Cyclic Flats of Matroids and their connections to Tutte Polynomials and Other Matroid Invariants

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These slides are available at

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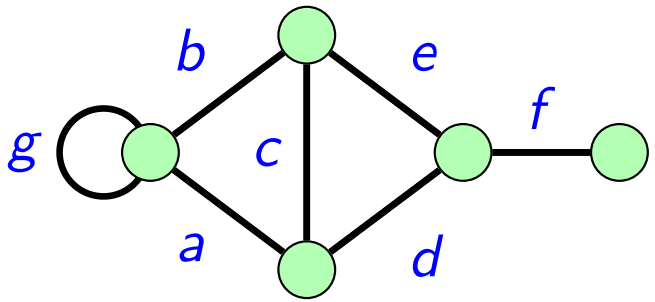
A matroid M consists of a finite set $E(M)$ and function $r : 2^{E(M)} \to \mathbb{Z}$ (the rank function) such that:

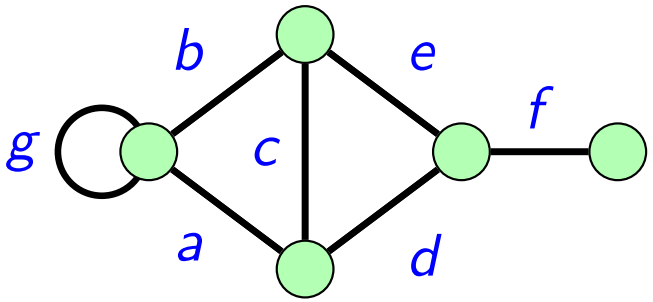
- $0 \leq r(X) \leq |X|$ for all $X \subseteq E(M)$,
- if $X \subseteq Y \subseteq E(M)$, then $r(X) \leq r(Y)$, and
- $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$ for all $X, Y \subseteq E(M)$. (submodularity)

E.g., for a graph (V, E) and set $X \subseteq E$, let $r(X)$ be the number of edges in a maximal forest in (V, X).



E.g., $r(c, e) = r(c, d, e, g) = 2$, $\quad r(a, b, e) = r(a, b, c, d, e, g) = 3$

Matroid basics: some terms



A set X is independent if $r(X) = |X|$. E.g., $\{b, f\}$ and $\{a, b, e\}$.

A circuit is a minimal dependent set. E.g., $\{g\}$, $\{a, b, c\}$, and $\{a, b, d, e\}$.

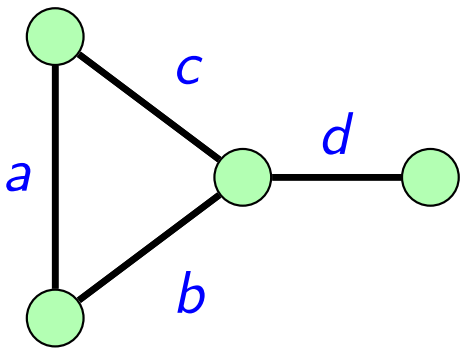
Loops are elements in singleton circuits. E.g., g.

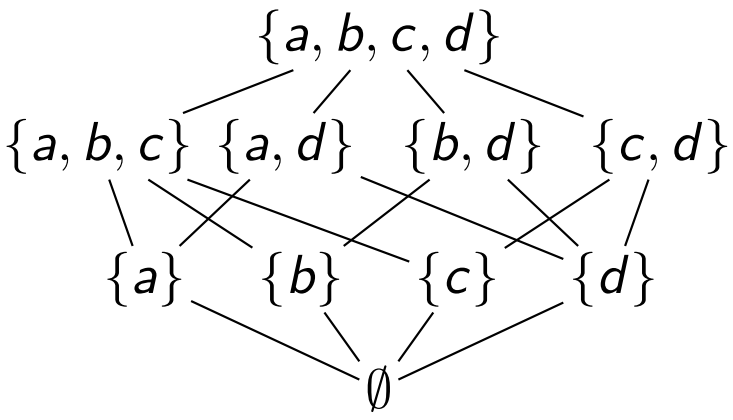
Coloops are elements that are in no circuits. E.g., f.

