Characterizations of Fundamental Transversal Matroids

Joseph E. Bonin

The George Washington University

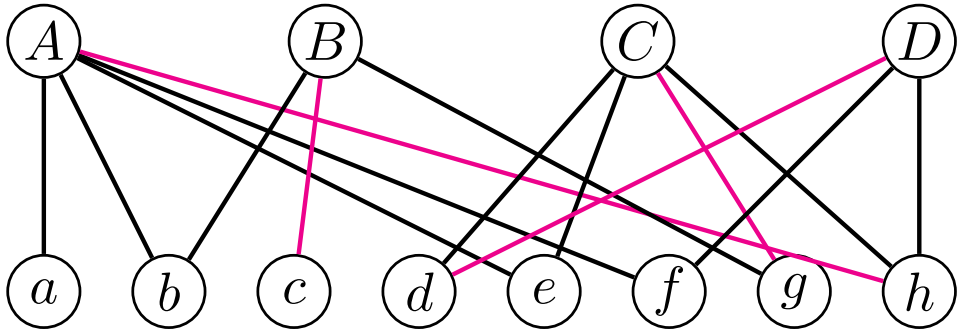
Joint work with
Anna de Mier (Universitat Politècnica de Catalunya)
and
Joseph Kung (University of North Texas)

These slides and the corresponding paper are available at
http://home.gwu.edu/∼jbonin/

A set system is a multiset $\mathcal{A} = (A_j : j \in J)$ of subsets of a set S.

A partial transversal of $\mathcal{A}$ is a subset I of S for which there is an injection $\phi : I \to J$ with $x \in A_{\phi(x)}$ for all $x \in I$.

E.g., $\mathcal{A} = (A, B, C, D)$,
$A = \{a, b, e, f, h\}$,
$B = \{b, c, g\}$,
$C = \{d, e, g, h\}$,
$D = \{d, f, h\}$



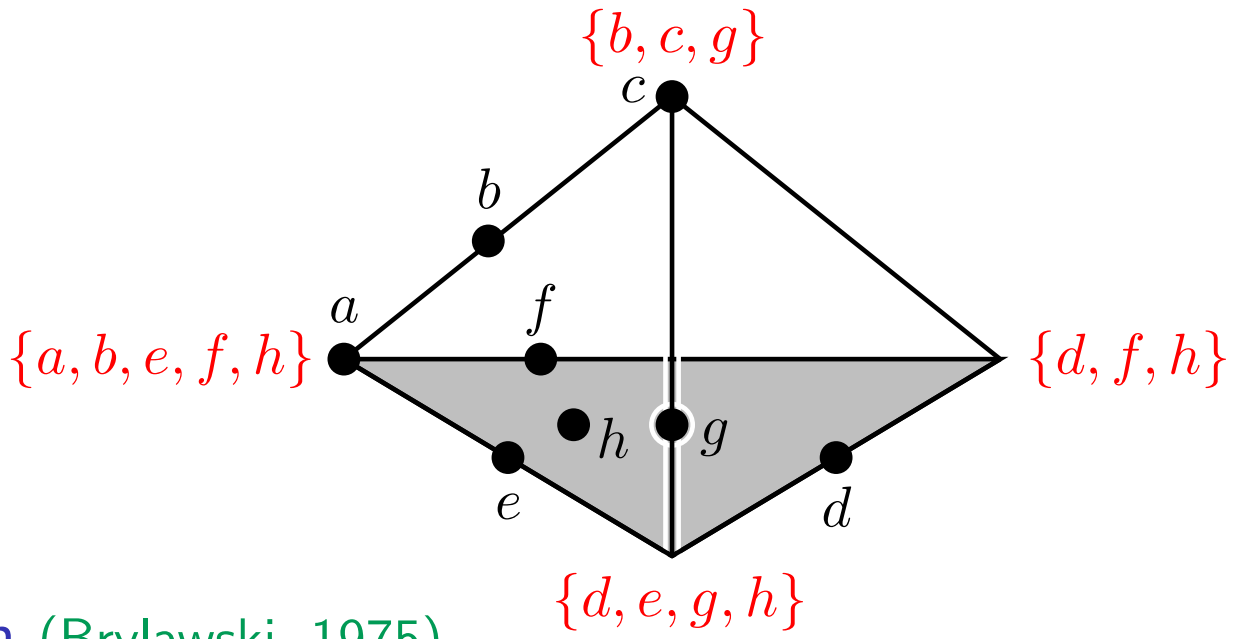
Theorem (Edmonds and Fulkerson, 1965)

The partial transversals of a set system $\mathcal{A}$ are the independent sets of a matroid on S.

