EXTENDING A MATROID BY A COCIRCUIT

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Abstract. Our main result describes how to extend a matroid so that its ground set is a modular hyperplane of the larger matroid. This result yields a new way to view Dowling lattices and new results about line-closed geometries. We complement these topics by showing that line-closure gives simple geometric proofs of the (mostly known) basic results about Dowling lattices. We pursue the topic of line-closure further by showing how to construct some line-closed geometries that are not supersolvable.

1. Introduction

Modular flats of matroids (that is, the flats X such that the modular equality $r(X)+r(Y)=r(X\cup Y)+r(X\cap Y)$ holds for every flat Y) have considerable structural significance. For instance, while relatively few results have been obtained in the general theory of amalgams of matroids, the special case of generalized parallel connections [5, 11], in which matroids are glued together along a modular flat, is well developed; in turn, for example, this operation leads to formulas that simplify the computation of Tutte polynomials [4]. A related property that is also relevant for this paper is supersolvability, that is, having a maximal chain of flats, all of which are modular. The characteristic polynomial of a supersolvable matroid has a linear factorization over the integers [12, 13] that reflects a factorization of the broken circuit complex [2].

Brylawski [5, Corollary 3.4] gave the following elegant characterization of the hyperplanes of a geometry (simple matroid) that are modular. (Results about modular flats of geometries have counterparts for general matroids. We focus on geometries since the statements of the results are slightly simpler in this context.)

Theorem 1.1. *A hyperplane H of a geometry M is modular if and only if H has nonempty intersection with every line of M.*

Our main result, Theorem 2.1, is a simple description of how to extend a given geometry on a set S to a geometry in which S is a modular hyperplane. Implications of this result in the case of line-closed geometries are presented in Section 3, which also treats several complementary results about line-closure. In Section 4, we apply Theorem 2.1 to the construction of Dowling lattices; we also prove some basic properties of Dowling lattices to demonstrate the insights that this perspective and line-closure offer. The last section develops and applies a technique to construct line-closed geometries that are not supersolvable.

We assume knowledge of matroid theory. Specialized topics are reviewed as they arise. We consider all matroids of finite rank, including infinite matroids. We recall the axioms for flats so that we may refer to them in the proof of the main result. Recall first that for X and Y in a collection $\mathcal{F}$ of sets, we say that Y *covers* X if

$X \subsetneq Y$ and there is no set Z in $\mathcal{F}$ with $X \subsetneq Z \subsetneq Y$. A matroid M is a set S and a collection $\mathcal{F}$ of subsets of S such that

(F1) S is in $\mathcal{F}$,
(F2) if X and Y are in $\mathcal{F}$, then so is $X \cap Y$,
(F3) for all $X \in \mathcal{F}$ and $a \notin X$, exactly one cover of X contains a, and
(F4) every chain in $\mathcal{F}$ is finite.

We use the terms points, lines, planes, and hyperplanes for the flats of ranks 1, 2, 3, and $n-1$ in a matroid of rank n. We let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

2. Extending a Matroid by a Cocircuit

To motivate Theorem 2.1 and see that its hypotheses are necessary, let S be a modular hyperplane of a geometry N on T. Note that N is determined by the restriction $N|S$ and the set $\mathcal{L}$ of lines of N that are not contained in S: since S is modular, for any flat Y of N of rank at least two, either Y is a flat of $N|S$ or, for the flat $Y \cap S$ of $N|S$ and any point a in $Y - S$, we have

$$Y = \bigcup_{y \in Y \cap S} \mathrm{cl}(\{a, y\}).$$

Note that each pair of elements of T, not both in S, is in exactly one line in $\mathcal{L}$. Since S is modular, each line of $\mathcal{L}$ intersects S in a point and each plane of N that is not contained in S intersects S in a line.

