Characterizations of Fundamental Transversal Matroids

Joseph E. Bonin

The George Washington University

Joint work with
Anna de Mier (Universitat Politècnica de Catalunya)
and
Joseph Kung (University of North Texas)

These slides and the corresponding paper are available at
http://home.gwu.edu/~jbonin/
Transversal Matroids

A set system is a multiset $\mathcal{A} = (A_j : j \in J)$ of subsets of a set $S$.

A partial transversal of $\mathcal{A}$ is a subset $I$ of $S$ for which there is an injection $\phi : I \rightarrow J$ with $x \in A_{\phi(x)}$ for all $x \in I$.

E.g., $\mathcal{A} = (A, B, C, D)$,

$A = \{a, b, e, f, h\}$,

$B = \{b, c, g\}$,

$C = \{d, e, g, h\}$,

$D = \{d, f, h\}$

Theorem (Edmonds and Fulkerson, 1965)

The partial transversals of a set system $\mathcal{A}$ are the independent sets of a matroid on $S$.

Such matroids are transversal matroids; $\mathcal{A}$ is a presentation.
A flat $F$ of $M$ is **cyclic** if $M|F$ has no coloops; i.e., $F$ is a union of circuits.

Let $\mathcal{Z}(M)$ be the set of cyclic flats of $M$.

**Theorem** (Brylawski, 1975)

A matroid $M$ is transversal iff it has an affine realization on a simplex $\Delta$ in which each $F \in \mathcal{Z}(M)$ is the set of points in some face of $\Delta$ with $r(F)$ vertices.
Maximal Presentations

Three representations of $U_{3,6}$ on a 3-vertex simplex.

A presentation $(A_1, \ldots, A_r)$ of $M$ is maximal if, whenever $(A'_1, \ldots, A'_r)$ is a presentation of $M$ with $A_i \subseteq A'_i$ for $i \in [r]$, then $A_i = A'_i$ for $i \in [r]$.

A transversal matroid $M$ has a unique maximal presentation with $r(M)$ sets.  

(Mason, 1971.)
A fundamental transversal matroid, FTM, is a transversal matroid $M$ that has an affine representation on a simplex in which each vertex has an element of $M$ placed at it.

A matroid $M$ is an FTM iff it has a basis $B$ (a fundamental basis) with $B \cap F$ spanning $F$ for each $F \in \mathcal{Z}(M)$.

Each transversal matroid extends to an FTM of the same rank.
The Inspiration for our Work

Theorem (Mason, 1971; refined by Ingleton, 1977)

* A matroid $M$ is transversal iff for all nonempty $\mathcal{F} \subseteq \mathcal{Z}(M)$,

$$r(\cap \mathcal{F}) \leq \sum_{\mathcal{F}' \subseteq \mathcal{F}} (-1)^{|\mathcal{F}'|+1} r(\cup \mathcal{F}')$$

* where $\cap \mathcal{F} = \bigcap_{X \in \mathcal{F}} X$ and $\cup \mathcal{F} = \bigcup_{X \in \mathcal{F}} X$.

Example: if $\mathcal{F}$ consists of three coplanar cyclic lines, then the right side is $3 \cdot 2 - 3 \cdot 3 + 3 = 0$, so for $M$ to be transversal, $r(\cap \mathcal{F})$ must be 0.
A Sketch of the Proof of the Necessity

A matroid $M$ is transversal iff for all nonempty $\mathcal{F} \subseteq \mathcal{Z}(M)$,

$$r(\cap \mathcal{F}) \leq \sum_{\mathcal{F}' \subseteq \mathcal{F}} (-1)^{|\mathcal{F}'|+1} r(\cup \mathcal{F}').$$

Extend $M$ to an FTM $M'$ with a fundamental basis $B'$.

Show that for $\mathcal{F} \subseteq \mathcal{Z}(M')$,

$$r'(\cup \mathcal{F}) = \left| B' \cap (\cup \mathcal{F}) \right| \quad \text{and} \quad r'(\cap \mathcal{F}) = \left| B' \cap (\cap \mathcal{F}) \right|.$$

For $M'$, inclusion-exclusion gives the inequality — indeed, equality.

The inequality for $M$ follows.

(For a proof that the inequality implies that $M$ is transversal, see the paper.)
The Main New Result: A Characterization of FTMs

**Theorem**

A matroid $M$ is an FTM iff for all nonempty $\mathcal{F} \subseteq Z(M)$,

$$ r(\cap \mathcal{F}) = \sum_{\mathcal{F}' \subseteq \mathcal{F}} (-1)^{|\mathcal{F}'|+1} r(\cup \mathcal{F}'). $$

The inclusion-exclusion argument in the last proof shows that the equality holds for all FTMs.
Strategy of the Proof of Sufficiency

By the Mason/Ingleton result, $M$ is transversal.