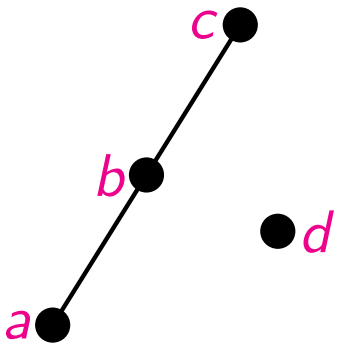
A set X is a flat of M if $r(X \cup y) > r(X)$ for all $y \in E(M) - X$.
E.g., $\{g\}$, $\{d, g\}$, $\{c, d, e, g\}$, and E.

A set X is a **flat** if $r(X \cup y) > r(X)$ for all $y \in E(M) - X$.

Under inclusion, the flats form a geometric lattice.
(atomic and semimodular)







The set of flats determines M.

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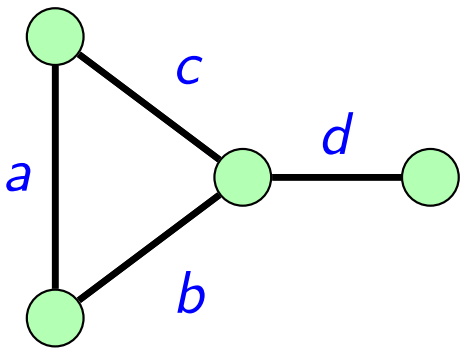
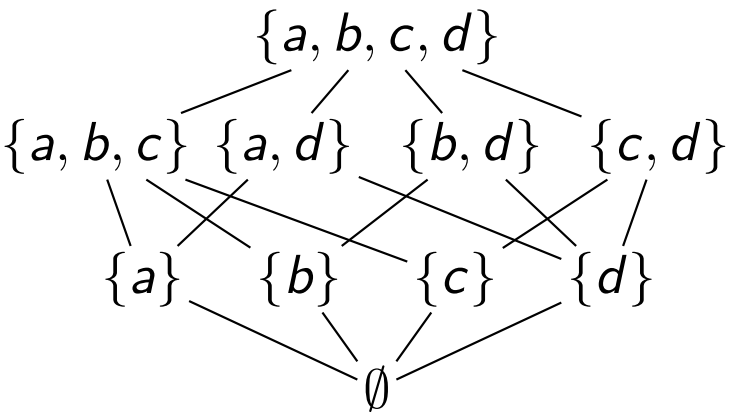
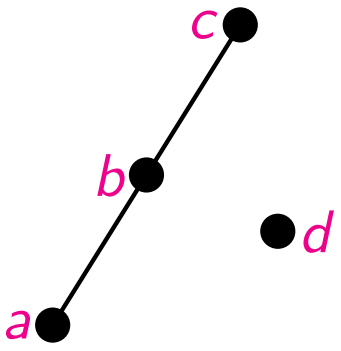






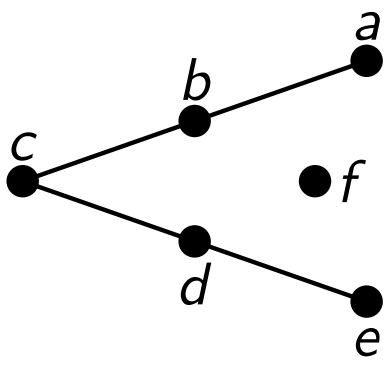
The set of flats determines M.

Equivalent formulations of the closure, $\mathrm{cl}(X)$, of $X \subseteq E(M)$:

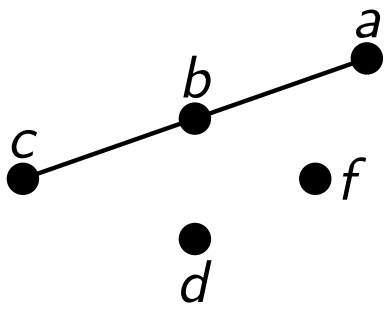
- the least flat that contains X,
- $\{y : r(X) = r(X \cup y)\}$,
- $X \cup \{y :$ there is a circuit C with $y \in C \subseteq X \cup y\}$.