Theorem 2.1. *Let M be a geometry on S, let T be a proper superset of S, and let $\mathcal{L}$ be a set of nonsingleton subsets of T such that the following properties hold.*

(E1) *Each pair x, y of points of T, not both in S, is in a unique set $\ell(x, y)$ of $\mathcal{L}$.*
(E2) *Each set in $\mathcal{L}$ contains exactly one element in S.*
(E3) *For $\ell \in \mathcal{L}$ and $a \notin S \cup \ell$, the points $\ell(a, y) \cap S$, with y in ℓ, are collinear.*

For each flat X of M and each element a of $T - S$, let X^a be given by

$$X^a = \begin{cases} \{a\} & \text{if } X = \emptyset; \\ \bigcup_{x \in X} \ell(x, a) & \text{otherwise.} \end{cases}$$

Then the flats of M together with the sets X^a are the flats of a geometry N on T. The restriction $N|S$ is M and S is a modular hyperplane of N.

Proof. By condition (E2), $T = S^a$ for any $a \notin S$, so property (F1) holds.

The following statement (an exchange property) yields the only nontrivial case of property (F2), namely, intersections $X^a \cap Y^b$ that are not contained in S.

(EX) *If b is in $X^a - X$, then $X^a = X^b$.*

To prove assertion (EX), we prove $X^a \subseteq X^b$; the other containment then follows since a is in $X^b - X$. Since $X^a \cap S$ is X, we need only consider elements c in $X^a - S$. If a, b, and c are collinear, then clearly c is in X^b, so assume this is not the case. Let $\ell(a, b)$, $\ell(a, c)$, and $\ell(b, c)$ meet S in x, y, and z, respectively, which, by property (E3) are collinear. By the definition of X^a, both x and y are in X. Therefore z is in X, so c is in X^b, as desired.

Assertion (EX) implies property (F3) when X is a flat of M and $a \notin S$. In the only other nontrivial case of property (F3), assertion (EX) implies that for $b \notin X^a$, the unique cover of X^a that contains b is $\big(\mathrm{cl}_M(X \cup c)\big)^a$ where c is $S \cap \ell(a, b)$.

Property (F4) holds since any chain among the sets X and X^a contains at most one more set than a chain of flats of M. That $N|S$ is M is transparent. The last assertion of the theorem follows from Theorem 1.1 and condition (E2). □

3. An Application to Line-Closed Geometries

This section treats the implications of Theorem 2.1 for line-closed geometries [9] and then presents results about line-closure that are used in later sections. A set Z in a geometry is *line-closed* if for every pair of points x and y in Z, the line $\mathrm{cl}(\{x, y\})$ is contained in Z. Flats are line-closed, but many geometries (e.g., the uniform matroid $U_{3,4}$) have line-closed sets that are not flats. A geometry is *line-closed* if the flats are precisely the line-closed sets.

Theorem 3.1. *Let S be a modular hyperplane of a geometry N. If $N|S$ is line-closed, then so is N.*

Proof. We must show that any line-closed set Z of N is a flat of N, that is, Z has one of the two forms discussed in and before Theorem 2.1. If $Z \subseteq S$, then Z, as a line-closed set of $N|S$, is a flat of $N|S$. For $Z \not\subseteq S$, let X be $Z \cap S$ and fix $a \in Z - X$. That Z is line-closed and S is a flat of N imply that X is line-closed in $N|S$, so X is a flat of $N|S$. Since Z is line-closed, $X^a \subseteq Z$. For $c \in Z - X$ with $c \neq a$, let $\mathrm{cl}(\{a, c\})$ intersect the modular hyperplane S in the point d. Since Z is line-closed, d is in Z. Therefore d is in X, so c is in X^a. Thus, Z is X^a, as needed. □

An immediate corollary of Theorem 3.1 is the following result of [9].

Corollary 3.2. *Supersolvable geometries are line-closed.*

The next theorem rests on the following lemma, which comes from the theory of matroid quotients [11, Section 7.3]. We offer a direct, elementary argument.

