X and Y of $E(M)$, if $r(X) = |s(X)|$ and $r(Y) = |s(Y)|$, then $r(X \cup Y) = |s(X \cup Y)|$.*

Proof. Let $M[\mathcal{A}^+]$, or M^+, be the fundamental transversal matroid on $E(M) \cup V$ where the set $V = \{v_1, v_2, \ldots, v_r\}$ is disjoint from $E(M)$ and $\mathcal{A}^+ = (A_i \cup v_i : i \in [r])$. For any basis B of X and matching $\phi : B \to [r]$, we have $\phi(B) = s(X)$ since $r(X) = |s(X)|$, so there is no matching of any set D with $B \subsetneq D \subseteq B \cup \{v_i : i \in s(X)\}$ into $\mathcal{A}^+$. Thus, $\{v_i : i \in s(X)\}$, which is independent and has size $r(X)$, is contained in $\mathrm{cl}_{M^+}(X)$ and so is a basis of $\mathrm{cl}_{M^+}(X)$. Likewise $\{v_i : i \in s(Y)\}$ is a basis of $\mathrm{cl}_{M^+}(Y)$. It follows that $\{v_i : i \in s(X \cup Y)\}$, which is $\{v_i : i \in s(X) \cup s(Y)\}$, is a basis of $\mathrm{cl}_{M^+}(X \cup Y)$. Thus, $r_M(X \cup Y) = r_{M^+}(X \cup Y) = |s(X \cup Y)|$. □

Lemma 4.3. *The modular cut that corresponds to the extension $M^I = M[\mathcal{A}^I]$ of M is*

$$\mathcal{M}^I = \{F \in \mathcal{F}(M) : \text{ for some } X \subseteq F,\ r(X) = |s(X)| \text{ and } I \subseteq s(X)\}.$$

Proof. The modular cut $\mathcal{M}^I$ consists of the flats F of M with $x \in \mathrm{cl}_{M^I}(F)$. First assume there is a subset X of F with $r(X) = |s(X)|$ and $I \subseteq s(X)$. Let ϕ be a matching of a basis B of X into $\mathcal{A}$. As above, we have $\phi(B) = s(X)$, so, since $I \subseteq s(X)$, there is no matching of $B \cup x$ into $\mathcal{A}^I$; thus, $x \in \mathrm{cl}_{M^I}(X)$, so $x \in \mathrm{cl}_{M^I}(F)$, so $F \in \mathcal{M}^I$. For the converse, for $F \in \mathcal{M}^I$, let X be a minimal subset of F with $x \in \mathrm{cl}_{M^I}(X)$. By minimality, X is independent and matchings of X into $\mathcal{A}$ do not extend to matchings of $X \cup x$ into $\mathcal{A}^I$, so $I \subseteq s(X)$. Also by minimality, $X \cup x$ is a circuit of M^I. Thus, the only subset of $X \cup x$ for which the inequality in Hall's condition fails is $X \cup x$ itself, so $|s_{\mathcal{A}^I}(X \cup x)| < |X \cup x|$ and $|s_{\mathcal{A}^I}(X)| \geq |X|$; therefore $|s_{\mathcal{A}^I}(X)| = |X|$, so $|s_{\mathcal{A}}(X)| = |X| = r(X)$, as needed. □

We now show that if two transversal extensions of M have presentations that extend the same presentation of M, then they have a join in $\mathcal{T}(M)$, and we identify their join.

Theorem 4.4. *The join of $M[\mathcal{A}^I]$ and $M[\mathcal{A}^J]$ in $\mathcal{E}(M)$ is $M[\mathcal{A}^{I \cup J}]$, which therefore is their join in $\mathcal{T}(M)$.*

Proof. The modular cut of $M[\mathcal{A}^I] \vee M[\mathcal{A}^J]$ is $\mathcal{M}^I \cap \mathcal{M}^J$, so, by Lemma 4.3, we must show that, for a flat F of M, the following two statements are equivalent:

(1) there are subsets X and Y of F with $r(X) = |s(X)|,\ r(Y) = |s(Y)|,\ I \subseteq s(X)$, and $J \subseteq s(Y)$
(2) there is a subset Z of F with $r(Z) = |s(Z)|$ and $I \cup J \subseteq s(Z)$.

