Not Sin' Wash in' ton

2021

Amusing Representations

2021

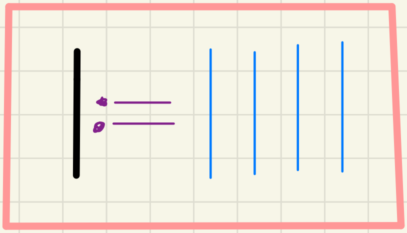
Scott

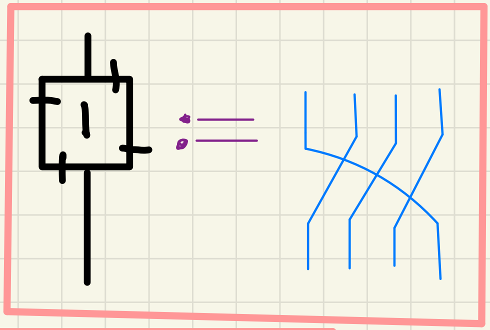
Carter

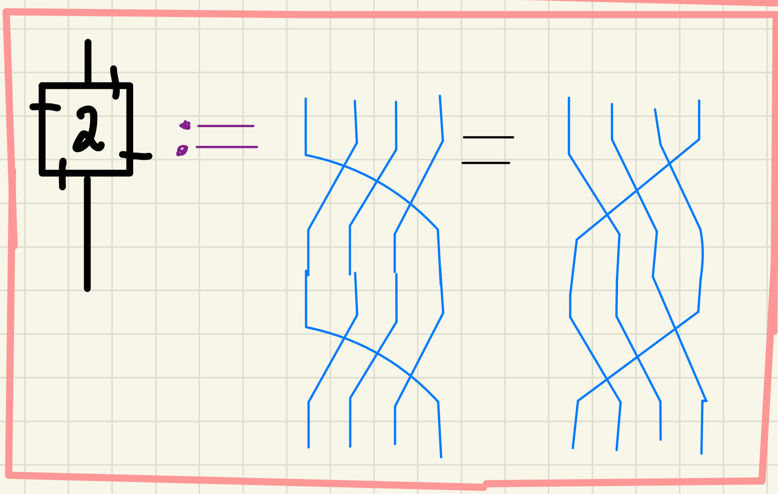
Based on joint work
with Yongju Bae
& Byoerhi Kim

Start by playing

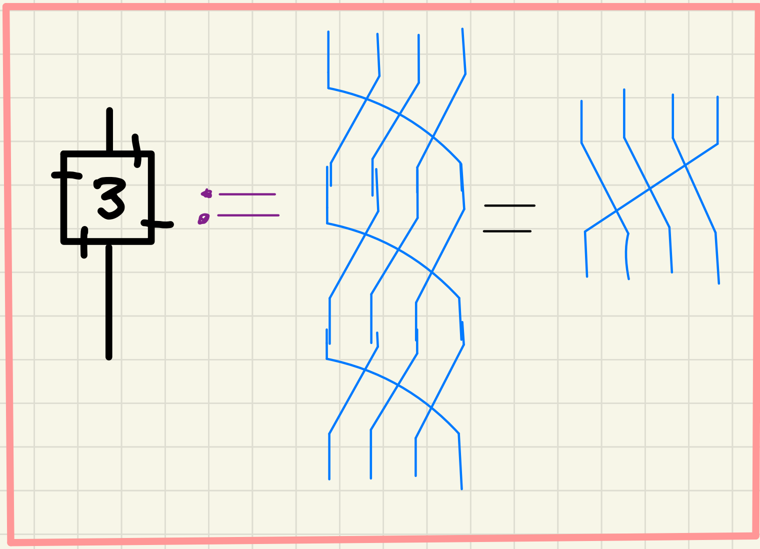
Dictionary:

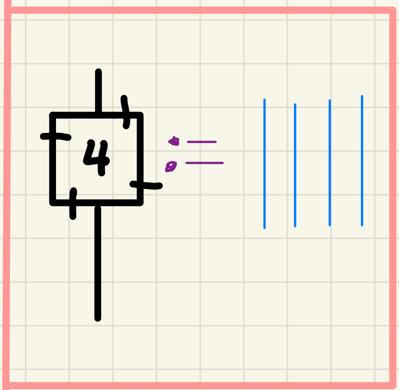






So just
$\mathbb{Z}/4$
w/ 4 half twists being none.

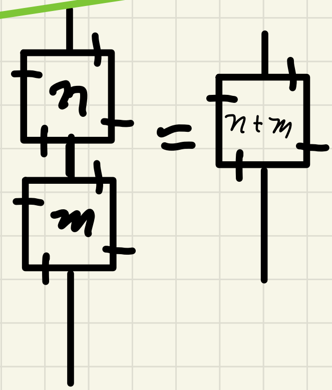




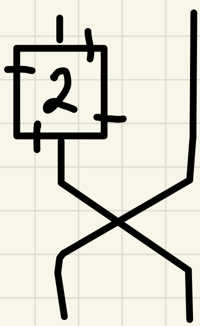
So just $\mathbb{Z}/4$ w/ 4 half twists being none.

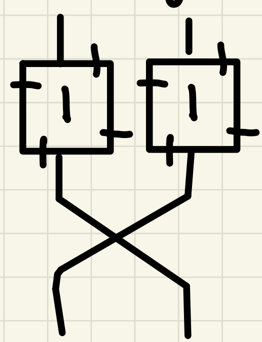
Cf.  & $\pi_1(SO(3)) = \mathbb{Z}/2$.

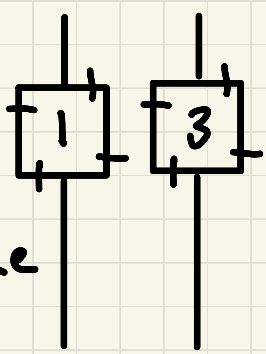
[After all we're interested in SU(2)]



Introduce:

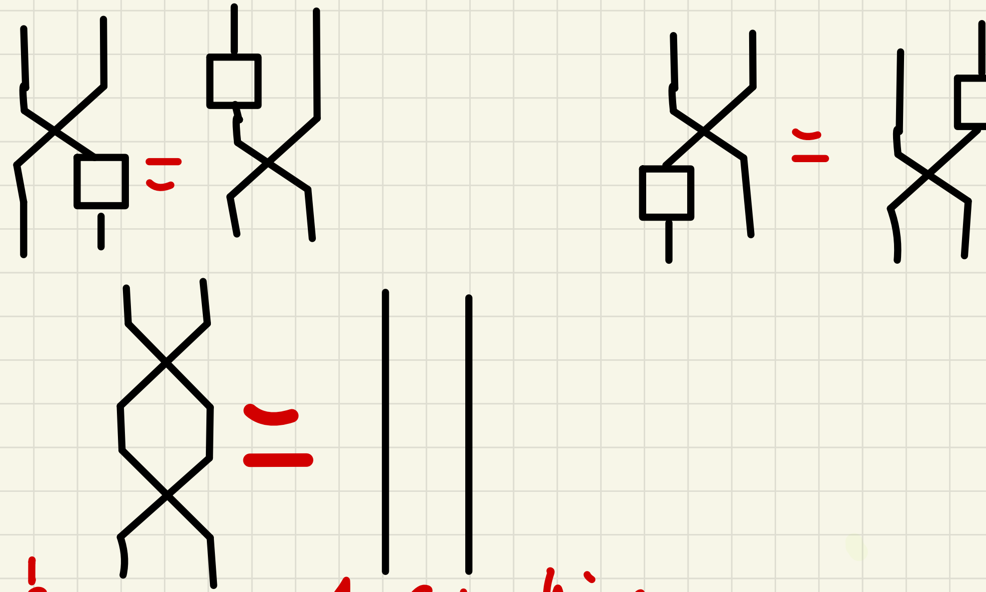
Irene 

Kathy 

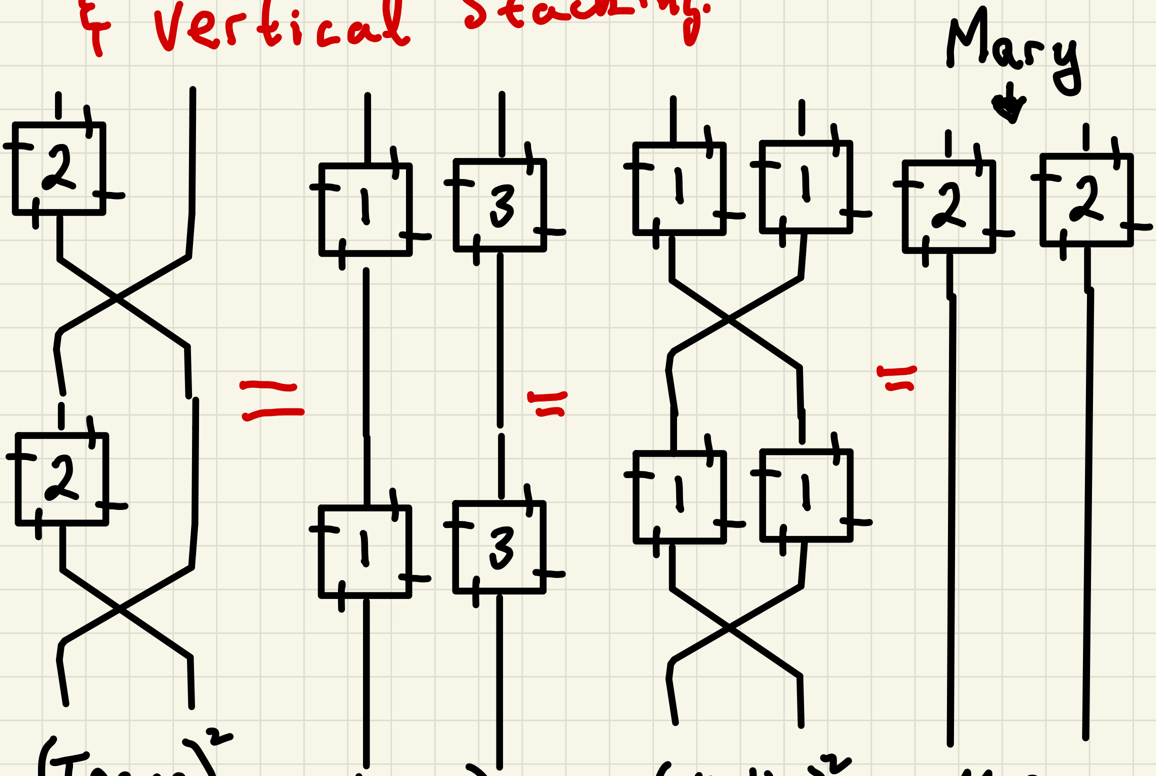


Jane

Rules:

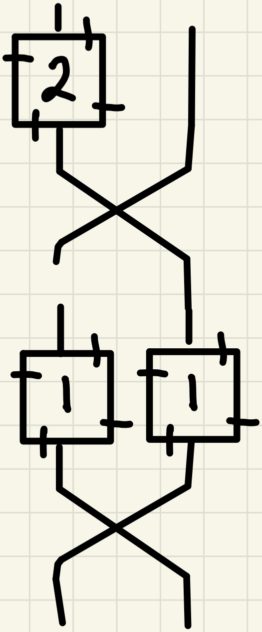


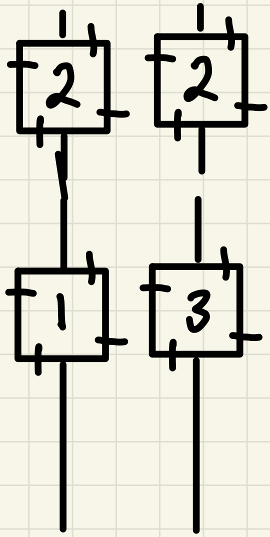
& vertical stacking.

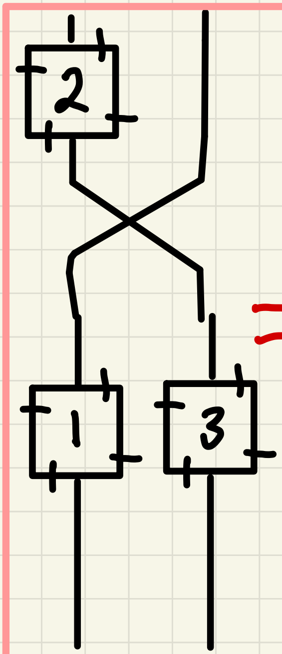


$(\text{Irene})^2 = (\text{Jane})^2 = (\text{Kathy})^2 = \text{Mary}$

Irene Jane Kathy Mary







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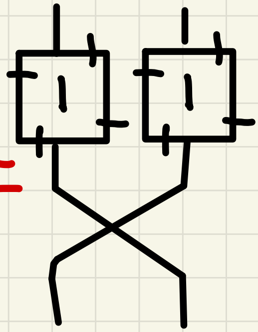
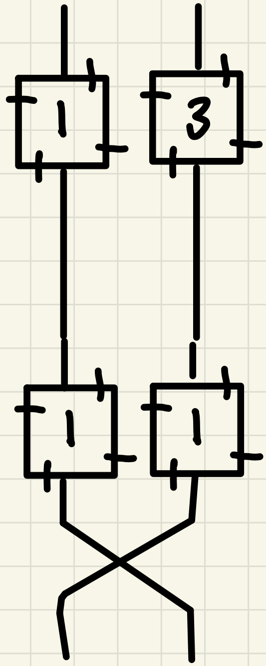


Irene · Jane
= Kathy



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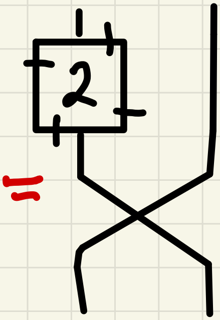
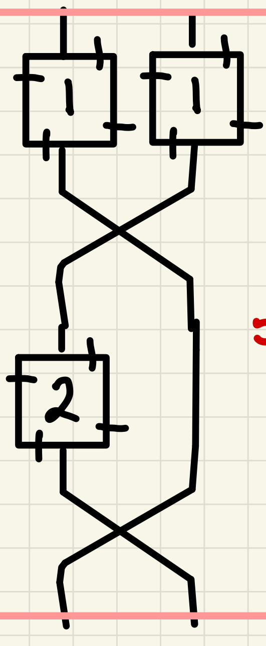


Jane · Kathy = Irene

Kathy Irene
= Jane



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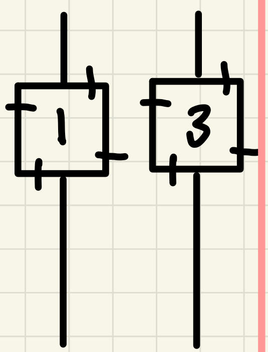
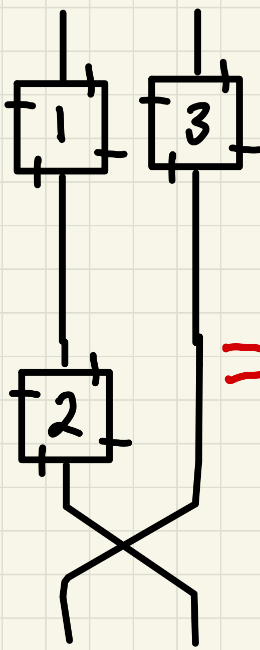


Jane Irene



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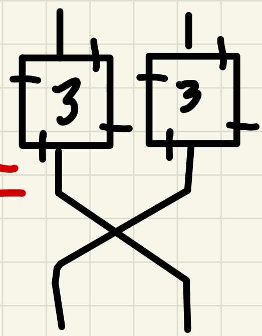
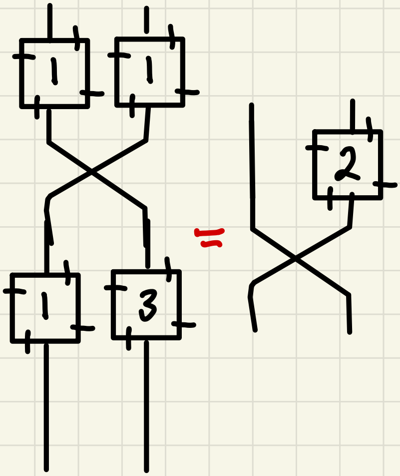


Mary
Kathy

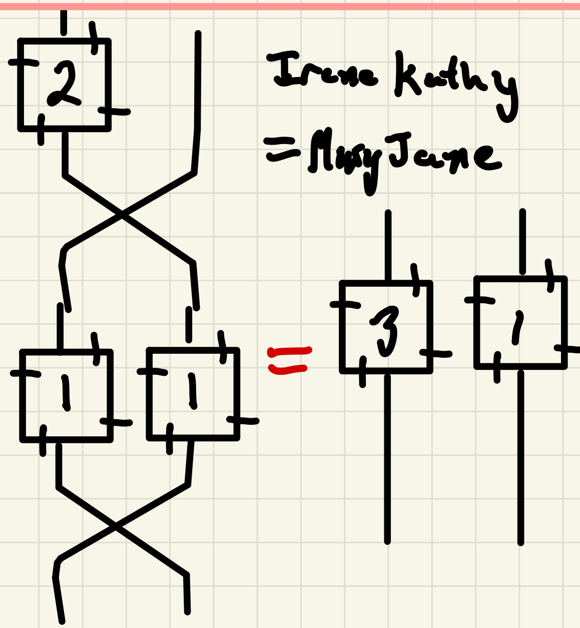


Kathy Jane

= Mary Irene

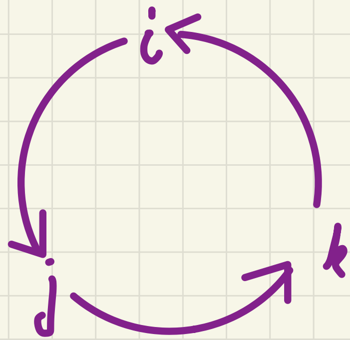
Irene Kathy

= Mary Jane



Summary:

$1, i, j, k, -1, -i, -j, -k$



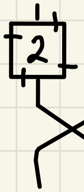
$ij = k = -ji$

$jk = i = -kj$

$ki = j = -ik$

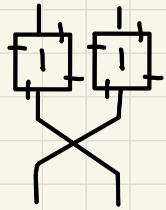
*Cue Song "Judy in disguise"

So Irene, Jane, & Kathy were unit quaternions in disguise.



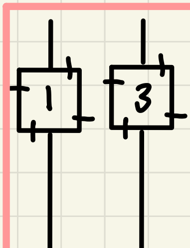
$$\longleftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

i



$$\longleftrightarrow \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

k



$$\longleftrightarrow \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

j

lie alg.

(2×2) trace$=0$, det$=1$ $\therefore$ basis for $su(2)$

Divide by $\frac{1}{2}$ (norm$=1$). Lie bracket
$[A,B]=AB-BA$ is the standard crossproduct
from cal III.