E.g., $\mathrm{cl}(a, b) = \{a, b, c\}$.

The restriction, $M|X$, consists of X and the restriction of r to its subsets.



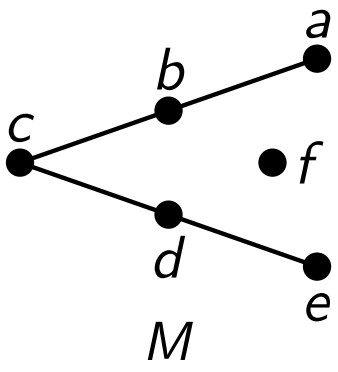
M

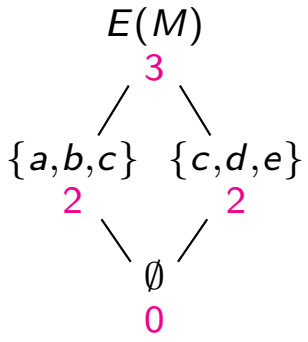


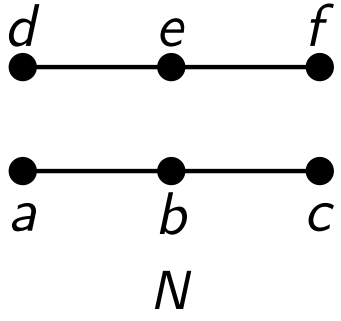
$M|\{a, b, c, d, f\}$

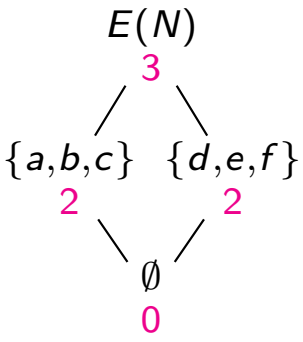
Cyclic flats

A set X in a matroid M is cyclic if X is a union of circuits of M, i.e., $M|X$ has no coloops.









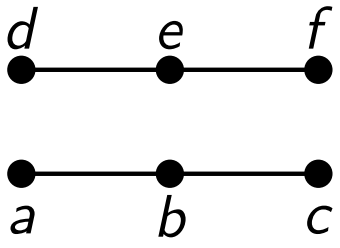
The set $\mathcal{Z}(M)$ of cyclic flats of M, ordered by inclusion, is a lattice.

Joins: $X \vee Y = \mathrm{cl}(X \cup Y)$.

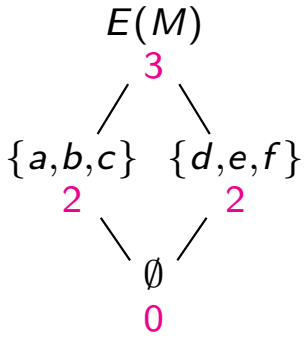
Meets: from $X \cap Y$, delete the coloops of $M|(X \cap Y)$.

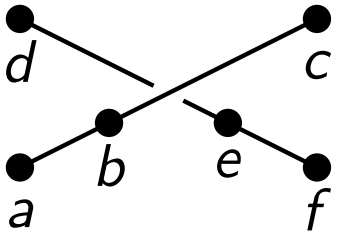
The cyclic flats alone do not determine a matroid

Many matroids may have the same cyclic flats, with different ranks.