Lemma 3.3. *Assume M and N are matroids of the same rank and on the same ground set. If every flat of N is a flat of M, then $M = N$.*

Proof. We need to show that all flats of M are flats of N. Since $r(M) = r(N)$, no chain of flats of M has more flats than a maximal chain of flats of N. Therefore the hypotheses have two consequences: (i) $\mathrm{cl}_N(\emptyset) = \mathrm{cl}_M(\emptyset)$ and (ii) if Y covers Z as flats of N, then Y covers Z as flats of M. Let X be a minimal flat of M that we do not yet know to be a flat of N. Since X is not $\mathrm{cl}_M(\emptyset)$, it covers some flat X' of M, which is a flat of N. For $a \in X - X'$, since $\mathrm{cl}_N(X' \cup a)$ is a flat of M that contains X' and a, we have $X' \subsetneq X = \mathrm{cl}_M(X' \cup a) \subseteq \mathrm{cl}_N(X' \cup a)$. Since $\mathrm{cl}_N(X' \cup a)$ covers X' as flats of N, conclusion (ii) shows that X is the flat $\mathrm{cl}_N(X' \cup a)$ of N. □

A line-closed geometry is determined by its points and lines since the flats are defined by the lines. The next theorem, which is a stronger statement of this type, is a mild but useful strengthening of [8, Theorem 2.2].

Theorem 3.4. *Assume geometries M and N have the same rank and ground set. If M is line-closed and each of its line is contained in a line of N, then $M = N$.*

Proof. The hypotheses imply that any flat X of N is line-closed in M; thus, since M is line-closed, X is a flat of M. The theorem now follows by Lemma 3.3. □

Theorem 3.4 yields a brief proof of the following theorem from [3].

Theorem 3.5. *Assume the geometry M has a basis $x_1, x_2, \ldots, x_n$ so that*

(i) *each line* $\mathrm{cl}(\{x_i, x_j\})$ *is a* 3*-point line, say* $\{x_i, x_j, a_{ij}\}$,
(ii) *every point is on such a* 3*-point line* $\{x_i, x_j, a_{ij}\}$, *and*
(iii) *for all* i, j, k, *the points* a_{ij}, a_{ik}, a_{jk} *are collinear.*

Then M *is isomorphic to* $M(K_{n+1})$, *the cycle matroid of the complete graph* K_{n+1}.

Proof. Identify the edge $\{i, n+1\}$ of K_{n+1} with x_i and, for $1 \leq i, j \leq n$, the edge $\{i, j\}$ of K_{n+1} with a_{ij}; this identifies the ground set of $M(K_{n+1})$ with that of M so that each line of $M(K_{n+1})$ is contained in a line of M. The conclusion follows from Theorem 3.4 since $M(K_{n+1})$ is supersolvable and so line-closed. □

4. An Application to Dowling Lattices

This section uses Theorems 2.1 and 3.4 to give a self-contained development of Dowling lattices [7] and their basic (mostly known) properties from the perspective of line-closure. We start with the construction. The Dowling lattice $Q_n(G)$ depends on a group G and a positive integer n. The ground set $S_n(G)$ of $Q_n(G)$ consists of

(i) the *joints*, denoted by $p_1, p_2, \ldots, p_n$, and
(ii) the *internal points*, denoted by a_{ij}; here, a ranges over G and the indices i, j are distinct and range over $[n]$; we identify a_{ji} and $(a^{-1})_{ij}$.

The lines of $Q_n(G)$ are of three types:

(i′) there are $\binom{n}{2}$ *coordinate lines* ℓ_{ij}, one for each pair i, j of distinct indices, given by $\ell_{ij} := \{p_i, p_j\} \cup \{a_{ij} : a \in G\}$,
(ii′) the *transversal lines* are the triples of the form $\{a_{ij}, b_{jk}, (ab)_{ik}\}$ as a and b range over G, and i, j, and k are distinct and range over $[n]$, and
(iii′) the *trivial* (two-point) *lines*, namely, the sets of the forms $\{p_i, a_{jk}\}$ where $|\{i, j, k\}| = 3$, and $\{a_{hi}, b_{jk}\}$ where $|\{h, i, j, k\}| = 4$.