Such matroids are transversal matroids; $\mathcal{A}$ is a presentation.

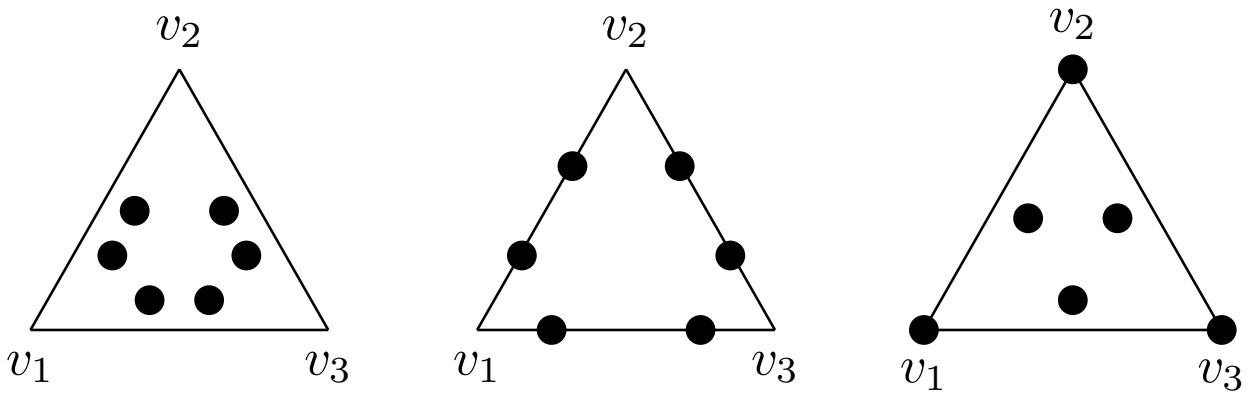
A flat F of M is cyclic if $M|F$ has no coloops;
i.e., F is a union of circuits.

Let $\mathcal{Z}(M)$ be the set of cyclic flats of M.



Theorem (Brylawski, 1975)

A matroid M is transversal iff it has an affine realization on a simplex Δ in which each $F \in \mathcal{Z}(M)$ is the set of points in some face of Δ with $r(F)$ vertices.



Three representations of $U_{3,6}$ on a 3-vertex simplex.

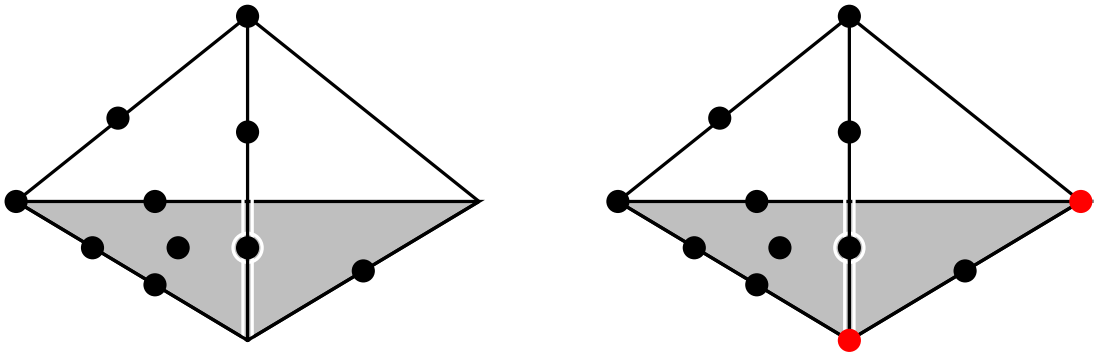
A presentation $(A_1, \ldots, A_r)$ of M is maximal if, whenever $(A'_1, \ldots, A'_r)$ is a presentation of M with $A_i \subseteq A'_i$ for $i \in [r]$, then $A_i = A'_i$ for $i \in [r]$.

A transversal matroid M has a unique maximal presentation with $r(M)$ sets. (Mason, 1971.)

A fundamental transversal matroid, **FTM**, is a transversal matroid M that has an affine representation on a simplex in which each vertex has an element of M placed at it.

A matroid M is an FTM iff it has a basis B (a fundamental basis) with $B \cap F$ spanning F for each $F \in \mathcal{Z}(M)$.

Each transversal matroid extends to an FTM of the same rank.



Theorem (Mason, 1971; refined by Ingleton, 1977)
A matroid M is transversal iff for all nonempty $\mathcal{F} \subseteq \mathcal{Z}(M)$,

$$r(\cap\mathcal{F}) \leq \sum_{\mathcal{F}' \subseteq \mathcal{F}} (-1)^{|\mathcal{F}'|+1} r(\cup\mathcal{F}'),$$

where $\cap\mathcal{F} = \bigcap_{X \in \mathcal{F}} X$ and $\cup\mathcal{F} = \bigcup_{X \in \mathcal{F}} X$.