Lemma 4.2 shows that statement (1) implies statement (2) where $Z = X \cup Y$. The converse is immediate upon taking X and Y to be Z. □

In particular, if M has only one minimal presentation, say $\mathcal{A}$, then, by Theorem 3.3, every matroid in $\mathcal{T}(M)$ has a presentation of the form $\mathcal{A}^I$, so Theorem 4.4 implies that $\mathcal{T}(M)$ is a lattice. (Recall that a join semilattice with a least element (the extension by a loop, in this case) is a lattice: if each pair of elements has a join, then each pair also has a meet.) By the next result, this lattice is isomorphic to a boolean lattice on $r(M)$ elements. Corollary 4.5 brings our results full circle, back to Theorem 3.1, by providing another characterization of minimal presentations.

Corollary 4.5. *Let $\mathcal{A}$ be a presentation of M with r sets, where $r = r(M)$. If $I \subseteq J \subseteq [r]$, then $M[\mathcal{A}^I] \leq_w M[\mathcal{A}^J]$. Furthermore, the converse holds if and only if the presentation $\mathcal{A}$ of M is minimal.*

Proof. Consider the following statements:

(i) $I \subseteq J$,
(ii) $I \cup J = J$,
(iii) $M[\mathcal{A}^{I\cup J}] = M[\mathcal{A}^J]$,
(iv) $M[\mathcal{A}^I] \vee M[\mathcal{A}^J] = M[\mathcal{A}^J]$,
(v) $M[\mathcal{A}^I] \leq_w M[\mathcal{A}^J]$.

Each statement is equivalent to the next, except, while statement (ii) implies statement (iii), by Theorem 3.1 the converse holds if $\mathcal{A}$ is minimal. By Theorem 3.1, if $\mathcal{A}$ is not minimal, then there are distinct sets I and J with $M[\mathcal{A}^I] = M[\mathcal{A}^J]$, yet either $I \not\subseteq J$ or $J \not\subseteq I$. □

We end by posing another problem that, if the answer is affirmative, would strengthen Theorem 3.3.

Problem 4.6. *If $M_1 \leq_w M_2$ with $M_1, M_2 \in \mathcal{T}(M)$, is there a minimal presentation $\mathcal{A}$ of M and sets $I_1 \subseteq I_2 \subseteq [r]$, where $r = r(M)$, with $M_1 = M[\mathcal{A}^{I_1}]$ and $M_2 = M[\mathcal{A}^{I_2}]$?*

Acknowledgments

We thank Nancy Ann Neudauer for asking a question that prompted the observations in Section 3.1. We also thank the referees for their comments and questions, which led to improvements in the exposition. The second author was supported by the Spanish and Catalan governments through projects MTM2011–24097 and DGR2009–SGR1040.

References

[1] J. A. Bondy, Presentations of transversal matroids, *J. London Math. Soc. (2)* **5** (1972) 289–292.
[2] J. A. Bondy and D. J. A. Welsh, Some results on transversal matroids and constructions for identically self-dual matroids, *Quart. J. Math. Oxford Ser.* **22** (1971) 435–451.
[3] R. Brualdi, Transversal matroids, in: *Combinatorial geometries, Encyclopedia Math. Appl., 29*, Cambridge Univ. Press, Cambridge, 1987, 72-97.
[4] R. Brualdi and G. Dinolt, Characterizations of transversal matroids and their presentations, *J. Combin. Theory Ser. B* **12** (1972) 268–286.
[5] H. H. Crapo, Single-element extensions of matroids, *J. Res. Natl. Bureau Standards Sect. B* **69** (1965) 55–65.
[6] J. Edmonds and D. R. Fulkerson, Transversals and matroid partition, *J. Res. Nat. Bur. Standards Sect. B* **69B** (1965) 147–153.
[7] M. Las Vergnas, Sur les systèmes de représentants distincts d'une famille d'ensembles, *C. R. Acad. Sci. Paris Sér. A-B* **270** (1970) A501–A503.
[8] J. Mason, *Representations of Independence Spaces,* (Ph.D. Dissertation, University of Wisconsin, Madison WI, 1969).
[9] J. G. Oxley, *Matroid Theory*, second edition (Oxford University Press, Oxford, 2011).

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