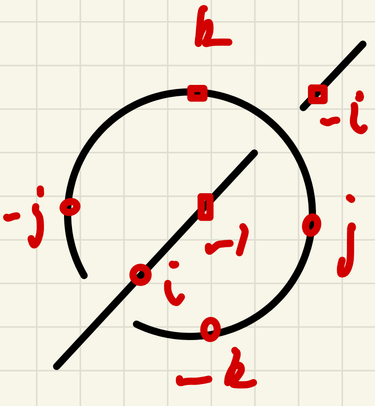
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & +i \\ i & 0 \end{bmatrix}$$

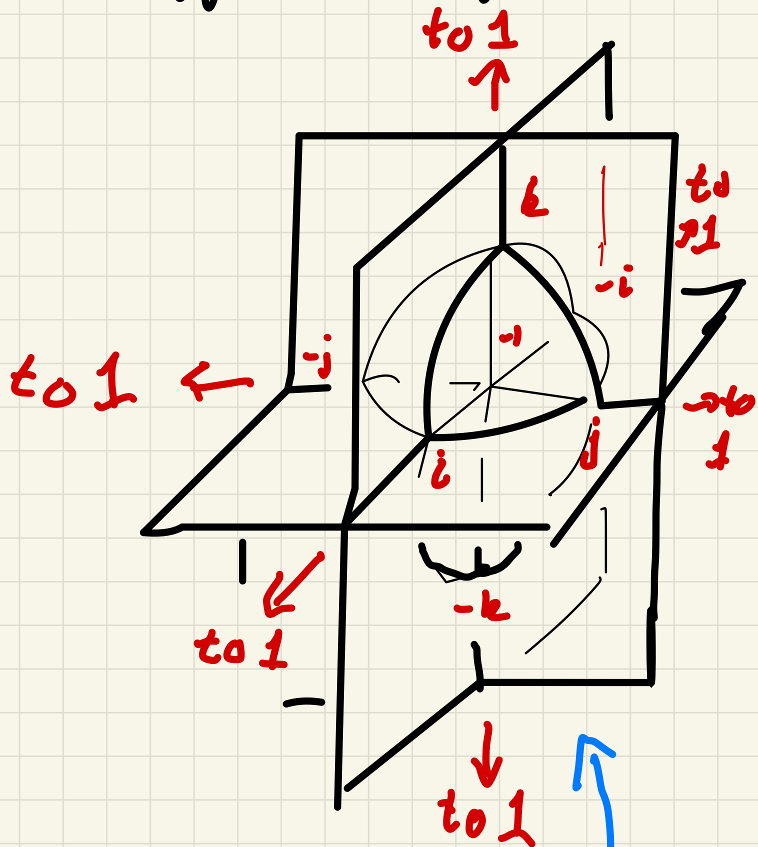
$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Reminder

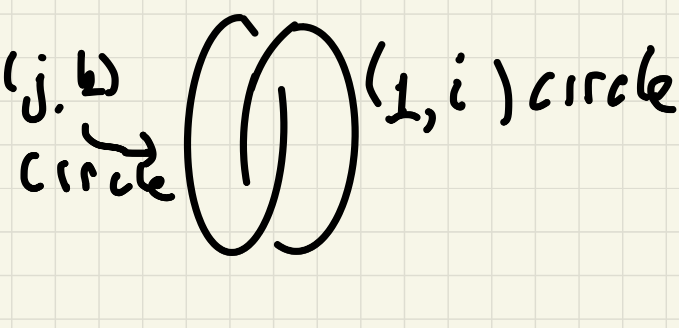
So su(2)-lie alg. is the tangent space at identity for SU(2) which topologically is $S^3 = \{w + xi + yj + zk : x^2+y^2+z^2+w^2=1\}$





↓ Hopf link

S^3 in stereographic projection

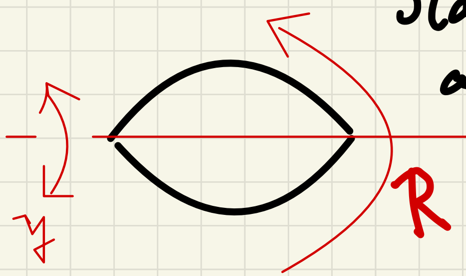


Meanwhile, $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is Dic_2, where

$$\mathrm{Dic}_n = \langle \rho, x : \rho^{2n} = 1,\ x^2 = \overset{\overset{-1}{''}}{\rho^n},\ \rho x = x\rho^{-1} \rangle.$$

is the 2-fold extension of the dihedral group — the group of symmetries of an n-gon.

Start from sym.s of a bigon



[leave the gratuitous pun in the chat]

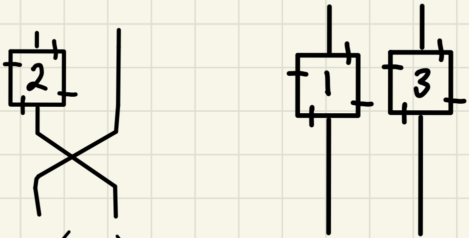
ρ projects to R

x " to y.

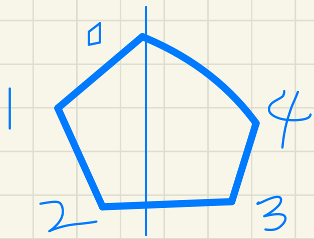
$$0 \to \mathbb{Z}/_2 \to \mathrm{Dic}_n \longrightarrow \mathrm{Dihedral}_n \to 0$$

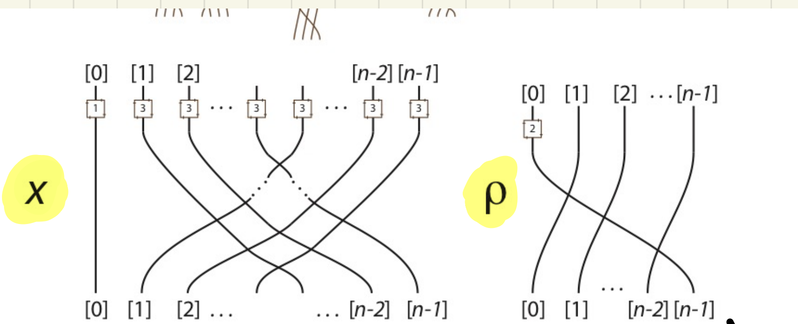
is given by the presentation

$$\mathrm{Dic}_n = \langle \rho, x : \rho^{2n} = 1,\ x^2 = \rho^n,\ \rho x = x\rho^{-1} \rangle.$$



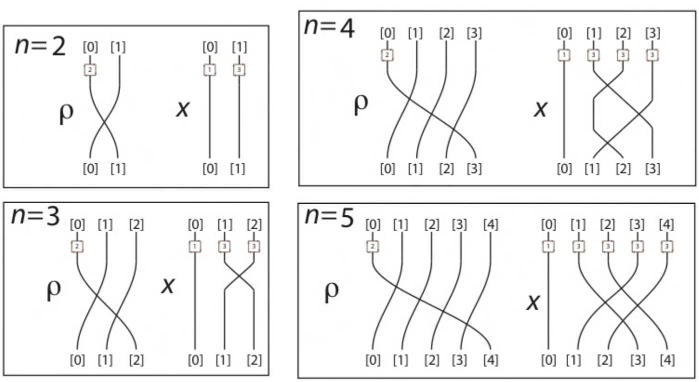
$i \leftrightarrow \rho$ $\qquad$ $j \leftrightarrow x$





Apparently projs to refl. Ditto rotation

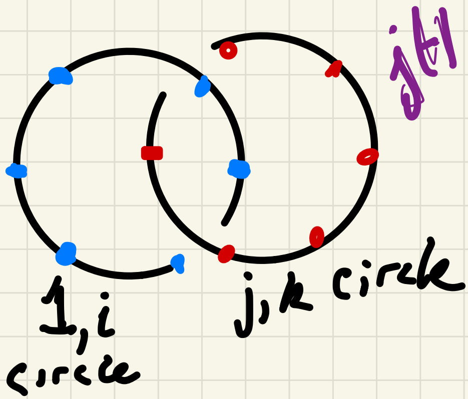
FIGURE 11. Generators of the dicyclic groups in permutation form



But there's a better presentation.

Rather than doing this in general, I'll look @ $n=3$.

$$\zeta = e^{\pi i/3}$$
$$= \cos\left(\tfrac{\pi}{3}\right) + i\sin\left(\tfrac{\pi}{3}\right)$$



$$H = (\zeta^l : l = 0, \dots, 5)$$

an ordered subgroup

— only ordered as a set.

$\rho \mapsto \zeta$
$x \mapsto j$

jH = a coset with induced order.

is given by the presentation

$$\mathrm{Dic}_n = \langle \rho, x : \rho^{2n} = 1,\ x^2 = \rho^n,\ \rho x = x\rho^{-1} \rangle.$$

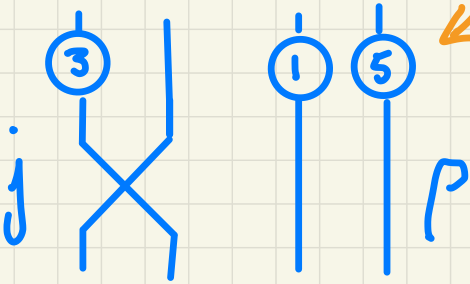
ρ rotates $(1,i)$ circle one way
& the (j,k) " the other way.

x swaps them, but

$$j(j\cos(\theta) + k\sin(\theta)) = i\sin(\theta) - \cos(\theta)$$

So...