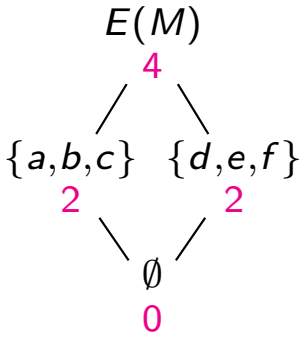


rank 3

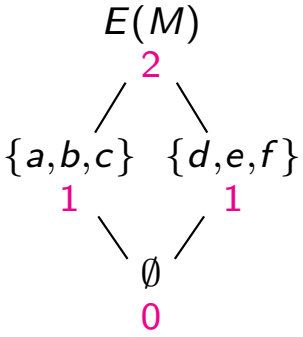




rank 4



rank 2



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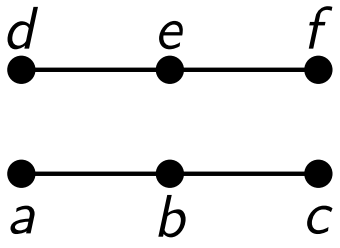

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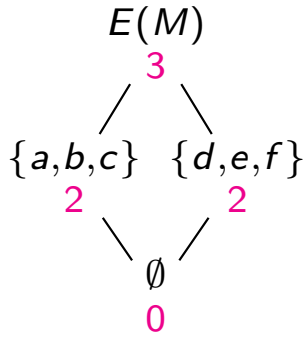
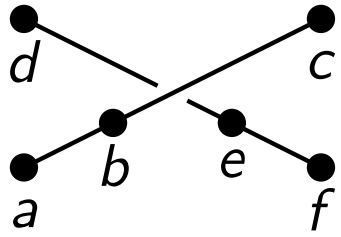
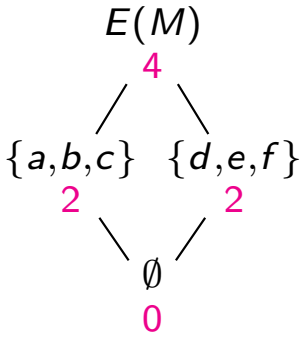




rank 4



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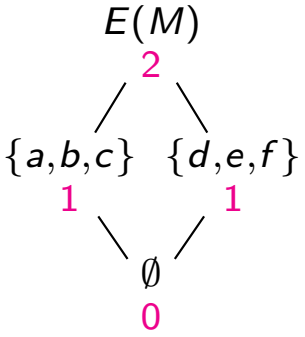


A matroid M is determined by $E(M)$ and the pairs $(X, r(X))$ with $X \in \mathcal{Z}(M)$.

(Brylawski, 1975)

Theorem

For $\mathcal{Z} \subseteq 2^E$ and $r : \mathcal{Z} \to \mathbb{Z}$, there is a matroid M on E with $\mathcal{Z} = \mathcal{Z}(M)$ and r is r_M restricted to $\mathcal{Z}$ iff

(Z0) $(\mathcal{Z}, \subseteq)$ *is a lattice,*

(Z1) $r(0_{\mathcal{Z}}) = 0$, *where $0_{\mathcal{Z}}$ is the least element of $\mathcal{Z}$,*

(Z2) $0 < r(Y) - r(X) < |Y - X|$ *for all $X, Y \in \mathcal{Z}$ with $X \subsetneq Y$,*

