Online Supplementary Material to

"Portfolio Selection with Mental Accounts: An Equilibrium Model with Endogenous Risk Aversion"

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The online supplementary material to the paper "Portfolio Selection with Mental Accounts: An Equilibrium Model with Endogenous Risk Aversion" published in the *Journal of Banking and Finance* involves four online appendices. Online Appendices A and B summarize, respectively, the notation and the implications of our model. Online Appendix C contains our proofs. Online Appendix D adds a risk-free asset to our model.

Online Appendix A: summary of notation

Panels A1, A2, A3, A4, and A5 refer to, respectively, assets, portfolios, the MV frontier, the MV agent, and the DMSS agent. Panels A6, A7, and A8 refer to single-agent economies with, respectively, an MV agent, a DMSS agent and a single account, and a DMSS agent with multiple accounts. Panel I refers to two-agent economies with MV and DMSS agents. Panel A9 refers to equilibrium expected returns.

Panel A1. Assets

J	Number of assets
$\mathbb{J}$	Set of assets
$(\boldsymbol{d},\boldsymbol{S})$	Vector of expected asset payoffs and variance-covariance matrix of asset payoffs
$\boldsymbol{p}$	Asset price vector

Panel A2. Portfolios

$\boldsymbol{q}$	Portfolio (quantities of asset shares)
$(d_{\boldsymbol{q}}, s_{\boldsymbol{q}})$	Expected payoff and payoff standard deviation of portfolio $\boldsymbol{q}$
$v_{1-\alpha,\boldsymbol{q}}$	Payoff VaR at confidence level $1-\alpha$ of portfolio $\boldsymbol{q}$
$p_{\boldsymbol{q}}$	Price of portfolio $\boldsymbol{q}$
$(r_{\boldsymbol{q}}, \sigma_{\boldsymbol{q}})$	Expected return and return standard deviation of portfolio $\boldsymbol{q}$
$V_{1-\alpha,\boldsymbol{q}}$	Return VaR at confidence level $1-\alpha$ of portfolio $\boldsymbol{q}$

Panel A3. MV frontier

A, B, C, D	Values used to characterize the MV frontier
$\boldsymbol{q}_{d,p}$	Portfolio on the MV frontier with expected payoff d and price p
$(\boldsymbol{q}_{A/C,1},\boldsymbol{q}_{B/A,1})$	Portfolios that span the MV frontier

Panel A4. MV agent

$\boldsymbol{q}_0$	Vector of asset endowments
γ_0	Risk aversion coefficient
$\boldsymbol{q}_0^*$	Optimal portfolio
(d_0^*, s_0^*)	Expected payoff and payoff standard deviation of the optimal portfolio

Panel A5. DMSS agent

M	Number of accounts
$\mathbb{M}$	Set of accounts
$\boldsymbol{q}_m$	Vector of asset endowments in account m
(α_m, H_m)	Threshold probability and return of account m
$(\underline{\alpha}, H_{\alpha_m})$	Bounds on thresholds for the existence of the optimal portfolio within account m
$\boldsymbol{q}_m^*$	Optimal portfolio within account m
γ_m^*	Implied risk aversion coefficient of the optimal portfolio within account m
(d_m^*, s_m^*)	Expected payoff and payoff standard deviation of the optimal portfolio within account m
$\boldsymbol{q}_a$	Vector of aggregate asset endowments
$\boldsymbol{q}_a^*$	Aggregate portfolio
γ_a^*	Implied risk aversion coefficient of the aggregate portfolio
(d_a^*, s_a^*)	Expected payoff and payoff standard deviation of the aggregate portfolio

Panel A6. Single-agent economies with an MV agent

$[(\boldsymbol{d},\boldsymbol{S}),(\boldsymbol{q}_0,\gamma_0)]$	Economy
Γ_0	Set of risk aversion coefficients of the MV agent
Θ_0	Set used to determine equilibria
θ_0	Element of Θ_0
$(\boldsymbol{p}^*,\boldsymbol{q}_0^*) = (\boldsymbol{p}_{\theta_0,\gamma_0},\boldsymbol{q}^*_{0,\theta_0,\gamma_0})$	Equilibrium

Panel A7. Single-agent economies with a DMSS agent and a single account

$[(\boldsymbol{d},\boldsymbol{S}),(\boldsymbol{q}_1,H_1,\alpha_1)]$	Economy
Γ_1	Set of implied risk aversion coefficients of the DMSS agent's optimal portfolio within account 1
γ_1	Element of Γ_1
$(\boldsymbol{p}^*,\boldsymbol{q}_1^*) = (\boldsymbol{p}_{\gamma_1},\boldsymbol{q}^*_{1,\gamma_1})$	Equilibrium