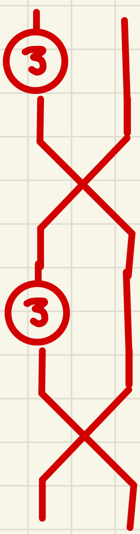
read mod 6



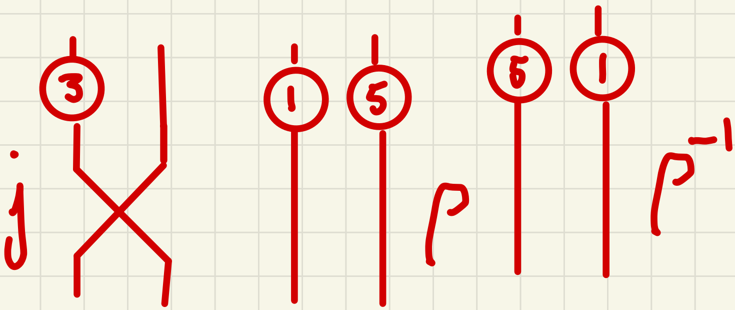
$$j \leftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\rho \leftrightarrow \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^5 \end{bmatrix}$$

Play area:

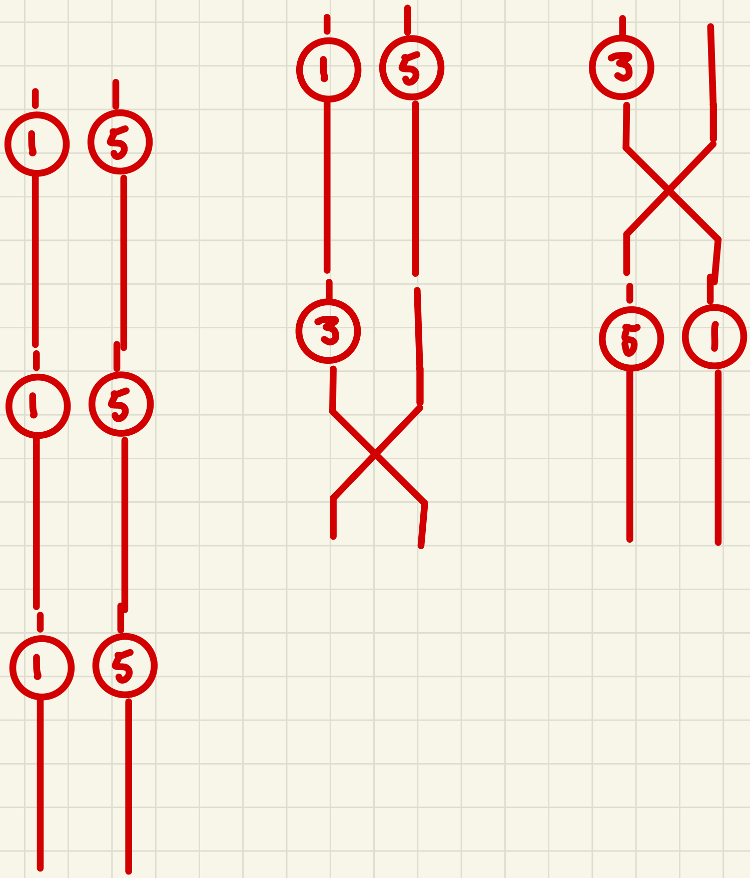


$x =$



is given by the presentation

$$\mathrm{Dic}_n = \langle \rho, x : \rho^{2n} = 1,\ x^2 = \rho^n,\ \rho x = x\rho^{-1} \rangle.$$

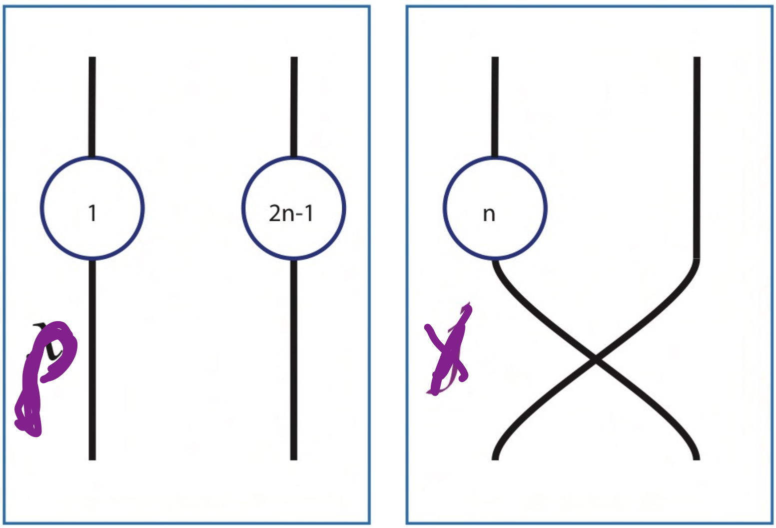

1
5
1
5
1
5
1
5
3
3
5
1

In general Dic_n, let

$\zeta = e^{\pi i/n}$

Since $\zeta = e^{(\pi \boldsymbol{\imath})/n}$, there is a corresponding matrix representation

$$\rho \leftrightharpoons \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}, \quad \boldsymbol{j} \leftrightharpoons \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$



& get these diagrams & matrices

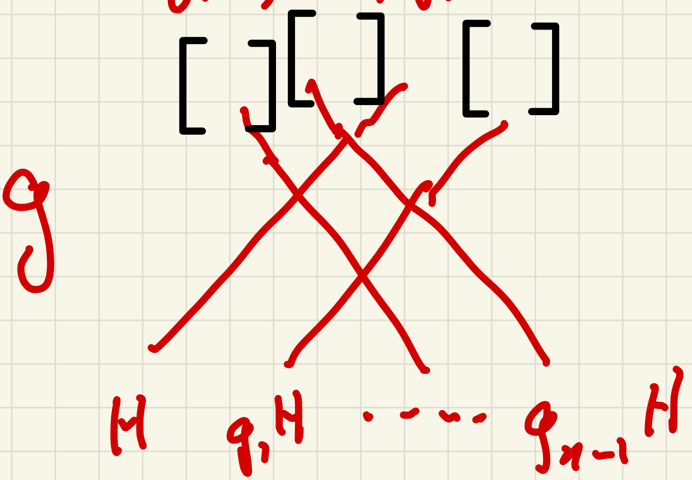
What's going on:

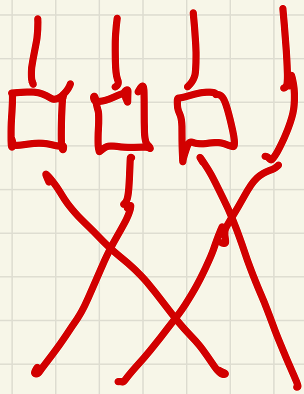
Krasner Kaloujine Theorem:

THEOREM 2. *Let G denote a finite group of order nk. Let H denote a subgroup of order k. Then there is an inclusion $G \subset H^n \rtimes \Sigma_n$ where the second factor permutes the coordinates of H^n.*

Sketch: Order $H = (h_1, \dots, h_k)$.

$(g_i H, \dots, g_j H)$



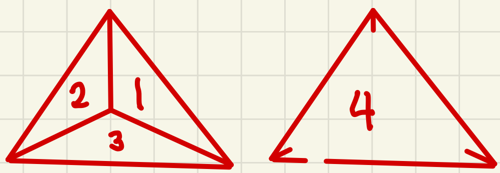


Goal to make use of This thm to give cool descriptions of favorite groups.

Digress:

$$A_4 = \mathrm{Inn}(Q(4,1))$$

$$\begin{array}{cccc} \big[(123), & (134), & (243), & (142)\big] \cong Q(4,1) \\ \| & \| & \| & \| \\ 4 & 2 & 1 & 3 \end{array}$$



Row ◁ Col

◁	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

$\mathrm{Inn}(Q(4,1)) = A_4$

There is a 2-fold extension of Vendramin's $Q(4,1)$: known as $Q(8,1)$.
$\mathrm{Inn}(Q(8,1)) = \tilde{A}_4$ the binary tetrahedral group.

HMM...

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & \tilde{A}_4 \arrow[r] \arrow[d] & A_4 \arrow[r] \arrow[d] & 0 \\
& \mathbb{Z}/2 \arrow[r, hook] & Q(8,1) \arrow[r] & Q(4,1) \arrow[r] & 0
\end{tikzcd}$$

But B4 that we had to Determine $\mathrm{Inn}(Q(8,1))$. It is generated by these permutations

$$
\begin{aligned}
a &\leftrightharpoons (04)(123567);\\
b &\leftrightharpoons (37)(052416);\\
c &\leftrightharpoons (15)(036472);\\
d &\leftrightharpoons (26)(075431).
\end{aligned}
$$

& $\mathrm{Inn}(Q(4,1))$ is

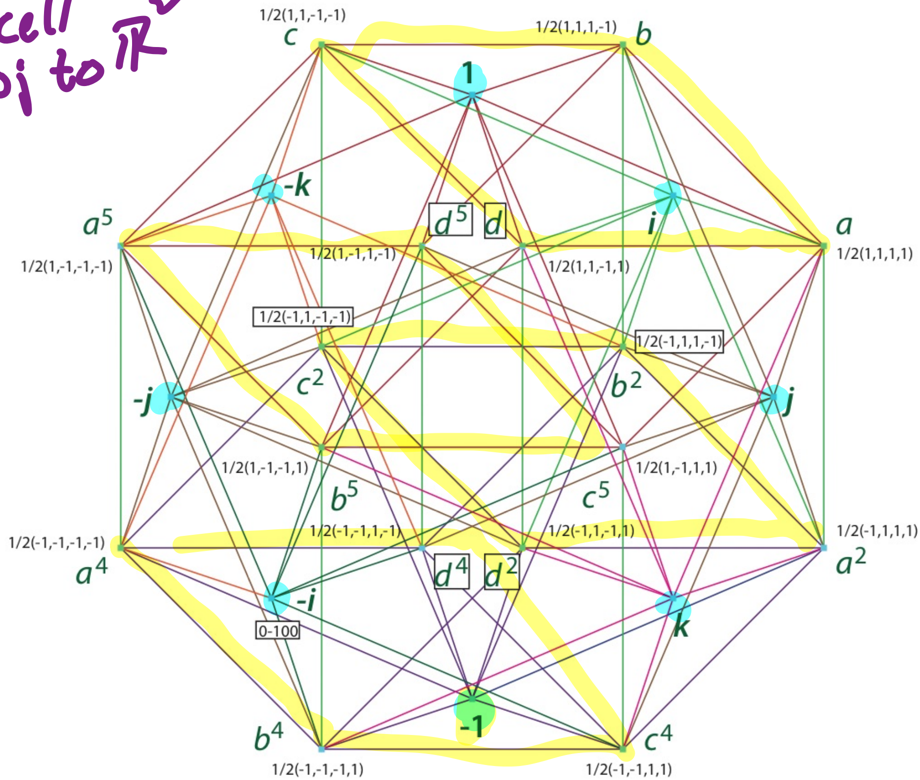
$$\langle (123),(032),(013),(021) \rangle$$

We looked @ subgroups gen by a,b,c,d. Saw they were $\mathbb{Z}/6$ + then looked up the group

$$SL_2(\mathbb{Z}/3) \simeq \tilde{A}_4$$

— the binary tetrahedral group.