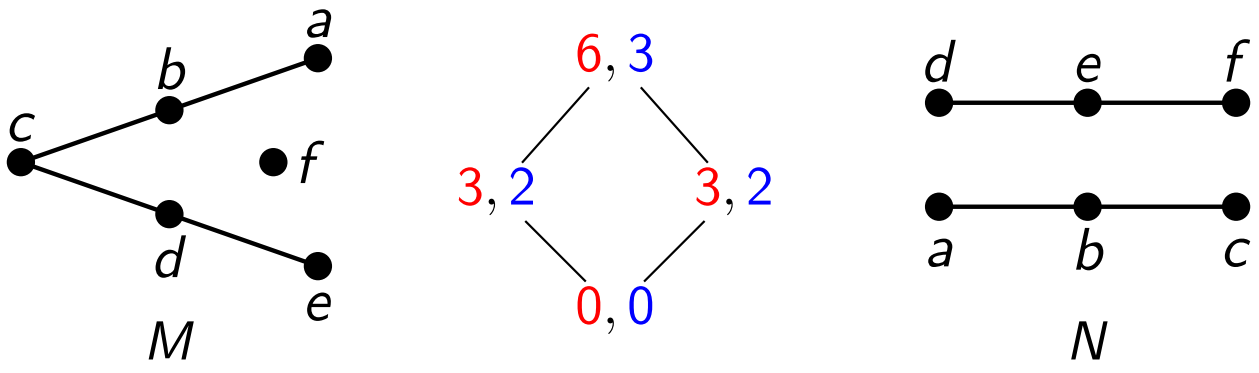
and

(Z3) $r(X) + r(Y) \geq r(X \vee Y) + r(X \wedge Y) + |(X \cap Y) - (X \wedge Y)|$ *for all $X, Y \in \mathcal{Z}$.*

(Sims, 1980; Bonin and de Mier, 2008)

The configuration of M is a 4-tuple $(L, s, \rho, |E(M)|)$ where

- L is an abstract lattice with $L \simeq \mathcal{Z}(M)$;
  say $x \mapsto X$ is an isomorphism of $L \to \mathcal{Z}(M)$,
- $s : L \to \mathbb{Z}$ with $s(x) = |X|$,
- $\rho : L \to \mathbb{Z}$ with $\rho(x) = r(X)$. (Eberhardt, 2014)



The cyclic flats themselves are not recorded,
so non-isomorphic matroids may have the same configuration.

Theorem

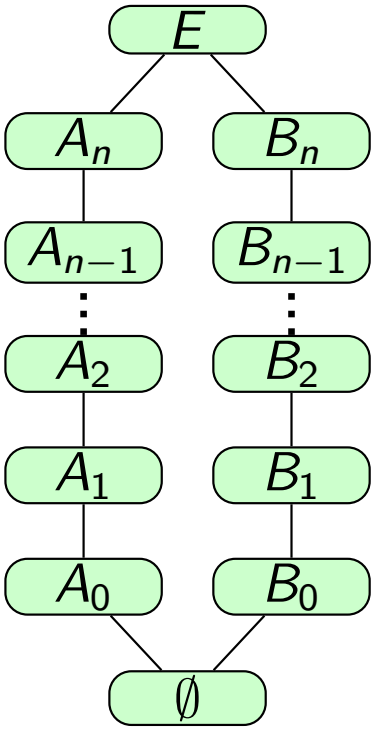
The Tutte polynomial of M is determined by its configuration.

(Eberhardt, 2014)

Many non-isomorphic matroids can have the same configuration.

O. Giménez' example:

Fix $\pi \in S_n$, the group of permutations of $\{1, 2, \ldots, n\}$.



Fix four disjoint sets

A_0 with $|A_0| = n + 2$ and $r(A_0) = n + 1$,
B_0 with $|B_0| = n + 3$ and $r(B_0) = n + 1$,
$\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$.

Let E be their union, so $|E| = 4n + 5$.

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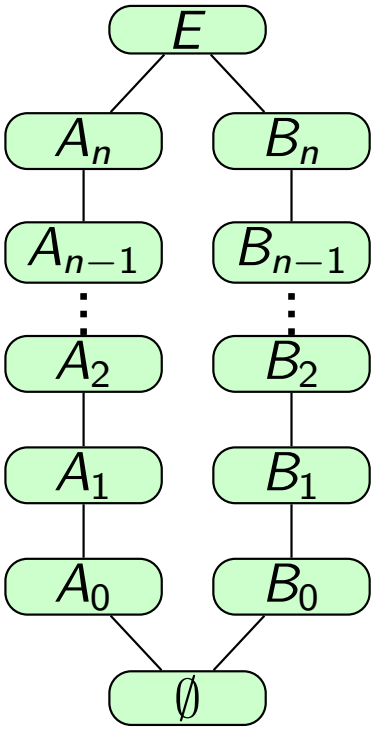

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For $i \in [n]$, set
$A_i = A_{i-1} \cup \{x_i, y_i\}$ with $r(A_i) = n + i + 1$,
$B_i = B_{i-1} \cup \{x_i, y_{\pi(i)}\}$ with $r(B_i) = n + i + 1$,
and set $r(E) = 2n + 2$.