Example: if $\mathcal{F}$ consists of three coplanar cyclic lines, then the right side is $3 \cdot 2 - 3 \cdot 3 + 3 = 0$, so for M to be transversal, $r(\cap\mathcal{F})$ must be 0.

A Sketch of the Proof of the Necessity

A matroid M is transversal iff for all nonempty $\mathcal{F} \subseteq \mathcal{Z}(M)$,

$$r(\cap\mathcal{F}) \leq \sum_{\mathcal{F}' \subseteq \mathcal{F}} (-1)^{|\mathcal{F}'|+1} r(\cup\mathcal{F}').$$

Extend M to an FTM M' with a fundamental basis B'.

Show that for $\mathcal{F} \subseteq \mathcal{Z}(M')$,

$$r'(\cup\mathcal{F}) = \left|B' \cap (\cup\mathcal{F})\right| \quad \text{and} \quad r'(\cap\mathcal{F}) = \left|B' \cap (\cap\mathcal{F})\right|.$$

For M', inclusion-exclusion gives the inequality — indeed, equality.

The inequality for M follows.

(For a proof that the inequality implies that M is transversal, see the paper.)

Theorem

A matroid M is an FTM iff for all nonempty $\mathcal{F} \subseteq \mathcal{Z}(M)$,

$$r(\cap\mathcal{F}) = \sum_{\mathcal{F}' \subseteq \mathcal{F}} (-1)^{|\mathcal{F}'|+1} r(\cup\mathcal{F}').$$

The inclusion-exclusion argument in the last proof shows that the equality holds for all FTMs.

Strategy of the Proof of Sufficiency

By the Mason/Ingleton result, M is transversal.

Start with the simplex realization of its maximal presentation.

Define the support of $x \in E(M)$: Let $\Delta(x)$ be the set of vertices of the simplex that span the face that x is freely placed in.

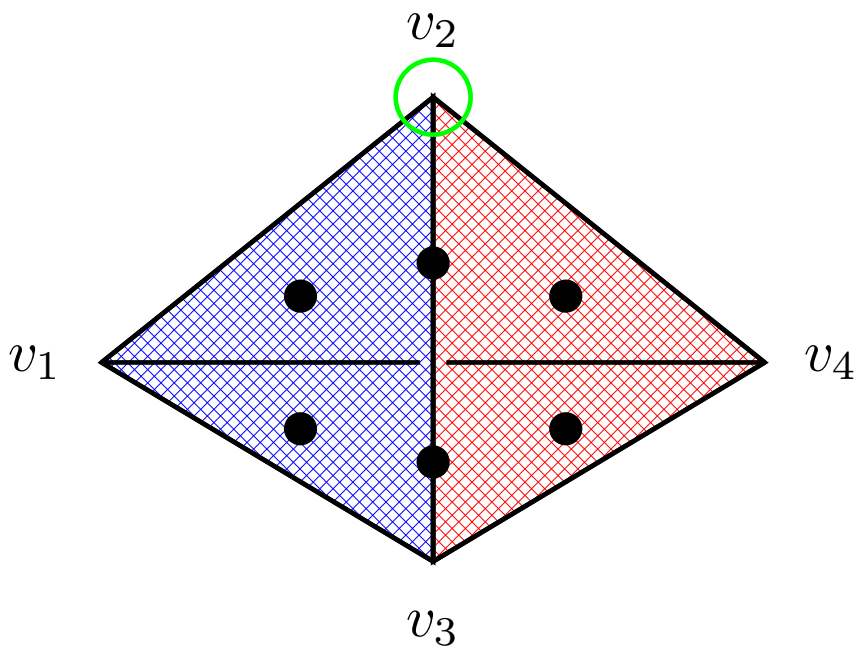
Extend to sets $Y \subseteq E(M)$ via unions: $\Delta(Y) = \bigcup_{x \in Y} \Delta(x)$.

For each vertex v_i at which no element of M has been placed, let $\mathcal{F}_i = \{F \in \mathcal{Z}(M) \,:\, v_i \in \Delta(F)\}$.

Show

(a) some $x \in E(M)$ with $v_i \in \Delta(x)$ is freely in $\Delta(\cap \mathcal{F}_i)$ and

(b) moving x to v_i preserves the circuits and so also realizes M.