24 cell proj to $\mathbb{R}^2$



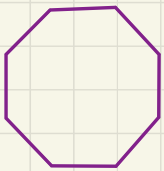
$$\pm 1, \pm \boldsymbol{i}, \pm \boldsymbol{j}, \pm \boldsymbol{k}, \frac{1}{2}(\pm 1 \pm \boldsymbol{i} \pm \boldsymbol{j} \pm \boldsymbol{k}).$$

$$a = \frac{1}{2}(1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}), \quad b = \frac{1}{2}(1 + \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}),$$

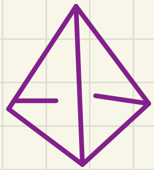
$$c = \frac{1}{2}(1 + \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}), \quad \text{and} \quad d = \frac{1}{2}(1 + \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$$

$$SU(2) \twoheadrightarrow SO(3)$$

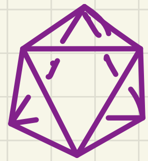
$SO(3)$ a group of rots of 3-D space.
Finite subgroups therein



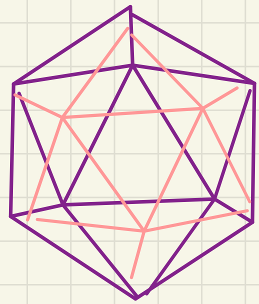
Dihedral: groups of Sym of polygons



tetrahedral
A_4



Octahedral
Σ_4



Icosahedral
A_5

120 cell

* 2-fold extensions inside $S^3 = SU(2)$

Dicylic

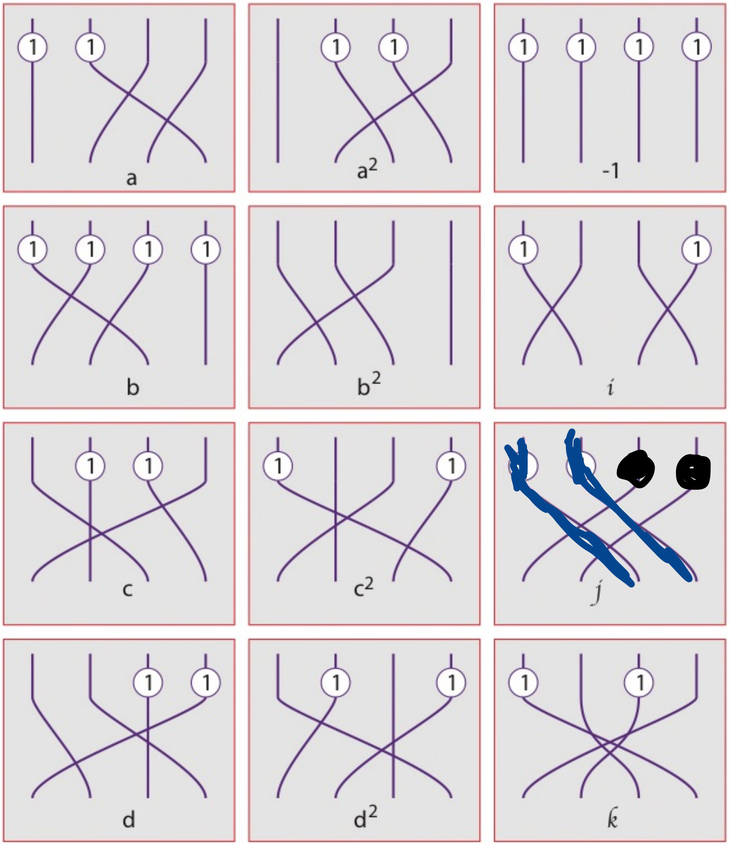
Binary tet.

Binary oct.

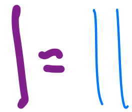
Binary icos.

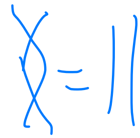
* $0 \to \mathbb{Z}/2 \to \tilde{A}_4 \to A_4 \to 0$

eg

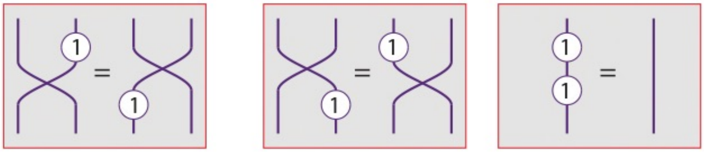


$\tilde{A}_4$





subject to the relations:



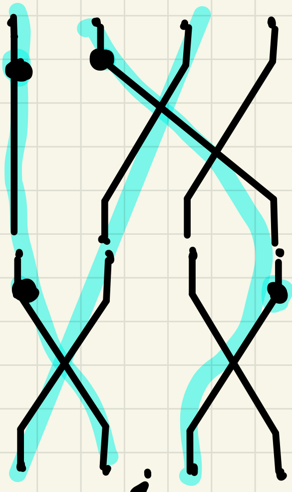
In terms of the

Krasner Kaloujine Theorem:

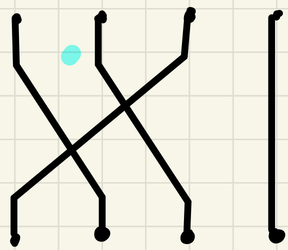
N is a subgroup. Cosets correspond as indicated

$0 \leftrightarrow N = (1, a^2, a^4)$	$4 \leftrightarrow aN = (a, -1, a^5)$
$1 \leftrightarrow \boldsymbol{i}N = (\boldsymbol{i}, b^4, d^5)$	$5 \leftrightarrow \boldsymbol{i}aN = (d^2, -\boldsymbol{i}, b)$
$2 \leftrightarrow \boldsymbol{j}aN = (b^2, -\boldsymbol{j}, c^5)$	$6 \leftrightarrow \boldsymbol{j}N = (\boldsymbol{j}, c^2, b^5)$
$3 \leftrightarrow \boldsymbol{k}N = (\boldsymbol{k}, d^4, c)$	$7 \leftrightarrow \boldsymbol{k}aN = (c^4, -\boldsymbol{k}, d)$

Play [toys below]



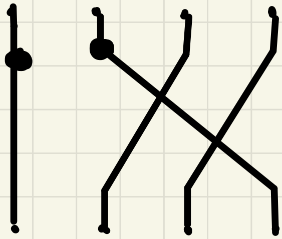
i



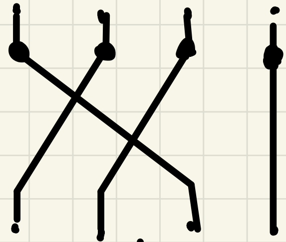
b^2

$ai = b^2$, $aj = -c$, $ak = d^2$

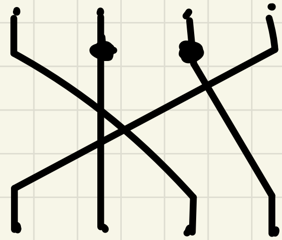
$ab = j$ $\quad ba = i$ $\quad aj = -c$ $\quad a^{-1}b = c$ $\quad a^{-1}i = d$