We first show, using Theorems 2.1 and Theorem 3.4, that these collections of points and lines uniquely determine a rank-n geometry.

Theorem 4.1. *For any group* G *and positive integer* n, *there is a unique geometry* $Q_n(G)$ *of rank* n *on* $S_n(G)$ *whose lines are given in* (i′)–(iii′) *above. Furthermore,* $Q_n(G)$ *is supersolvable.*

Proof. We induct on n. The case $n = 1$ is trivial. It suffices to show that the lines in (i′)–(iii′) that are contained in $S_n(G)$ but not in $S_{n-1}(G)$ satisfy properties (E1)–(E3) of Theorem 2.1, for then, besides having a well-defined geometry $Q_n(G)$, this geometry is supersolvable and so line-closed, so Theorem 3.4 gives uniqueness. The only part that is not completely obvious is property (E3) in the case of a transversal line, say $\{(ab^{-1})_{ij}, b_{jn}, a_{in}\}$, and an internal point, say c_{kn}, where i, j, k, and n are distinct (Fig. 1). By (ii′), the lines spanned by $\{c_{kn}, a_{in}\}$, $\{c_{kn}, b_{jn}\}$, and $\{c_{kn}, (ab^{-1})_{ij}\}$, meet S in the points $(ac^{-1})_{ik}$, $(bc^{-1})_{jk}$, and $(ab^{-1})_{ij}$, respectively. By (ii′) and the associative law, these points are collinear, as needed. □

The simple computation at the end of this proof is the sole use of the associative law, and this issue arises only for $n > 3$. As observed in [7], rank-3 Dowling lattices can be defined over Latin squares (quasi-groups). An immediate consequence of Theorems 4.1 and 3.5 is the following result from [7]: the rank-n Dowling lattice based on the trivial group is isomorphic to $M(K_{n+1})$.

Theorem 2.1 gives the flats of the extension N of M. This theorem and the inductive construction of Dowling lattices yield the following result.

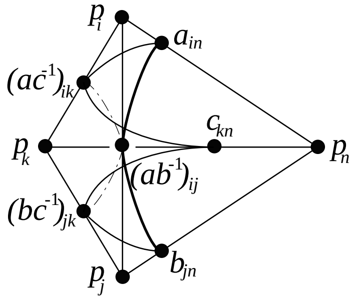

Figure 1. The sole nontrivial case when checking property (E3) of Theorem 2.1 for Dowling lattices.

Theorem 4.2. *The flats of $Q_n(G)$ are the sets X for which there are disjoint subsets B_0 (possibly empty) and $B_1, B_2, \dots, B_t$ of $[n]$, with $|B_h| \geq 2$ for $1 \leq h \leq t$, such that the elements of X are as follows:*

(i) *for $1 \leq i \leq n$, the joint p_i is in X if and only if i is in B_0,*
(ii) *for all i, j in B_0 and all a in G, the internal point a_{ij} is in X, and*
(iii) *for h with $1 \leq h \leq t$ and each pair i, j in B_h, there is exactly one element a in G with a_{ij} in X, and these points satisfy the condition:*
for h with $1 \leq h \leq t$ and each triple i, j, k in B_h, the points a_{ij}, b_{jk}, c_{ik} of X are collinear.

Note that for $X_0 = \{p_i : i \in B_0\} \cup \{a_{ij} : a \in G \text{ and } i, j \in B_0\}$, the restriction $Q_n(G)|X_0$ is isomorphic to $Q_{|B_0|}(G)$. By Theorem 3.5, a restriction $Q_n(G)|X_h$, where X_h is $\{a_{ij} : a_{ij} \in X \text{ and } i, j \in B_h\}$, is isomorphic to the cycle matroid $M(K_{|B_h|})$. Thus, the restriction of $Q_n(G)$ to any flat is isomorphic to a direct sum of a Dowling lattice, $Q_m(G)$, and cycle matroids of complete graphs.