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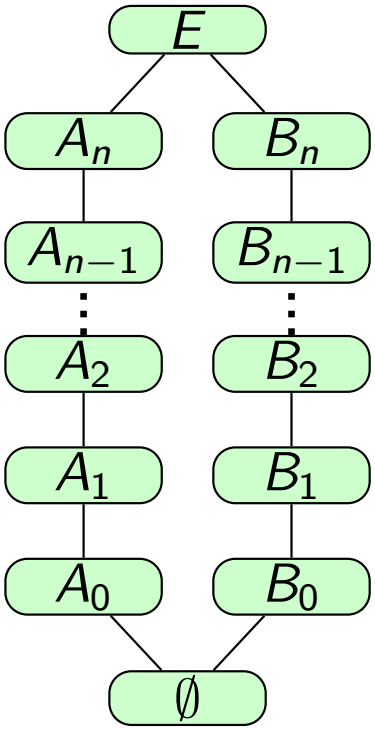

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and set $r(E) = 2n + 2$.

These $n!$ matroids of rank $2n + 2$ are non-isomorphic, have the same configuration, and so have the same Tutte polynomial.

$$T(M;x,y) = \sum_{A \subseteq E(M)} (x-1)^{r(M)-r(A)}(y-1)^{|A|-r(A)}$$
$$= \sum_{i \leq j} \left|E_{i,j}\right| (x-1)^{r(M)-i}(y-1)^{j-i}.$$

where $E_{i,j} = \{X \subseteq E(M) : r(X) = i, |X| = j\}$.

Goal: show that $|E_{i,j}|$ is determined by the configuration.

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It suffices to treat matroids without loops and coloops.

Induct on $|\mathcal{Z}(M)|$. The case $|\mathcal{Z}(M)| = 2$ is easy (M is uniform: $r(X) = \min(|X|, r)$), so assume $|\mathcal{Z}(M)| > 2$.

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Reduction: treat only $i < \min(r, j)$ since

$$|E_{r,j}| = \binom{n}{j} - \sum_{i<r} |E_{i,j}| \qquad \text{and} \qquad |E_{j,j}| = \binom{n}{j} - \sum_{i<j} |E_{i,j}|.$$

Let $\mathcal{Z}'(M) = \mathcal{Z}(M) - \{\mathrm{cl}(\emptyset), E(M)\}$.

For $F \in \mathcal{Z}'(M)$ and $i < \min(r, j)$, set
$D_F = \{X : X \in E_{i,j} \text{ and } \mathrm{cl}(X \cap F) = F\}$.

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Since $i < \min(r, j)$, any $X \in E_{i,j}$ contains a circuit C with $\mathrm{cl}(C) \in \mathcal{Z}'(M)$, so

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So, by inclusion/exclusion,

$$\left|E_{i,j}\right| = \sum_{S \subseteq \mathcal{Z}'(M),\, S \neq \emptyset} (-1)^{|S|+1} \left| \bigcap_{F \in S} D_F \right|.$$

Sketch of a new proof of Eberhardt's result: simplification to chains

Key: $F_1, F_2 \in \mathcal{Z}'(M)$ are incomparable, then $D_{F_1} \cap D_{F_2} \subseteq D_{F_1 \vee F_2}$.

If $X \in D_{F_1} \cap D_{F_2}$, then $\mathrm{cl}(X \cap F_1) = F_1$ and $\mathrm{cl}(X \cap F_2) = F_2$, so $\mathrm{cl}(X \cap \mathrm{cl}(F_1 \cup F_2)) = \mathrm{cl}(F_1 \cup F_2)$.