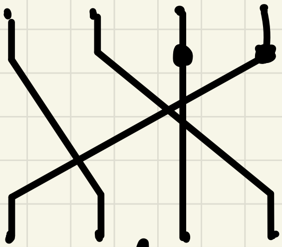
a



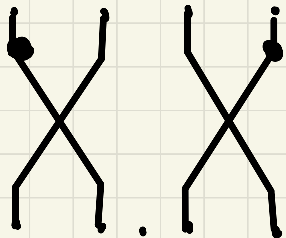
b



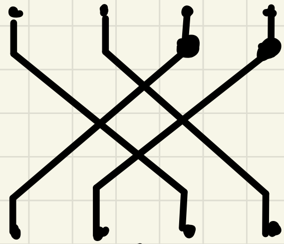
c



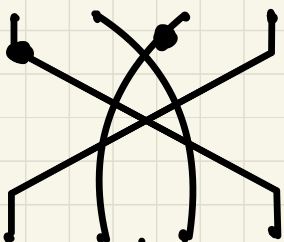
d



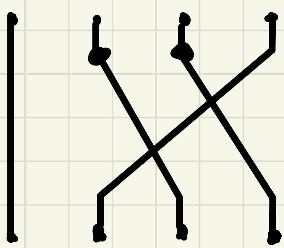
i



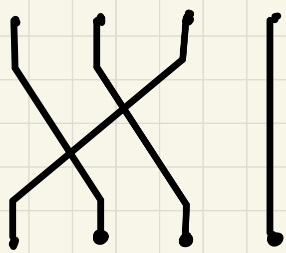
j



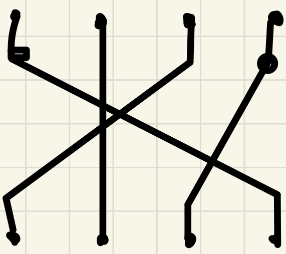
k



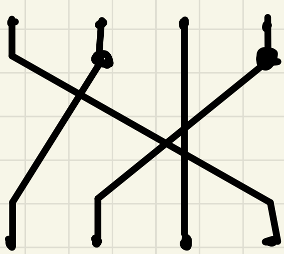
a^2



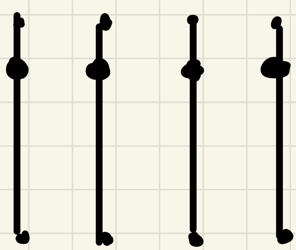
b^2



c^2

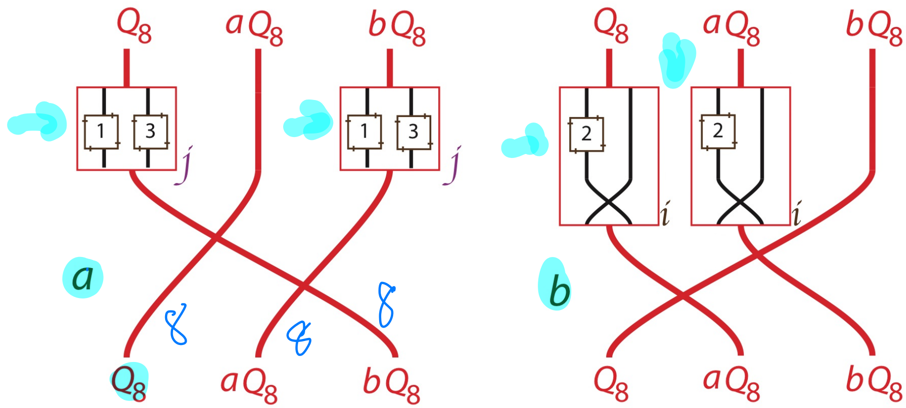


d^2

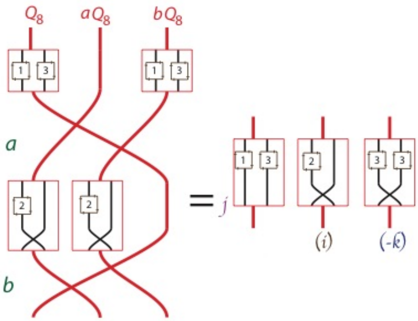


-1

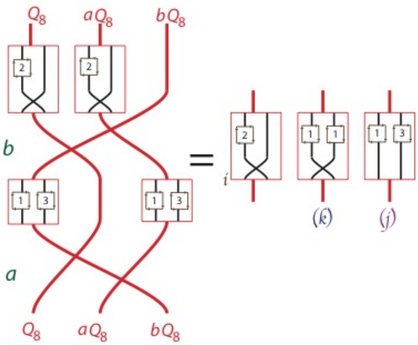
Other pictures:



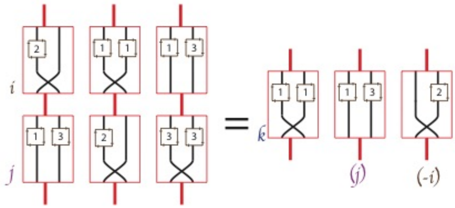
$Q_8 = S \cup iS = ((-1, -j, 1, j), (-i, -k, i, k))$



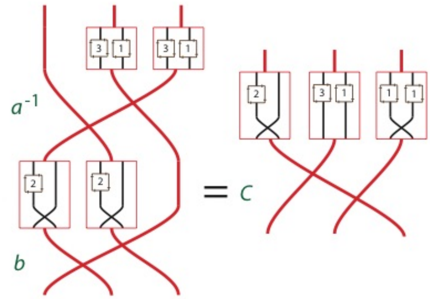
Q8
aQ8
bQ8
a
b
=
j
(i)
(-k)



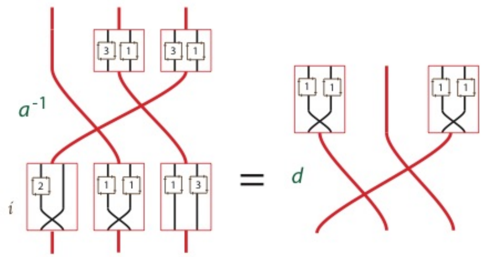
Q8
aQ8
bQ8
b
a
=
i
(k)
(j)
Q8
aQ8
bQ8



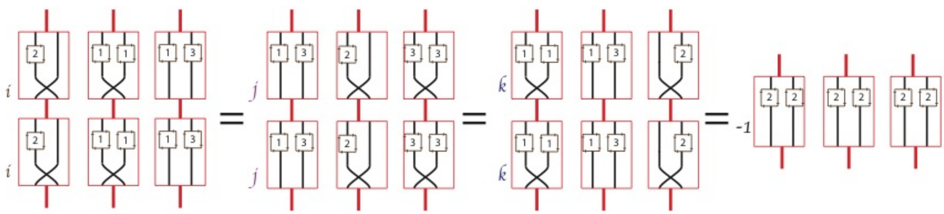
i
j
=
k
(j)
(-i)



a-1
b
=
c



a-1
i
=
d



i
i
=
j
j
=
k
k
=
-1

One more repr. of $\tilde{A}_4$ that uses the ordered subgroup: $A = \langle a \rangle = (1, a, a^2, -1, -a, -a^2)$

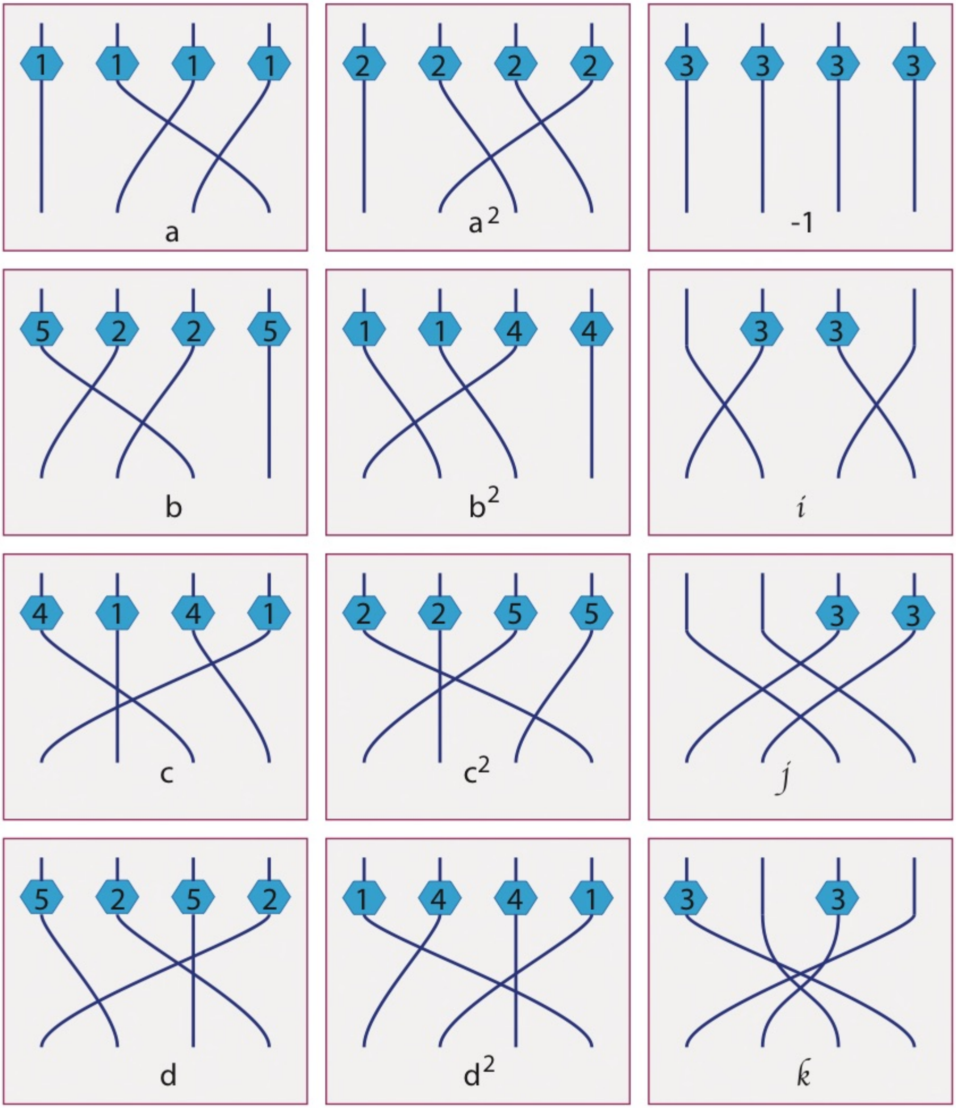
The action of a upon these ordered sets is as follows:

$$\begin{aligned}
aA &= (a, a^2, -1, -a, -a^2, 1) = [1/6, A] \\
a(\boldsymbol{i}A) &= (b^2, c^2, -\boldsymbol{j}, -b^2, -c^2, \boldsymbol{j}) = [1/6, \boldsymbol{j}A] \\
a(\boldsymbol{j}A) &= (-c, -d, -\boldsymbol{k}, c, d, \boldsymbol{k}) = [1/6, \boldsymbol{k}A] \\
a(\boldsymbol{k}A) &= (d^2, -b, -\boldsymbol{i}, -d^2, b, \boldsymbol{i}) = [1/6, \boldsymbol{i}A].
\end{aligned}$$

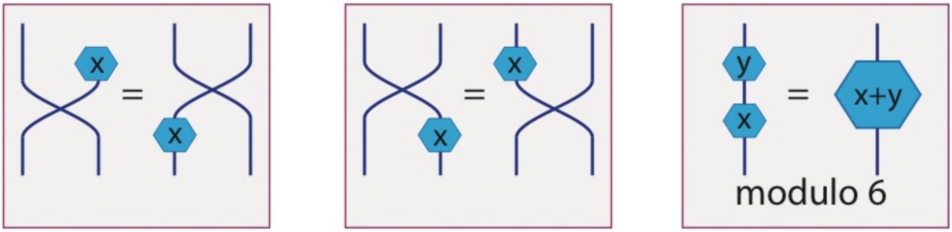
And here is the action of b:

$$\begin{aligned}
bA &= (b, \boldsymbol{i}, d^2, -b, -\boldsymbol{i}, -d^2) = [5/6, \boldsymbol{i}A] \\
b(\boldsymbol{i}A) &= (c^2, -\boldsymbol{j}, -b^2, -c^2, \boldsymbol{j}, b^2) = [1/3, \boldsymbol{j}A] \\
b(\boldsymbol{j}A) &= (a^2, -1, -a, -a^2, 1, a) = [1/3, A] \\
b(\boldsymbol{k}A) &= (d, \boldsymbol{k}, -c, -d, -\boldsymbol{k}, c) = [5/6, \boldsymbol{k}A].
\end{aligned}$$

This gives the following set of pictures:



subject to the relations:



In these, we see that a, b, c, & d project to 3cycles. & Q_8 projects to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ — the klein 4 group.