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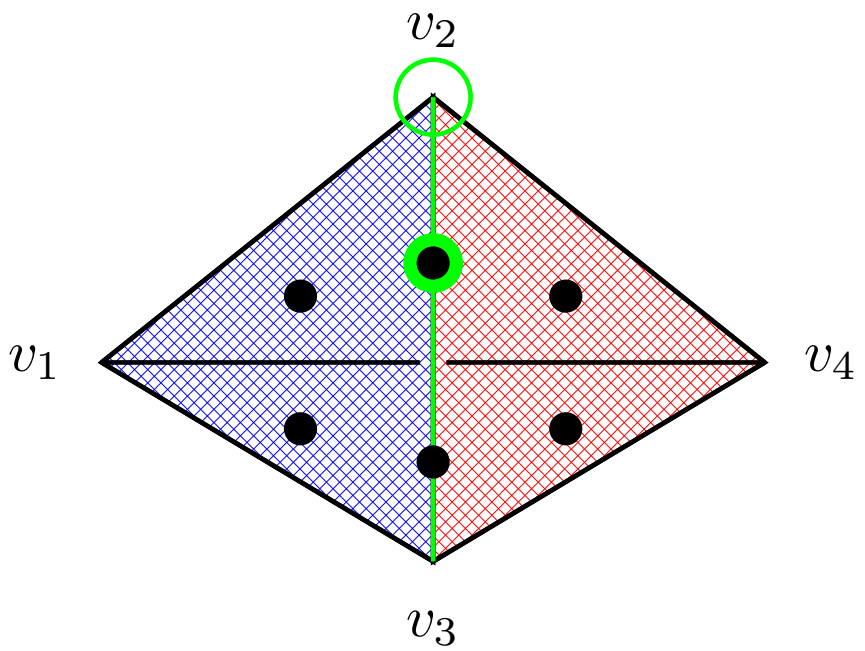

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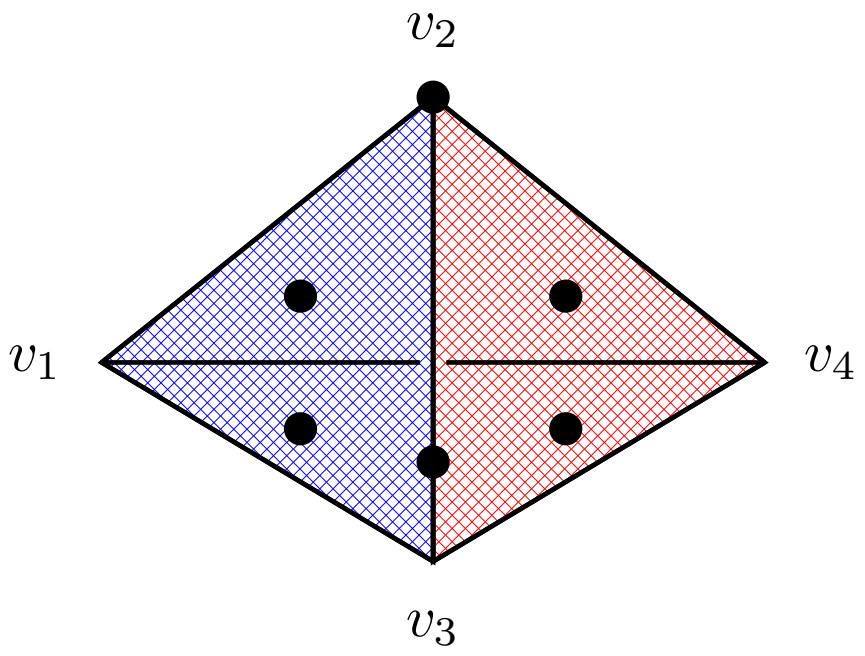