We now show that our geometric view of the construction of Dowling lattices does not complicate other parts of the basic theory. In what follows, we offer geometric insights into, and mild extensions of, some results from [7].

Let J be the set of joints of $Q_n(G)$. Each point of $Q_n(G)$ is in the complement of one or two hyperplanes of the form $\mathrm{cl}(J - p_i)$. By the symmetry of the construction, each hyperplane $\mathrm{cl}(J - p_i)$ is modular and each restriction $Q_n(G)|\mathrm{cl}(J - p_i)$ is isomorphic to $Q_{n-1}(G)$. Using Theorem 1.1, it follows that the simplification of any single-element contraction of $Q_n(G)$ is isomorphic to $Q_{n-1}(G)$. Since arbitrary contractions are iterated single-element contractions, we get the following result.

Theorem 4.3. *The simplification of any rank-r contraction of $Q_n(G)$ is isomorphic to $Q_r(G)$.*

The next result, which completes [7, Theorem 4] by providing a converse, identifies the modular flats of Dowling lattices. The modular flats of $M(K_n)$ are well known (see, e.g., [11, Corollary 6.9.11]), so we focus on nontrivial groups.

Theorem 4.4. *Assume G is not trivial. The modular flats of $Q_n(G)$ are the points and the flats $\mathrm{cl}(A)$ for subsets A of $J = \{p_1, p_2, \dots, p_n\}$.*

Proof. Let X be a modular flat of rank $m > 1$ in $Q_n(G)$ and let B be a basis of $Q_n(G)$ for which $B \cap X$ is a basis of X. Since X is modular, all elements in the contraction $Q_n(G)/(B-X)$ are in or parallel to elements of X, so by Theorem 4.3, it follows that $Q_n(G)|X$ is isomorphic to $Q_m(G)$. Thus, by Theorem 4.2, X is $\mathrm{cl}(A)$ for some subset A of J. The remaining implications are consequences of the following results: points are modular in any matroid; the hyperplanes $\mathrm{cl}(J-p_i)$ are modular; and intersections of modular flats are modular [11, Corollary 6.9.8]. $\square$

Theorem 4.5 is a mild extension of results in [7]; the proof follows those in [7].

Theorem 4.5. *For $n \geq 3$, the Dowling lattice $Q_n(G)$ is isomorphic to a restriction of $Q_n(G')$ if and only if G is isomorphic to a subgroup of G'.*

Proof. If G is isomorphic to a subgroup G^* of G', then the restriction of $Q_n(G')$ to the set $S = \{p_1, p_2, \dots, p_n\} \cup \{a_{ij} : a \in G^*, 1 \leq i, j \leq n\}$ gives a rank-n geometry whose points and lines can be identified with those of $Q_n(G)$. Thus, by Theorem 4.1, $Q_n(G')|S$ is isomorphic to $Q_n(G)$.

The converse is obvious if G is the trivial group, so assume G is not trivial and Φ is an isomorphism of $Q_n(G)$ onto a restriction of $Q_n(G')$. Only joints are on more than one line with four or more points, so Φ maps the joints of $Q_n(G)$ onto those of $Q_n(G')$; we may assume $\Phi(p_i) = p_i$ for $i = 1, 2, \dots, n$. Therefore Φ induces three maps $\phi_{12}, \phi_{13}, \phi_{23} : G \to G'$ defined by $\Phi(a_{ij}) = \big(\phi_{ij}(a)\big)_{ij}$. Since we identify a_{ij} and $(a^{-1})_{ji}$, we have the equalities $\big(\phi_{ij}(1)\big)_{ij} = \Phi(1_{ij}) = \Phi(1_{ji}) = \big(\phi_{ji}(1)\big)_{ji}$, so