$a \mapsto (234)$

$b \mapsto (123)$

$c \mapsto (143)$

$d \mapsto (142)$

8. The binary octahedral group

The binary octahedral group, $\widetilde{\Sigma_4}$ is a 2-fold extension of the permutation group Σ_4. It is given via the presentation

$$\widetilde{\Sigma_4} = \langle a, f : a^3 = f^4 = (af)^2 \rangle.$$

$S = (-1, -j, 1, j)$. $a = \frac{1}{2}(1 + i + j + k)$

$f = (1 + i)/\sqrt{2}$

~~pose the binary octahedral group into (cyclically) ordered cosets of the subgroup~~, $S = (-1, -\boldsymbol{j}, 1, \boldsymbol{j})$. Of course the cosets $\boldsymbol{i}S$, aS, $a\boldsymbol{i}S$, bS and $b\boldsymbol{i}S$ are as before. In $\widetilde{\Sigma_4}$, we have the six additional cosets:

$$fS = ((-1 - \boldsymbol{i}), (-\boldsymbol{j} - \boldsymbol{k}), (1 + \boldsymbol{i}), (\boldsymbol{j} + \boldsymbol{k}))/\sqrt{2},$$

$$f\boldsymbol{i}S = ((1 - \boldsymbol{i}), (\boldsymbol{j} - \boldsymbol{k}), (-1 + \boldsymbol{i}), (-\boldsymbol{j} + \boldsymbol{k}))/\sqrt{2},$$

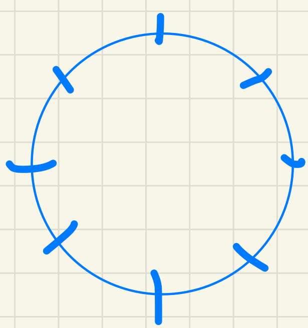
$$faS = ((-\boldsymbol{i} - \boldsymbol{k}), (\boldsymbol{i} - \boldsymbol{k}), (\boldsymbol{i} + \boldsymbol{k}), (-\boldsymbol{i} + \boldsymbol{k}))/\sqrt{2},$$

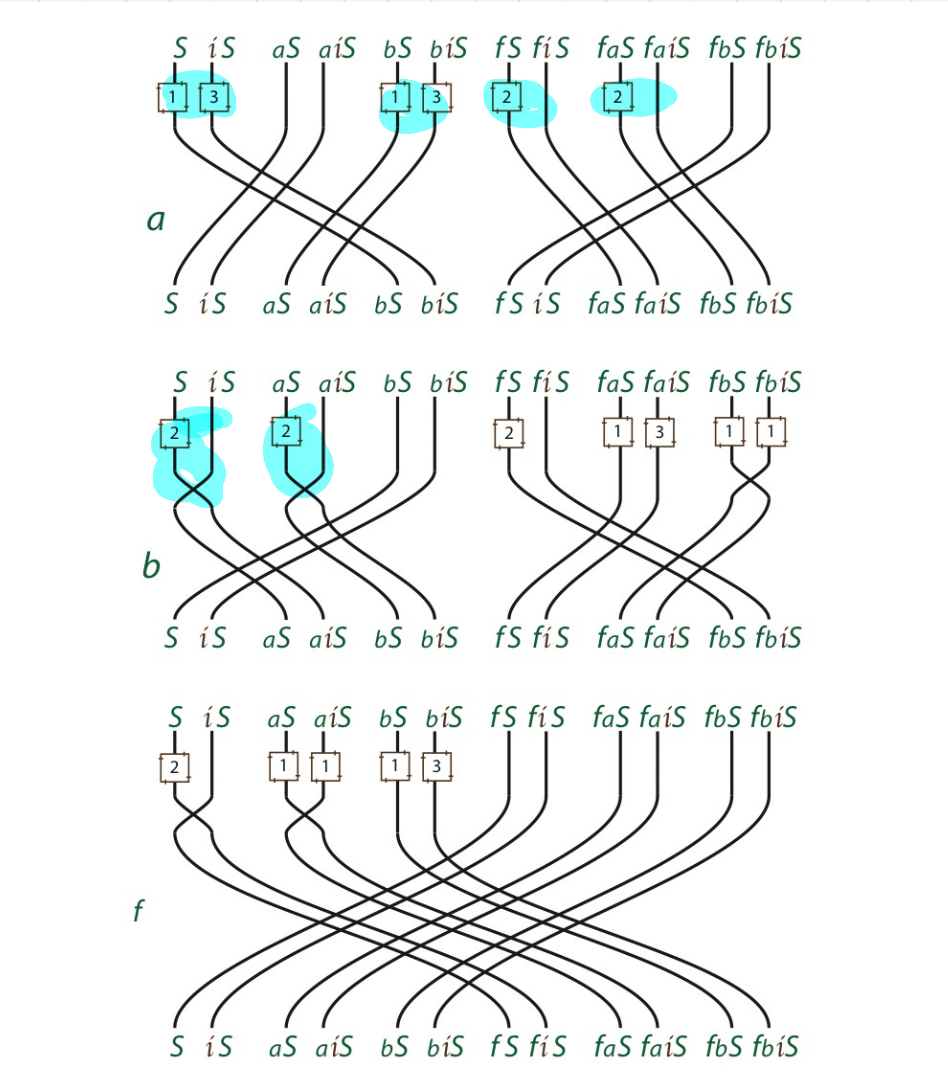
$$fa\boldsymbol{i}S = ((1 - \boldsymbol{j}), (1 + \boldsymbol{j}), (-1 + \boldsymbol{j}), (-1 - \boldsymbol{j}))/\sqrt{2},$$

$$fbS = ((-\boldsymbol{i} - \boldsymbol{j}), (1 - \boldsymbol{k}), (\boldsymbol{i} + \boldsymbol{j}), (-1 + \boldsymbol{k}))/\sqrt{2},$$

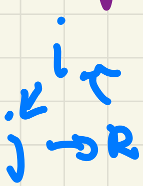
and

$$fb\boldsymbol{i}S = ((1 + \boldsymbol{k}), (-\boldsymbol{i} + \boldsymbol{j}), (-1 - \boldsymbol{k}), (\boldsymbol{i} - \boldsymbol{j}))/\sqrt{2}.$$





WIP: Rewrite in terms of quaternions.