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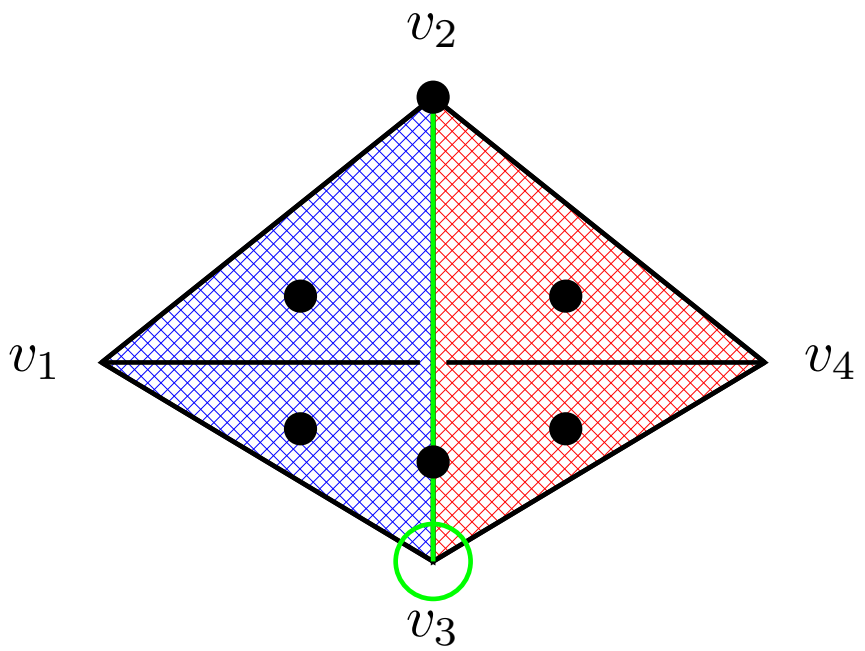

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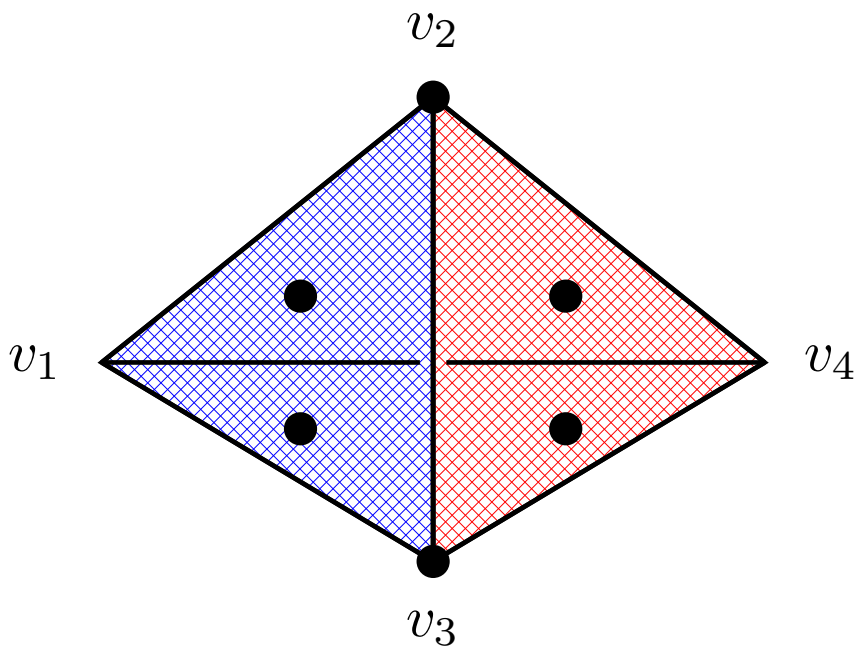

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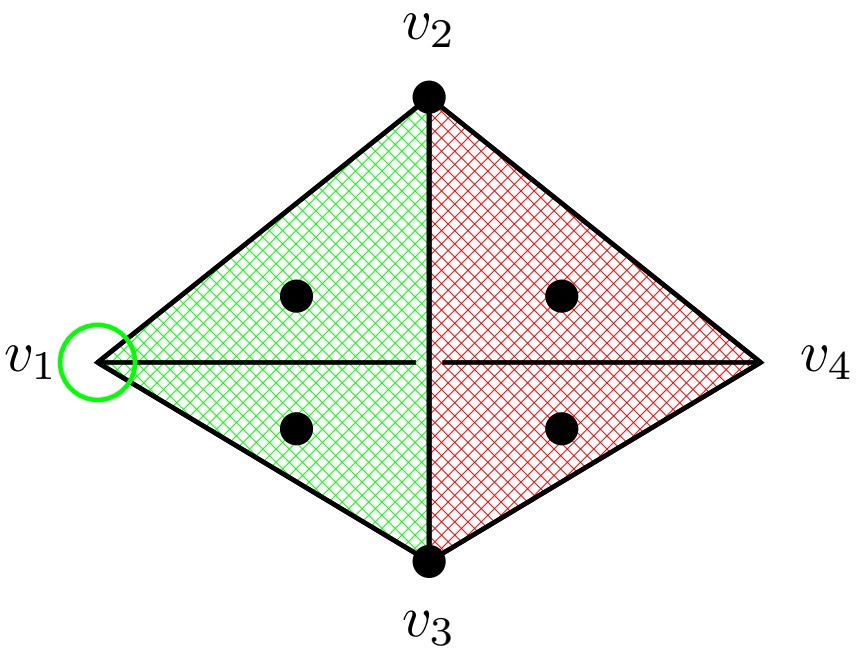