Thus, if $F_1, F_2 \in S \subseteq \mathcal{Z}'(M)$ and $S' = S \triangle \{F_1 \vee F_2\}$, then

$$\bigcap_{F \in S} D_F = \bigcap_{F \in S'} D_F.$$

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Since $(-1)^{|S|} = -(-1)^{|S'|}$, the corresponding terms could cancel in the sum from PIE.

We can match such potential cancellations, leaving

$$\left|E_{i,j}\right| = \sum_{\substack{\text{nonempty chains} \\ S \subseteq \mathcal{Z}'(M)}} (-1)^{|S|+1} \left| \bigcap_{F \in S} D_F \right|.$$

Sketch of a new proof of Eberhardt's result: treat chains

$$\left|E_{i,j}\right| = \sum_{\substack{\text{nonempty chains} \\ S \subseteq \mathcal{Z}'(M)}} (-1)^{|S|+1} \left| \bigcap_{F \in S} D_F \right|.$$

For a chain $S = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_p\}$, we can compute $\left|\bigcap_{F \in S} D_F\right|$ using the minors

$$M|F_1, \qquad M|F_k/F_{k-1}, \quad \text{and} \quad M/F_p$$

that correspond to the intervals

$$[\emptyset, F_1], \qquad [F_{k-1}, F_k], \quad \text{and} \quad [F_p, E(M)]$$

in $\mathcal{Z}(M)$, where $2 \leq k \leq p$.

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in $\mathcal{Z}(M)$, where $2 \leq k \leq p$.

Each of these minors has fewer cyclic flats than M.
We get their configurations from that of M.
That allows us to find $|E_{i,j}|$ from the configuration of M. $\square$

Let M be a rank-r matroid on $\{1, 2, \ldots, n\}$.

For $\pi \in S_n$, the rank sequence $\underline{r}(\pi) = (r_1, r_2, \ldots, r_n)$ is given by $r_1 = r(\{\pi(1)\})$ and for $j \geq 2$,

$$r_j = r(\{\pi(1), \pi(2), \ldots, \pi(j)\}) - r(\{\pi(1), \pi(2), \ldots, \pi(j-1)\}).$$

So $r_j \in \{0, 1\}$ and $\{\pi(j) : r_j = 1\}$ is a basis of M.

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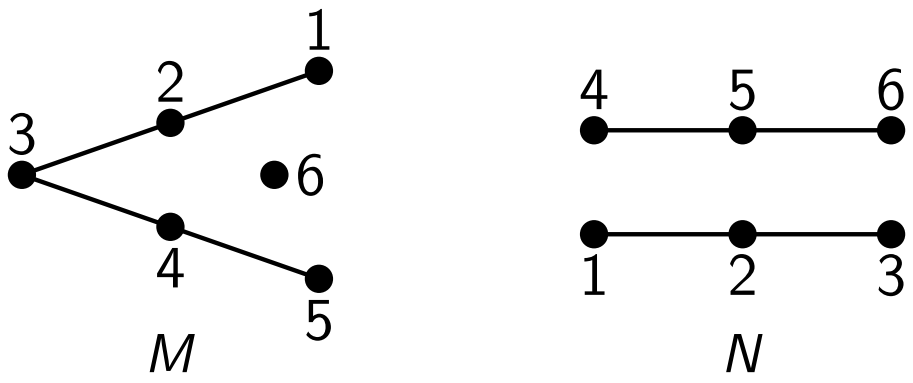
So $r_j \in \{0, 1\}$ and $\{\pi(j) : r_j = 1\}$ is a basis of M.

To a sequence $\underline{r}$ of r 1's and $n - r$ 0's, associate a variable $[\underline{r}]$.

The $\mathcal{G}$-invariant is $\mathcal{G}(M) = \sum_{\pi \in S_n} [\underline{r}(\pi)]$. (A reformulation.)