$$\frac{(i-j)}{\sqrt{2}}\frac{(i-j)}{\sqrt{2}} = \frac{i^2 + ij - ij + j^2}{2}$$

$$= -1$$

Define a subgroup $C \leq \tilde{\Sigma}_4$

$$C = \left(-1, \frac{i-j}{\sqrt{2}}, 1, \frac{j-i}{\sqrt{2}}\right)$$

Then

$$aC = \left(-a, \frac{i-k}{\sqrt{2}}, a, \frac{k-i}{\sqrt{2}}\right)$$

$$a^2C = \left(-a^2, j-k, a^2, k-j\right)$$

$$D = \left[C, aC, a^2C\right]$$ is isom. to the dicyclic group Dic_3

given by the presentation

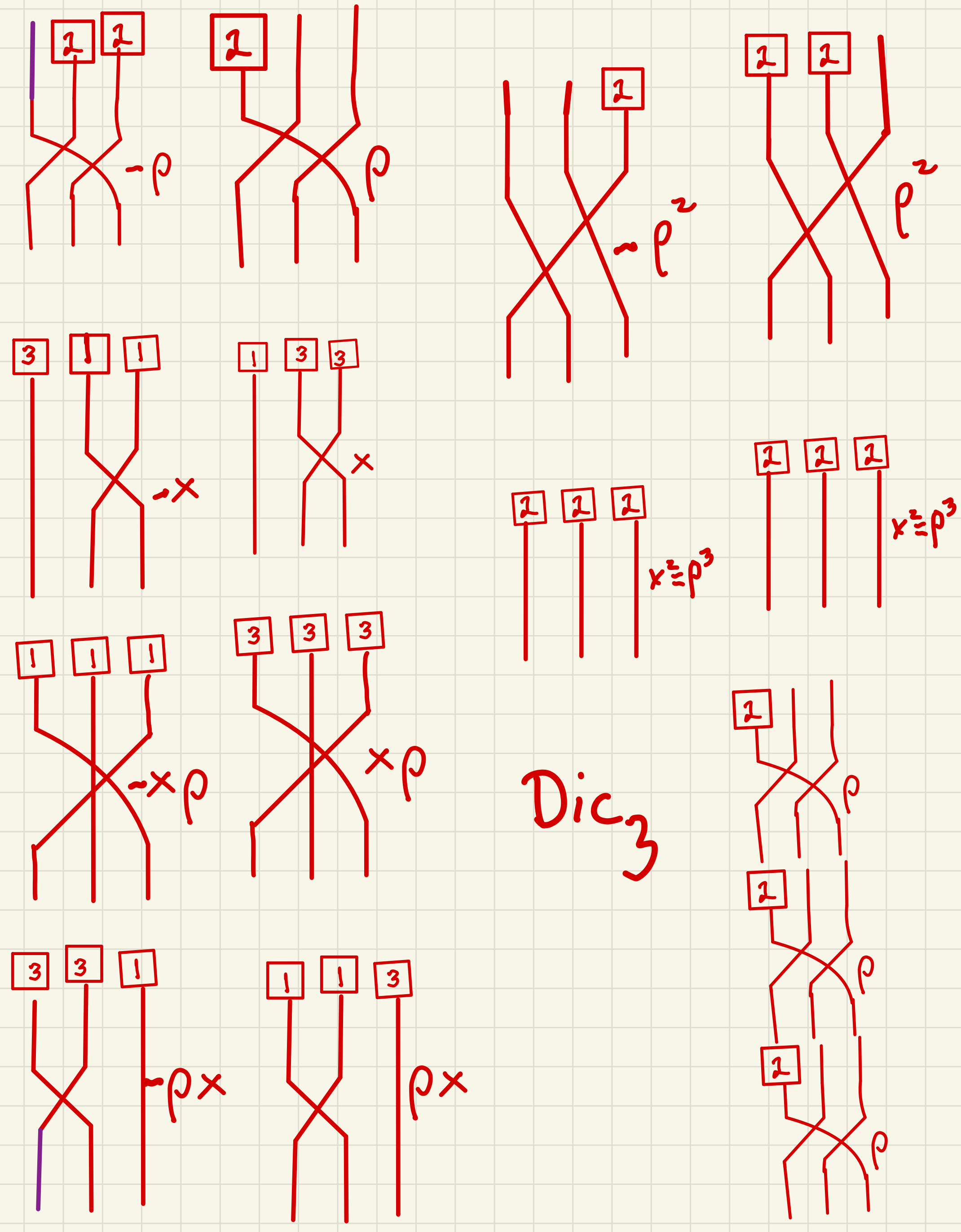
-1

$\wr\wr$

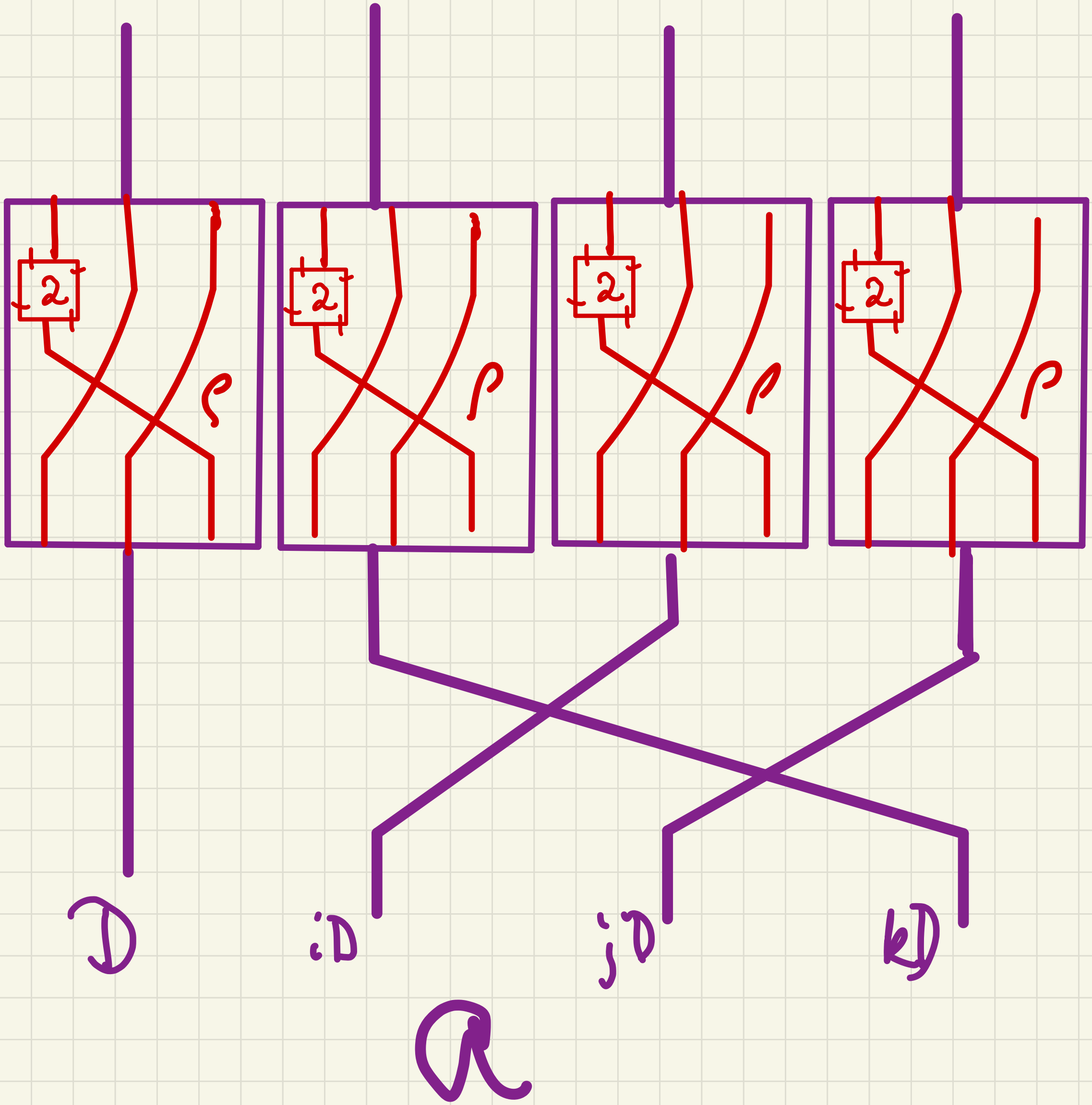
$$\mathrm{Dic}_n = \langle \rho, x : \rho^{2n} = 1,\ x^2 = \rho^n,\ \rho x = x\rho^{-1} \rangle.$$

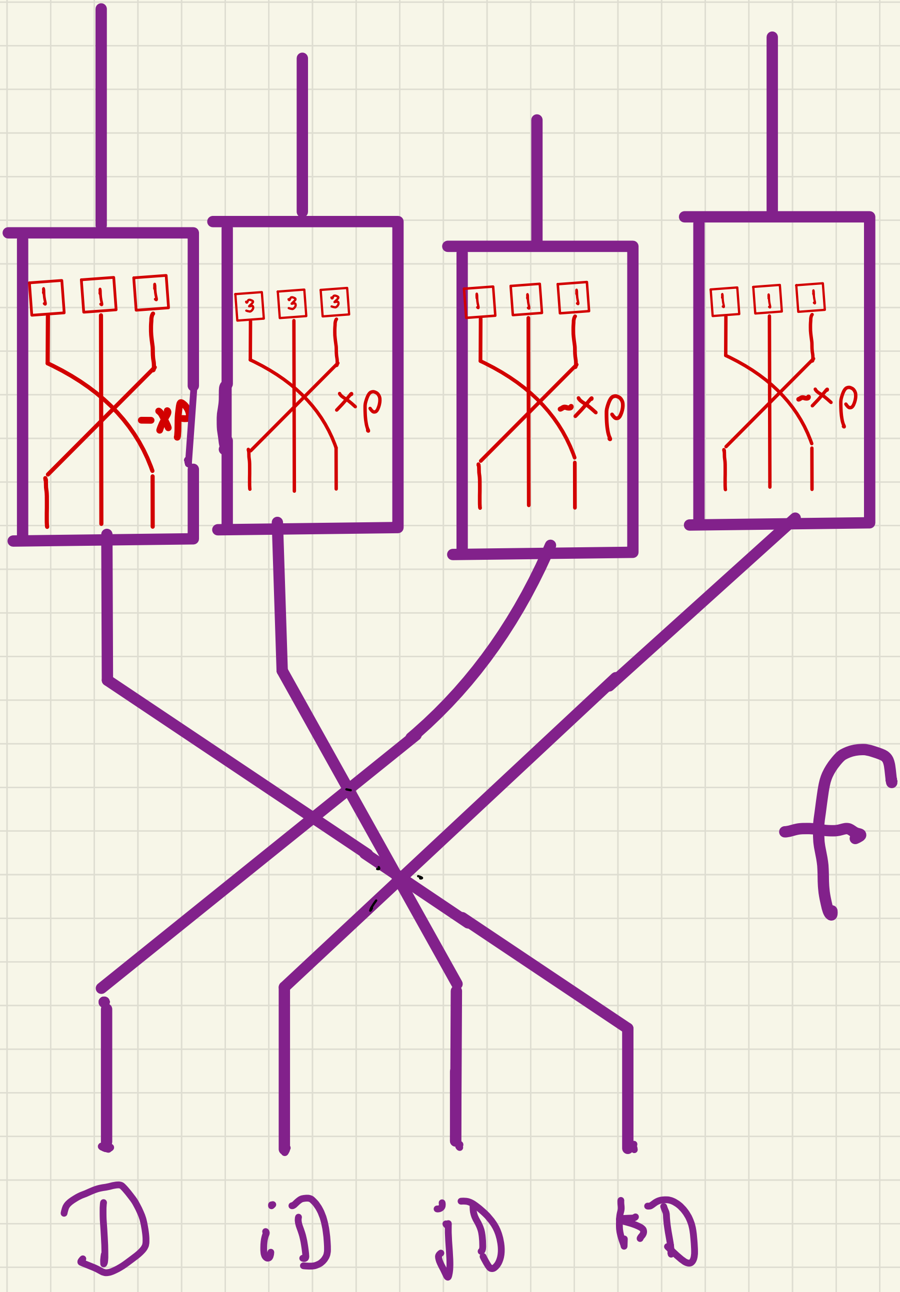
Dic_3

ρ has order 6



2 2
$-\rho$
2
ρ
2
$-\rho^2$
2 2
ρ^2
3 1 1
$-x$
1 3 3
x
2 2 2
$x^2 = \rho^3$
2 2 2
$x^2 = \rho^3$
1 1 1
$-x\rho$
3 3 3
$x\rho$
Dic_3
2
ρ
2
ρ
2
ρ
3 3 1
$-\rho x$
1 1 3
ρx


2
2
2
2
ρ
ρ
ρ
ρ
D
iD
jD
kD
α



1 1 1
-×ρ
3 3 3
×ρ
1 1 1
-×ρ
1 1 1
-×ρ
f
D
iD
jD
kD

$$\mathbb{Z}/6 \subset \tilde{A}_4 \subset \tilde{A}_5$$

$$Q_8 \subset \tilde{A}_4$$

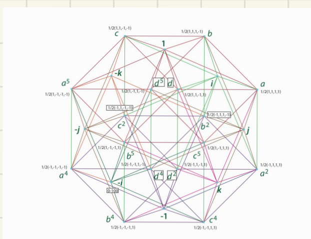
$\tilde{A}_4$: binary tetrahedral group

order 24

$\tilde{A}_5$: binary Icosahedral group

order 120

So have a 5 string picture for $\tilde{A}_5$ with "beads"



That's my story, and
I'm sticking to it!

Thank you!