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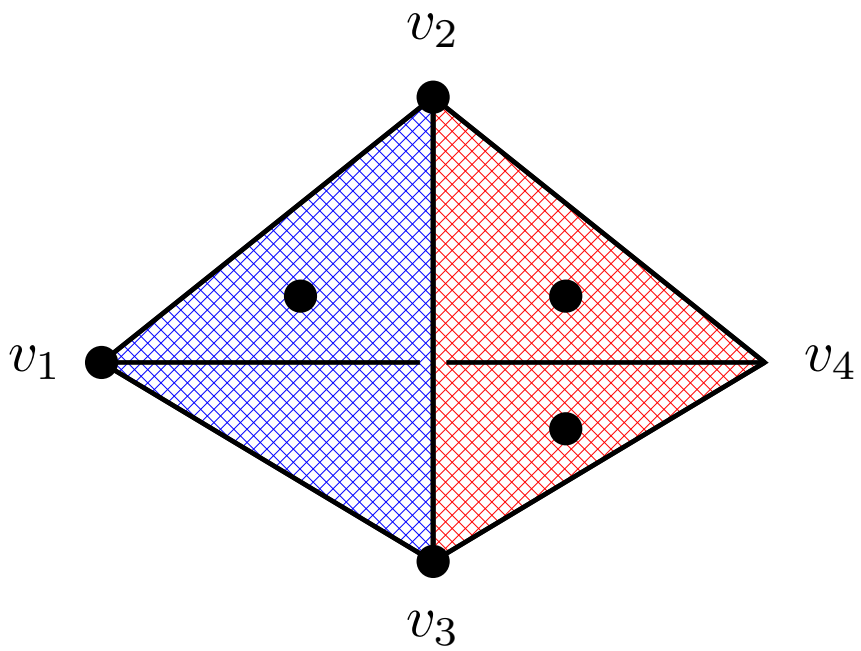

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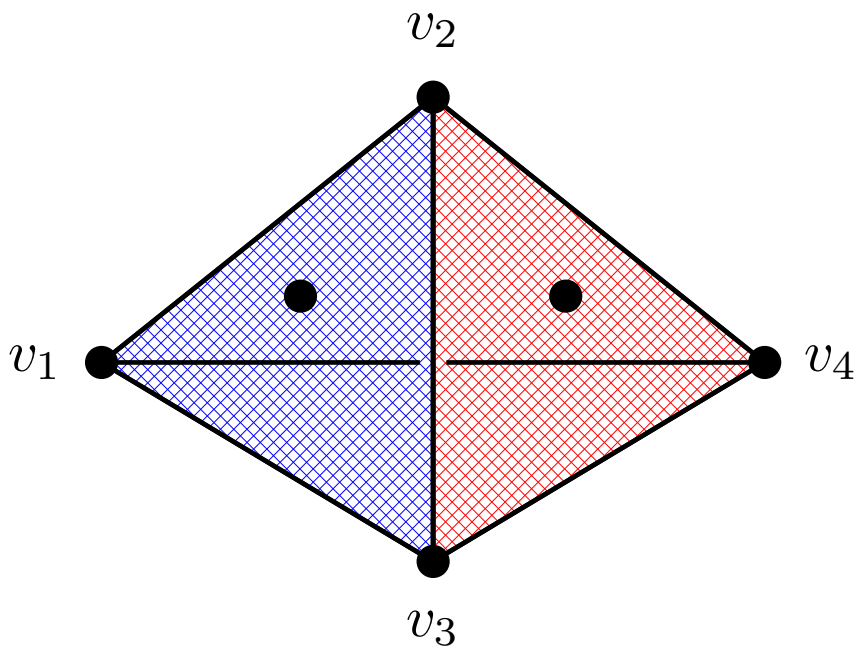
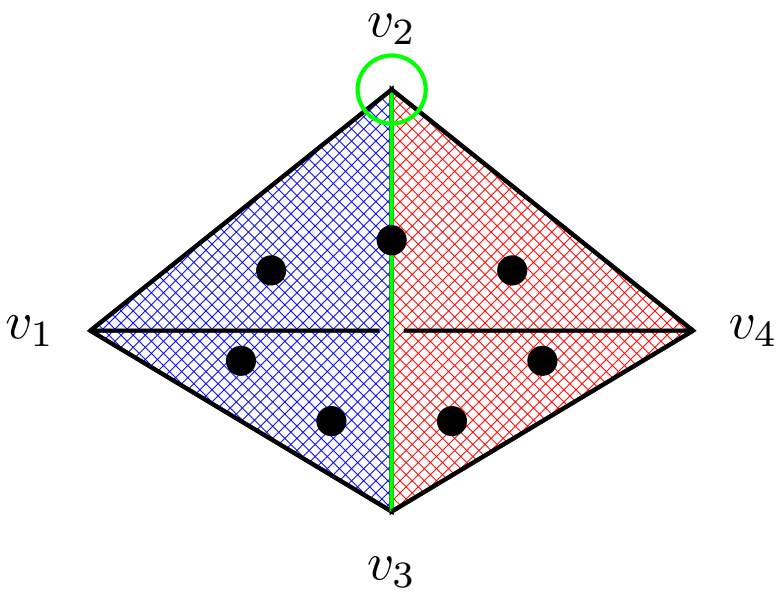