Start with the simplex realization of its maximal presentation.

Define the support of $x \in E(M)$: Let $\Delta(x)$ be the set of vertices of the simplex that span the face that $x$ is freely placed in.

Extend to sets $Y \subseteq E(M)$ via unions: $\Delta(Y) = \bigcup_{x \in Y} \Delta(x)$.

For each vertex $v_i$ at which no element of $M$ has been placed, let $\mathcal{F}_i = \{ F \in \mathcal{Z}(M) : v_i \in \Delta(F) \}$.

Show

(a) some $x \in E(M)$ with $v_i \in \Delta(x)$ is freely in $\Delta(\cap \mathcal{F}_i)$ and

(b) moving $x$ to $v_i$ preserves the circuits and so also realizes $M$. 
Strategy of the Proof of Sufficiency

By the Mason/Ingleton result, $M$ is transversal.

Start with the simplex realization of its maximal presentation.

**Define the support of $x \in E(M)$**: Let $\Delta(x)$ be the set of vertices of the simplex that span the face that $x$ is freely placed in.

**Extend to sets $Y \subseteq E(M)$ via unions**: $\Delta(Y) = \bigcup_{x \in Y} \Delta(x)$.

For each vertex $v_i$ at which no element of $M$ has been placed, let $\mathcal{F}_i = \{ F \in \mathcal{Z}(M) : v_i \in \Delta(F) \}$.

Show

(a) some $x \in E(M)$ with $v_i \in \Delta(x)$ is freely in $\Delta(\cap \mathcal{F}_i)$ and  
(b) moving $x$ to $v_i$ preserves the circuits and so also realizes $M$. 

![Diagram of a simplex with vertices and faces indicated for the support definition.](image)
By the Mason/Ingleton result, $M$ is transversal.

Start with the simplex realization of its maximal presentation.

**Define the support of** $x \in E(M)$: Let $\Delta(x)$ be the set of vertices of the simplex that span the face that $x$ is freely placed in.

**Extend to sets** $Y \subseteq E(M)$ via unions: $\Delta(Y) = \bigcup_{x \in Y} \Delta(x)$.

For each vertex $v_i$ at which no element of $M$ has been placed, let $\mathcal{F}_i = \{F \in \mathcal{Z}(M) : v_i \in \Delta(F)\}$.

Show

(a) some $x \in E(M)$ with $v_i \in \Delta(x)$ is freely in $\Delta(\cap \mathcal{F}_i)$ and

(b) moving $x$ to $v_i$ preserves the circuits and so also realizes $M$. 

\[ \begin{array}{c}
\text{Diagram Image} \\
\end{array} \]
Strategy of the Proof of Sufficiency

By the Mason/Ingleton result, \( M \) is transversal.

Start with the simplex realization of its maximal presentation.

**Define the support of** \( x \in E(M) \): Let \( \Delta(x) \) be the set of vertices of the simplex that span the face that \( x \) is freely placed in.

**Extend to sets** \( Y \subseteq E(M) \) **via unions**: \( \Delta(Y) = \bigcup_{x \in Y} \Delta(x) \).

For each vertex \( v_i \) at which no element of \( M \) has been placed, let
\[
\mathcal{F}_i = \{ F \in \mathcal{Z}(M) : v_i \in \Delta(F) \}.
\]

Show

(a) some \( x \in E(M) \) with \( v_i \in \Delta(x) \) is freely in \( \Delta(\cap \mathcal{F}_i) \) and

(b) moving \( x \) to \( v_i \) preserves the circuits and so also realizes \( M \).
Strategy of the Proof of Sufficiency

By the Mason/Ingleton result, $M$ is transversal.

Start with the simplex realization of its maximal presentation.

**Define the support of** $x \in E(M)$: Let $\Delta(x)$ be the set of vertices of the simplex that span the face that $x$ is freely placed in.

**Extend to sets** $Y \subseteq E(M)$ **via unions**: $\Delta(Y) = \bigcup_{x \in Y} \Delta(x)$.

For each vertex $v_i$ at which no element of $M$ has been placed, let $\mathcal{F}_i = \{ F \in \mathcal{Z}(M) : v_i \in \Delta(F) \}$.

Show

(a) some $x \in E(M)$ with $v_i \in \Delta(x)$ is freely in $\Delta(\cap \mathcal{F}_i)$ and

(b) moving $x$ to $v_i$ preserves the circuits and so also realizes $M$. 
By the Mason/Ingleton result, $M$ is transversal.