$$(1) \qquad \big(\phi_{ij}(1)\big)^{-1} = \phi_{ji}(1).$$

For a, b in G, the points $\big(\phi_{12}(a)\big)_{12}$ and $\big(\phi_{23}(b)\big)_{23}$ are collinear with $\big(\phi_{12}(a)\phi_{23}(b)\big)_{13}$ as well as with $\big(\phi_{13}(ab)\big)_{13}$ (since $a_{12}, b_{23}, (ab)_{13}$ are collinear in $Q_n(G)$), so

$$(2) \qquad \phi_{13}(ab) = \phi_{12}(a)\phi_{23}(b).$$

Define $\tau : G \to G'$ by $\tau(a) = \big(\phi_{12}(1)\big)^{-1}\phi_{13}(a)\big(\phi_{23}(1)\big)^{-1}$. Since ϕ_{13} is injective, τ is injective. An easy computation using Eqns. (1) and (2) shows that τ is a homomorphism, which completes the proof. $\square$

Note that in the last proof, if Φ is onto, so is ϕ_{13} and hence τ. Thus we get [7, Theorem 8]: for $n \geq 3$, the matroids $Q_n(G)$ and $Q_n(G')$ are isomorphic if and only if the groups G and G' are isomorphic. The next two results are also from [7].

Theorem 4.6. *Let $x_1, x_2, \dots, x_n$ be a basis of $\mathrm{PG}(n-1, F)$, the rank-n projective geometry over the division ring F, and let S be $\bigcup\{\mathrm{cl}(\{x_i, x_j\}) : 1 \leq i < j \leq n\}$. The restriction $\mathrm{PG}(n-1, F)|S$ is isomorphic to $Q_n(F^*)$.*

Proof. By viewing $\mathrm{PG}(n-1, F)$ as the simplification of the matroid on an n-dimensional (left) vector space over F, we can treat elements of S as vectors, take linear combination such as $x_i + ax_j$, and identify $x_i + ax_j$ and $x_j + a^{-1}x_i$.

The elements of S are $x_1, x_2, \dots, x_n$ and $x_i + ax_j$ for a in F^*. Thus, the map $\phi : \mathrm{PG}(n-1, F)|S \to Q_n(F^*)$ given by $\phi(x_i) = p_i$ and $\phi(x_i + ax_j) = (-a)_{ij}$ is a bijection on the ground sets. By the uniqueness assertion in Theorem 4.1, to show that ϕ is an isomorphism it suffices to show that ϕ identifies the lines of the two geometries. This is easy to see since the lines of $\mathrm{PG}(n-1, F)|S$ are given as follows:

(1) $\{x_i, x_j\} \cup \{x_i + ax_j : a \in F^*\}$ for $\{i, j\} \subseteq [n]$,
(2) $\{x_i + ax_j, x_j + bx_k, x_i - abx_k\}$ for $a, b \in F^*$ and $\{i, j, k\} \subseteq [n]$,
(3) $\{x_i, x_j + ax_k\}$ for $a \in F^*$ and $\{i, j, k\} \subseteq [n]$,
(4) $\{x_h + ax_i, x_j + bx_k\}$ for $a, b \in F^*$ and $\{h, i, j, k\} \subseteq [n]$. $\square$

Corollary 4.7. (i) *For $n \geq 3$, the Dowling lattice $Q_n(G)$ is representable over a division ring F if and only if G is isomorphic to a subgroup of F^*.*
(ii) *Let G be a finite group. For $n \geq 3$, the Dowling lattice $Q_n(G)$ is representable over some field if and only if G is cyclic.*

The final (apparently new) result uses the Sylvester-Gallai theorem: every finite simple rank-3 orientable matroid has a trivial line. (See Proposition 6.1.1 of [1].)