An example where the strategy fails when the equality fails:



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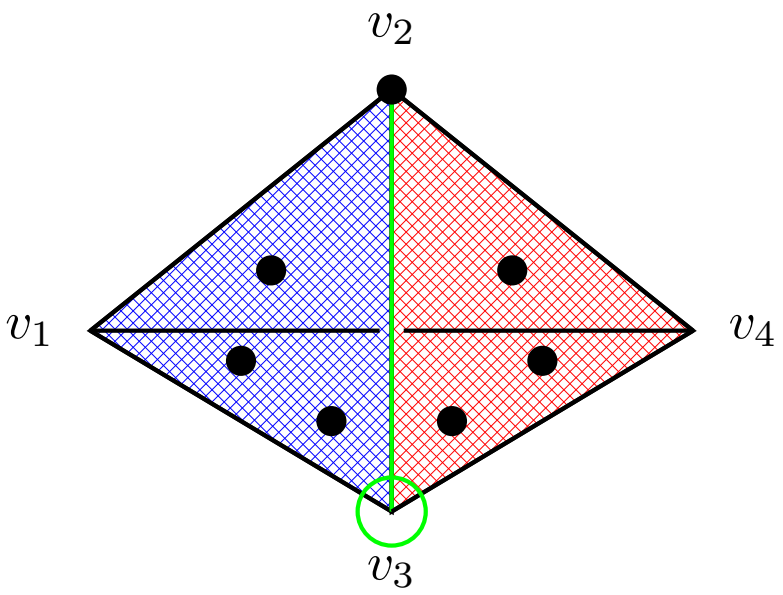

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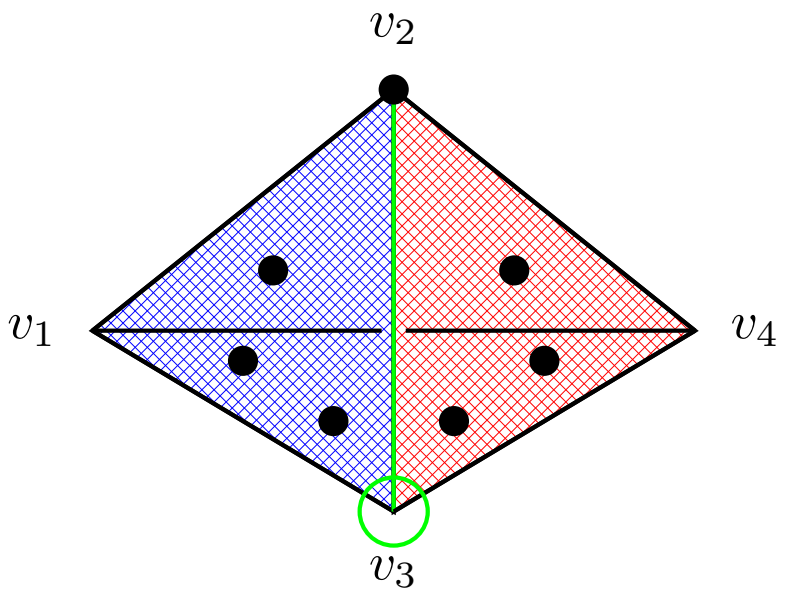

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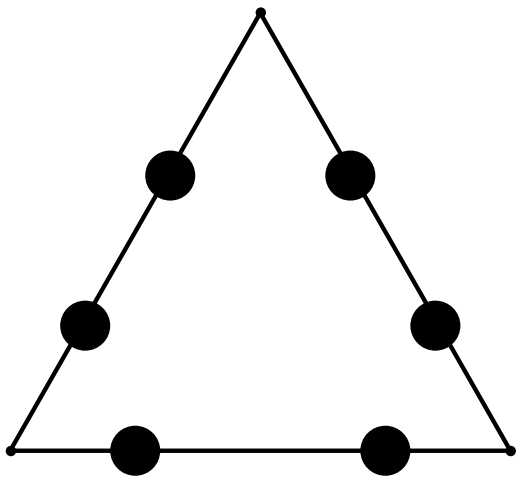
A hint of how the assumed equality is used:

We can extend M to an FTM M' of the same rank.