An example



Two rank sequences:

111000 if $\{\pi(1), \pi(2), \pi(3)\}$ a basis;

there are $\left(\binom{6}{3} - 2\right) \cdot 3! \cdot 3! = 648$ such $\pi \in S_6$;

110100 otherwise;

there are $2 \cdot 3! \cdot 3! = 72$ such $\pi \in S_6$.

Thus, $\mathcal{G}(M) = \mathcal{G}(N) = 648\,[111000] + 72\,[110100]$.

Theorem

The Tutte polynomial $T(M; x, y)$ is a specialization of $\mathcal{G}(M)$.

(Derksen, 2009)

A flag of a rank-r matroid is a maximal chain of flats

$$\mathrm{cl}(\emptyset) = X_0 < X_1 < X_2 < \cdots < X_{r-1} < X_r = E(M),$$

where $r(X_j) = j$.

Theorem

$\mathcal{G}(M)$ is equivalent to the data

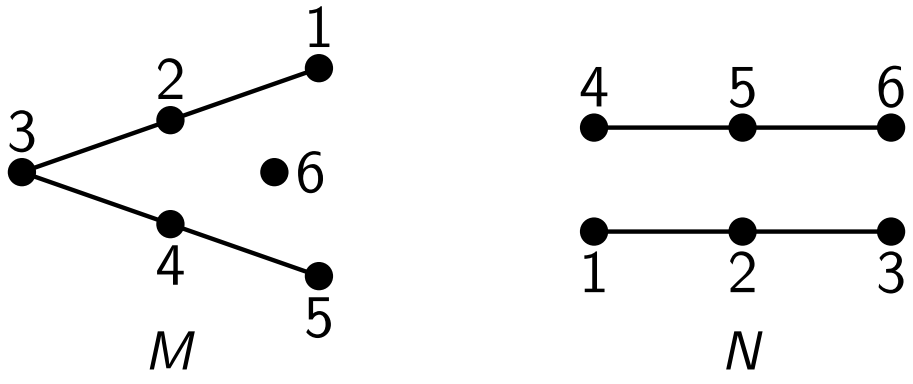
$$|X_0|, |X_1 - X_0|, |X_2 - X_1|, \ldots, |X_r - X_{r-1}|$$

for all flags of M. *(Kung, 2015)*

An example

Having $\mathcal{G}(M)$ is equivalent to having all the flag data

$$|X_0|,\ |X_1 - X_0|,\ |X_2 - X_1|, \ldots,\ |X_r - X_{r-1}|.$$



Two sequences arise from flags:

$0, 1, 1, 4$ arises from chains that contain a 2-point line;
there are $9 \cdot 2$ such chains;

$0, 1, 2, 3$ arises from chains that contain a 3-point line;
there are $2 \cdot 3$ such chains.

Theorem

From the configuration of M, one can compute the data

$$|X_0|, |X_1 - X_0|, |X_2 - X_1|, \ldots, |X_r - X_{r-1}|$$

for all flags of M, and so its $\mathcal{G}$-invariant. *(Bonin and Kung, 2015)*

The proof is an inclusion/exclusion argument, akin to our proof of Eberhardt's result.

Theorem

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The proof is an inclusion/exclusion argument, akin to our proof of Eberhardt's result.

Dowling matroids of the same rank $r > 3$ over non-isomorphic groups of the same order show that matroids with different configurations can share the same flag data, and so have the same $\mathcal{G}$-invariant.

Cyclic flats (fully dependent flats) play many important roles in the theory of transversal matroids. (Ingleton, Brualdi, Mason, Brylawski, ...)

Other areas of application:

- constructing infinite sets of intertwines for most pairs of matroids (Bonin, 2010)
- progress on the sticky matroid conjecture (Bonin, 2011)
- matroid constructions (the free product, Crapo and Schmitt, 2005; matroid splicing, Bonin and Schmitt, 2011; semidirect sums of matroids, Bonin and Kung, 2015)
- describing some excluded minors for base-orderable and strongly base-orderable matroids (Bonin and Savitsky, 2015).

Thank you for listening.