Start with the simplex realization of its maximal presentation.

**Define the support of** $x \in E(M)$: Let $\Delta(x)$ be the set of vertices of the simplex that span the face that $x$ is freely placed in.

**Extend to sets** $Y \subseteq E(M)$ via unions: $\Delta(Y) = \bigcup_{x \in Y} \Delta(x)$.

For each vertex $v_i$ at which no element of $M$ has been placed, let $\mathcal{F}_i = \{ F \in \mathcal{Z}(M) : v_i \in \Delta(F) \}$.

**Show**

(a) some $x \in E(M)$ with $v_i \in \Delta(x)$ is freely in $\Delta(\cap \mathcal{F}_i)$ and

(b) moving $x$ to $v_i$ preserves the circuits and so also realizes $M$. 

![Diagram](diagram.png)
**Strategy of the Proof of Sufficiency**

By the Mason/Ingleton result, $M$ is transversal.

Start with the simplex realization of its maximal presentation.

Define the support of $x \in E(M)$: Let $\Delta(x)$ be the set of vertices of the simplex that span the face that $x$ is freely placed in.

Extend to sets $Y \subseteq E(M)$ via unions: $\Delta(Y) = \bigcup_{x \in Y} \Delta(x)$. For each vertex $v_i$ at which no element of $M$ has been placed, let $\mathcal{F}_i = \{F \in \mathcal{Z}(M) : v_i \in \Delta(F)\}$.

Show

(a) some $x \in E(M)$ with $v_i \in \Delta(x)$ is freely in $\Delta(\bigcap \mathcal{F}_i)$ and

(b) moving $x$ to $v_i$ preserves the circuits and so also realizes $M$. 
Strategy of the Proof of Sufficiency

By the Mason/Ingleton result, $M$ is transversal.

Start with the simplex realization of its maximal presentation.

Define the support of $x \in E(M)$: Let $\Delta(x)$ be the set of vertices of the simplex that span the face that $x$ is freely placed in.

Extend to sets $Y \subseteq E(M)$ via unions: $\Delta(Y) = \bigcup_{x \in Y} \Delta(x)$.

For each vertex $v_i$ at which no element of $M$ has been placed, let $\mathcal{F}_i = \{ F \in \mathcal{Z}(M) : v_i \in \Delta(F) \}$.

Show

(a) some $x \in E(M)$ with $v_i \in \Delta(x)$ is freely in $\Delta(\cap \mathcal{F}_i)$ and

(b) moving $x$ to $v_i$ preserves the circuits and so also realizes $M$. 

\[
\begin{array}{c}
\text{v1} \\
\text{v2} \\
\text{v3} \\
\text{v4}
\end{array}
\]

\[
\begin{array}{c}
\text{v1} \\
\text{v2} \\
\text{v3} \\
\text{v4}
\end{array}
\]
An example where the strategy fails when the equality fails:
**The Role of the Hypothesis**

An example where the strategy fails when the equality fails:

No $x \in E(M)$ is freely in $\Delta(\cap \mathcal{F}_3)$.
An example where the strategy fails when the equality fails:

No \( x \in E(M) \) is freely in \( \Delta(\cap \mathcal{F}_3) \).

A hint of how the assumed equality is used:

We can extend \( M \) to an FTM \( M' \) of the same rank.

\( X \mapsto \text{cl}_{M'}(X) \) embeds \( \mathcal{Z}(M) \) in \( \mathcal{Z}(M') \) and preserves rank, so \( M' \) yields a set of equalities that essentially includes those for \( M \).

We exploit this to deduce information about \( M \) from the FTM \( M' \).

E.g., we compare \( M \) and \( M' \) to show that some \( x \in E(M) \) with \( v_i \in \Delta(x) \) is freely in \( \Delta(\cap \mathcal{F}_i) \).
The Importance of Starting with the Maximal Presentation
The Dual Result

The class of fundamental transversal matroids is dual-closed (Las Vergnas, 1970), so the next result follows from duality.

**Theorem**

A matroid $M$ is an FTM iff for all $\mathcal{F} \subseteq \mathcal{Z}(M)$,

$$ r(\cup \mathcal{F}) = \sum_{\mathcal{F}' \subseteq \mathcal{F}, \mathcal{F}' \neq \emptyset} (-1)^{|\mathcal{F}'|+1} r(\cap \mathcal{F}'). $$
Brylawski’s Characterization of FTM is a Corollary

**Theorem** (Brylawski, 1975)

A matroid $M$ is an FTM iff for all families $\mathcal{F}$ of intersections of cyclic flats,

$$r(\cup \mathcal{F}) \geq \sum_{\mathcal{F}' \subseteq \mathcal{F} : \mathcal{F}' \neq \emptyset} (-1)^{|\mathcal{F}'|+1} r(\cap \mathcal{F}')$$

or, equivalently, equality holds in this inequality.