$X \mapsto \mathrm{cl}_{M'}(\mathrm{X})$ embeds $\mathcal{Z}(M)$ in $\mathcal{Z}(M')$ and preserves rank, so M' yields a set of equalities that essentially includes those for M.

We exploit this to deduce information about M from the FTM M'.

E.g., we compare M and M' to show that some $x \in E(M)$ with $v_i \in \Delta(x)$ is freely in $\Delta(\cap\mathcal{F}_i)$



The class of fundamental transversal matroids is dual-closed (Las Vergnas, 1970), so the next result follows from duality.

Theorem

A matroid M is an FTM iff for all $\mathcal{F} \subseteq \mathcal{Z}(M)$,

$$r(\cup\mathcal{F}) = \sum_{\mathcal{F}' \subseteq \mathcal{F}, \mathcal{F}' \neq \emptyset} (-1)^{|\mathcal{F}'|+1} r(\cap\mathcal{F}').$$

Theorem (Brylawski, 1975)

A matroid M is an FTM iff for all families $\mathcal{F}$ of intersections of cyclic flats,

$$r(\cup\mathcal{F}) \geq \sum_{\mathcal{F}' \subseteq \mathcal{F} : \mathcal{F}' \neq \emptyset} (-1)^{|\mathcal{F}'|+1} r(\cap\mathcal{F}'),$$

or, equivalently, equality holds in this inequality.

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Sketch of the Proof.

The necessity of the condition follows from inclusion-exclusion.

For the sufficiency, induct on $|\mathcal{F}|$ to show that equality holds in the inequality, so the condition in our characterization of FTMs holds.

Completion of the Sketch of the Proof

To induct, fix $X \in \mathcal{F}$ and let $\mathcal{F}_{\widehat{X}} = \mathcal{F} - \{X\}$.

The sets $\mathcal{F}'$ in the sum on the right side are of three types:
(i) $\mathcal{F}' = \{X\}$, (ii) $\mathcal{F}' \subseteq \mathcal{F}_{\widehat{X}}$, and (iii) $\{X\} \subset \mathcal{F}'$.

With the inductive hypothesis, we can evaluate the contribution of each type, giving

$$r(\cup\mathcal{F}) \geq r(X) + r(\cup\mathcal{F}_{\widehat{X}}) - r(X \cap (\cup\mathcal{F}_{\widehat{X}})),$$

so semimodularity finishes the argument. $\square$

The following is a counterpart of a characterization of transversal matroids by Mason (refined by Ingleton).

Let $2^{[r]}$ be the lattice of subsets of $[r] = \{1, 2, \ldots, r\}$.

Theorem

A matroid M of rank r is an FTM iff there is an injection $\phi : \mathcal{Z}(M) \to 2^{[r]}$ with

1. $|\phi(F)| = r(F)$ *for all* $F \in \mathcal{Z}(M)$,
2. $\phi(\mathrm{cl}(F \cup G)) = \phi(F) \cup \phi(G)$ *for all* $F, G \in \mathcal{Z}(M)$, *and*
3. $r(\cap \mathcal{F}) = |\cap \{\phi(F) : F \in \mathcal{F}\}|$ *for every* $\mathcal{F} \subseteq \mathcal{Z}(M)$.

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3. $r(\cap\mathcal{F}) = |\cap\{\phi(F) : F \in \mathcal{F}\}|$ for all $\mathcal{F} \subseteq \mathcal{Z}(M)$.

If $\mathcal{A} = (A_1, A_2, \ldots, A_r)$ is a presentation of M, then a natural choice of ϕ is $\phi(X) = \{i : X \cap A_i \neq \emptyset\}$, which, up to permuting $[r]$, does not depend on $\mathcal{A}$.

Corollary

If $\mathcal{A}$ is any presentation of a transversal matroid M, then M is an FTM iff $r(\cap\mathcal{F}) = |\cap\{\phi(F) : F \in \mathcal{F}\}|$ for all $\mathcal{F} \subseteq \mathcal{Z}(M)$.

Open Problem

For any polytope P other than a simplex, is there a way to characterize (perhaps using a set of inequalities) when a matroid M has an affine representation by points in P in which each cyclic flat F of M spans a face of P of dimension $r(F) - 1$?