Theorem 4.8. *Let G be a finite group. For $n \geq 3$, the Dowling lattice $Q_n(G)$ is orientable if and only if G has at most two elements.*

Proof. If G has at most two elements, then G is isomorphic to a subgroup of $\mathbb{R}^*$; thus, $Q_n(G)$ is representable over $\mathbb{R}$ and so is orientable. If G has more elements, then $Q_3(G)\backslash\{p_1, p_2, p_3\}$ has no trivial lines and so, by the Sylvester-Gallai theorem, is not orientable; thus, $Q_n(G)$ is not orientable. $\square$

5. Line-Closed Geometries that are not Supersolvable

By Corollary 3.2, supersolvable geometries are line-closed. This section focuses on constructing line-closed geometries that are not supersolvable. The geometries constructed are deletions of other line-closed geometries. We use Lemma 5.1 to show that certain deletions of a line-closed geometry are line-closed.

Lemma 5.1. *Let M be a line-closed geometry on S. For a subset Z of S, the deletion $M\backslash Z$ is line-closed if the following properties hold:*
(a) *no line of M contains exactly one point of $S - Z$, and*
(b) *$M|P$ is line-closed for each plane P of $M\backslash Z$.*

Proof. Let X be line-closed in $M\backslash Z$ and let X^+ be $\bigcup\{\mathrm{cl}_M(\{y,z\}) \,:\, y,z \in X\}$. Since $X^+ - Z$ is X, to show that X is a flat of $M\backslash Z$ it suffices to show that X^+ is line-closed in M. Let a and b be in $X^+ - X$ and let c be in $\mathrm{cl}_M(\{a,b\})$; say $a \in \mathrm{cl}_M(\{a',a''\})$ and $b \in \mathrm{cl}_M(\{b',b''\})$ with $a',a'',b',b'' \in X$. By property (a), $\mathrm{cl}_M(\{a'',b\})$ and $\mathrm{cl}_M(\{a'',c\})$ each contain additional points of $S-Z$, say w and u, respectively. By property (b), if $x \notin Z$ is in the closure of three noncollinear points of X, then x is in X; thus, since $w \in \mathrm{cl}_M(\{a'',b',b''\})$ and $u \in \mathrm{cl}_M(\{a',a'',w\})$, we get $w \in X$, and so $u \in X$, and therefore, as needed, $c \in X^+$. A similar (simpler) argument applies for $a \in X^+ - X$ and $b \in X$, and so completes the proof. $\square$

By Theorem 1.1, a geometry is not supersolvable if, for every hyperplane, some line is disjoint from it. This observation yields the following lemma.

Lemma 5.2. *A restriction of $\mathrm{PG}(n-1,q)$ in which the simplification of every single-element contraction is $\mathrm{PG}(n-2,q)$ and in which every hyperplane has fewer points than $\mathrm{PG}(n-2,q)$ is not supersolvable.*

It follows from Lemma 5.1 that, as Halsey showed in [9], the affine geometry $\mathrm{AG}(n-1,q)$, for $q \neq 2$, is line-closed but not supersolvable. We extend this result.

Theorem 5.3. *Let $q \neq 2$ be a prime power. The geometry $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ is line-closed for $1 \leq k \leq n-1$ and not supersolvable for $2 \leq k \leq n-1$.*

Proof. The first assertion follows by checking conditions (a) and (b) in Lemma 5.1 where Z is a rank-k flat of $\mathrm{PG}(n-1,q)$. Condition (a) holds since each line of $\mathrm{PG}(n-1,q)$ contains either 0, 1, or $q+1$ points of Z. The geometry $M|P$ in condition (b) is $\mathrm{PG}(2,q)$, a single-element deletion of $\mathrm{PG}(2,q)$, or $\mathrm{AG}(2,q)$, all of which are line-closed. The last assertion follows from Lemma 5.2. □

The uniform matroid $U_{3,4}$, i.e., $\mathrm{AG}(2,2)$, is not line-closed. Since any deletion $\mathrm{PG}(n-1,2)\backslash\mathrm{PG}(k-1,2)$ with $k>1$ has $U_{3,4}$ as the restriction to some flat, the assumption $q \neq 2$ in the last result is necessary.