---

A matroid $M$ is an FTM iff for all $\mathcal{F} \subseteq \mathcal{Z}(M)$,

$$r(\cup \mathcal{F}) = \sum_{\mathcal{F}' \subseteq \mathcal{F} : \mathcal{F}' \neq \emptyset} (-1)^{|\mathcal{F}'|+1} r(\cap \mathcal{F}')$$
Brylawski’s Characterization of FTMs is a Corollary

**Theorem** (Brylawski, 1975)

A matroid $M$ is an FTM iff for all families $\mathcal{F}$ of intersections of cyclic flats,

$$r(\cup \mathcal{F}) \geq \sum_{\mathcal{F}' \subseteq \mathcal{F} : \mathcal{F}' \neq \emptyset} (-1)^{|\mathcal{F}'|+1} r(\cap \mathcal{F}') ,$$

or, equivalently, equality holds in this inequality.

**Sketch of the Proof.**

The necessity of the condition follows from inclusion-exclusion.

For the sufficiency, induct on $|\mathcal{F}|$ to show that equality holds in the inequality, so the condition in our characterization of FTMs holds.
To induct, fix $X \in \mathcal{F}$ and let $\mathcal{F}_{\tilde{X}} = \mathcal{F} - \{X\}$.

The sets $\mathcal{F}'$ in the sum on the right side are of three types: (i) $\mathcal{F}' = \{X\}$, (ii) $\mathcal{F}' \subseteq \mathcal{F}_{\tilde{X}}$, and (iii) $\{X\} \subset \mathcal{F}'$.

With the inductive hypothesis, we can evaluate the contribution of each type, giving

$$r(\bigcup \mathcal{F}) \geq r(X) + r(\bigcup \mathcal{F}_{\tilde{X}}) - r(X \cap (\bigcup \mathcal{F}_{\tilde{X}})),$$

so semimodularity finishes the argument. □
Another Characterization of FTMs

The following is a counterpart of a characterization of transversal matroids by Mason (refined by Ingleton).

Let $2^{[r]}$ be the lattice of subsets of $[r] = \{1, 2, \ldots, r\}$.

**Theorem**

A matroid $M$ of rank $r$ is an FTM iff there is an injection $\phi : \mathcal{Z}(M) \to 2^{[r]}$ with

1. $|\phi(F)| = r(F)$ for all $F \in \mathcal{Z}(M)$,
2. $\phi(\text{cl}(F \cup G)) = \phi(F) \cup \phi(G)$ for all $F, G \in \mathcal{Z}(M)$, and
3. $r(\bigcap \mathcal{F}) = |\bigcap \{\phi(F) : F \in \mathcal{F}\}|$ for every $\mathcal{F} \subseteq \mathcal{Z}(M)$. 

A matroid $M$ of rank $r$ is an FTM iff there is an injection

$\phi : \mathcal{Z}(M) \rightarrow 2^{[r]}$ with

1. $|\phi(F)| = r(F)$ for all $F \in \mathcal{Z}(M)$,
2. $\phi(\text{cl}(F \cup G)) = \phi(F) \cup \phi(G)$ for all $F, G \in \mathcal{Z}(M)$, and
3. $r(\cap F) = |\cap \{\phi(F) : F \in \mathcal{F}\}|$ for all $\mathcal{F} \subseteq \mathcal{Z}(M)$.

If $\mathcal{A} = (A_1, A_2, \ldots, A_r)$ is a presentation of $M$, then a natural choice of $\phi$ is $\phi(X) = \{i : X \cap A_i \neq \emptyset\}$, which, up to permuting $[r]$, does not depend on $\mathcal{A}$.

**Corollary**

*If $\mathcal{A}$ is any presentation of a transversal matroid $M$, then $M$ is an FTM iff* $r(\cap \mathcal{F}) = |\cap \{\phi(F) : F \in \mathcal{F}\}|$ for all $\mathcal{F} \subseteq \mathcal{Z}(M)$. 
For any polytope $P$ other than a simplex, is there a way to characterize (perhaps using a set of inequalities) when a matroid $M$ has an affine representation by points in $P$ in which each cyclic flat $F$ of $M$ spans a face of $P$ of dimension $r(F) - 1$?