For the next result, recall from the theory of the critical problem [6, 10] that the image of a geometry M that can be embedded in $\mathrm{PG}(n-1,q)$ is disjoint from some hyperplane of $\mathrm{PG}(n-1,q)$ if and only if q is not a root of the characteristic polynomial of M; thus, whether such hyperplanes exist depends only on M, not on the embedding. Such geometries are *affine over* $\mathrm{GF}(q)$.

Theorem 5.4. *Let Z be a set of points in $\mathrm{PG}(n-1,q)$. Assume*
(i) *each line of $\mathrm{PG}(n-1,q)|Z$ has at most $q-1$ points,*
(ii) *each plane of $\mathrm{PG}(n-1,q)\backslash Z$ has a line with $q+1$ points, and*
(iii) *$\mathrm{PG}(n-1,q)|Z$ is not affine over $\mathrm{GF}(q)$.*
Then the deletion $\mathrm{PG}(n-1,q)\backslash Z$ is line-closed and not supersolvable.

Proof. Condition (i) implies condition (a) of Lemma 5.1. Since lines with $q+1$ points are modular in restrictions of $\mathrm{PG}(n-1,q)$, rank-3 restrictions with such lines are supersolvable, so condition (b) of Lemma 5.1 follows from condition (ii) and Corollary 3.2. Thus, $\mathrm{PG}(n-1,q)\backslash Z$ is line-closed. That $\mathrm{PG}(n-1,q)\backslash Z$ is not supersolvable follows from conditions (i) and (iii), along with Lemma 5.2. □

For restrictions of $\mathrm{PG}(n-1,q)$ that are graphic, we get the following result.

Corollary 5.5. *If q exceeds three and $\mathrm{PG}(n-1,q)|Z$ is isomorphic to the cycle matroid of a graph with chromatic number greater than q, then $\mathrm{PG}(n-1,q)\backslash Z$ is line-closed and not supersolvable.*

This corollary applies, in particular, if $\mathrm{PG}(n-1,q)|Z$ is isomorphic to $M(K_{m+1})$, and if $4 \leq q \leq m$. Similarly, restrictions of representable Dowling lattices can be used in place of graphic matroids. We close with one such result. The proof uses the characteristic polynomial of a Dowling lattice, which, by [7, Theorem 5], is

$$\chi\big(Q_n(G);\lambda\big) = \prod_{i=0}^{n-1}\big(\lambda - (1+i|G|)\big). \tag{3}$$

Corollary 5.6. *Let G be a subgroup of $\mathrm{GF}^*(q)$ with $3|G|+1 \leq q$. Fix an integer m with $q \leq (m-1)|G|+1$. If $\mathrm{PG}(n-1,q)|Z$ is isomorphic to $Q_m(G)$, then $\mathrm{PG}(n-1,q)\backslash Z$ is line-closed and not supersolvable.*

Proof. We check conditions (i)–(iii) in Theorem 5.4. The inequality $q-1 \geq |G|+2$ follows since $q \geq 3|G|+1$, so condition (i) holds. The inequalities $3|G|+1 \leq q$ and $q \leq (m-1)|G|+1$ along with Eqn. (3) imply that $Q_3(G)$ is affine over $\mathrm{GF}(q)$ while $Q_m(G)$ is not. In particular, condition (iii) holds. Theorem 4.2 implies that each plane of $\mathrm{PG}(n-1,q)\backslash Z$ is isomorphic to an extension of a geometry $\mathrm{PG}(2,q)\backslash W$ where $\mathrm{PG}(2,q)|W$ is isomorphic to $Q_3(G)$; since $Q_3(G)$ is affine over $\mathrm{GF}(q)$, each such plane has a line with $q+1$ points, so condition (ii) holds. □

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