Panel A8. Single-agent economies with a DMSS agent and multiple accounts

$[(\boldsymbol{d},\boldsymbol{S}),\{(\boldsymbol{q}_m,H_m,\alpha_m)\}_{m\in\mathbb{M}}]$	Economy
Γ_a	Set of implied risk aversion coefficients of the DMSS agent's aggregate portfolio
γ_a	Element of Γ_a
Θ_a	Set used to determine equilibria
θ_a^*	Element of Θ_a
$(\boldsymbol{p}^*,\{\boldsymbol{q}_m^*\}_{m\in\mathbb{M}}) = (\boldsymbol{p}_{\theta_a^*,\gamma_a},\{\boldsymbol{q}^*_{m,\theta_a^*,\gamma_a}\}_{m\in\mathbb{M}})$	Equilibrium

Panel A9. Two-agent economies with MV and DMSS agents

$[(\boldsymbol{d},\boldsymbol{S}),(\boldsymbol{q}_0,\gamma_0),\{(\boldsymbol{q}_m,H_m,\alpha_m)\}_{m\in\mathbb{M}}]$	Economy
$\Gamma_{0,a}$	Set of implied risk aversion coefficients of the DMSS agent's aggregate portfolio
$\gamma_{0,a}$	Element of $\Gamma_{0,a}$
$\varphi_{0,a}$	Value that depends on γ_0 and $\gamma_{0,a}$
$\Theta_{0,a}$	Set used to determine equilibria
$\theta_{0,a}^*$	Element of $\Theta_{0,a}$
$[\boldsymbol{p}^*,(\boldsymbol{q}_0^*,\{\boldsymbol{q}_m^*\}_{m\in\mathbb{M}})] = [\boldsymbol{p}_{\theta_{0,a}^*,\varphi_{0,a}},(\boldsymbol{q}^*_{0,\theta_{0,a}^*,\varphi_{0,a}},\{\boldsymbol{q}^*_{m,\theta_{0,a}^*,\varphi_{0,a}}\}_{m\in\mathbb{M}})]$	Equilibrium

Panel A10. Equilibrium expected returns, betas, zero-covariance portfolio, and market portfolio

r_j^*	Expected return of asset j
β_j^*	Beta of asset j
$\boldsymbol{q}_{zc}$	Zero-covariance portfolio
$r^*_{\boldsymbol{q}_{zc}}$	Expected return of the zero-covariance portfolio
$\mathbf{1}$	Market portfolio
$r^*_{\mathbf{1}}$	Expected return of the market portfolio

Online Appendix B: summary of the implications of our model

Panel B1 compares (i) a single-agent economy with an MV agent, (ii) a single-agent economy with a DMSS agent, and (iii) a two-agent economy with MV and DMSS agents when the moments of the distribution of asset returns are endogenous. Panel B2 compares portfolio selection, risk aversion, and asset pricing with accounts in the cases where the moments of the distribution of asset returns are either endogenous or exogenous.

Panel B1. Comparing economies with either an MV agent, a DMSS agent, or both agents

Economy	Primitives		Risk aversion coefficient of the MV agent	Implied risk aversion coefficient of the DMSS agent's aggregate portfolio	Vector of relative asset prices (given the primitives of the economy)
	MV agent	DMSS agent			
Single-agent economy with an MV agent	Risk aversion coefficient	–	Exogenous	–	Unique
Single-agent economy with a DMSS agent	–	Thresholds	–	Endogenous interval	Depends on the implied risk aversion coefficient of the DMSS agent's aggregate portfolio
Two-agent economy with MV and DMSS agents	Risk aversion coefficient	Thresholds	Exogenous	Endogenous interval	Depends on the implied risk aversion coefficient of the DMSS agent's aggregate portfolio

Panel B2. Comparing portfolio selection, risk aversion, and asset pricing with accounts when the moments of the distribution of asset returns are either endogenous or exogenous

Moments of the distribution of asset returns	Composition of the DMSS agent's optimal portfolios within accounts (given the thresholds)	Implied risk aversion coefficient of the DMSS agent's aggregate portfolio (given the thresholds)	Effect of the size of the thresholds of any given account on the composition of the DMSS agent's optimal portfolios within accounts
Endogenous (depend on the thresholds and the implied risk aversion coefficient of the DMSS agent's aggregate portfolio)	Depends on the thresholds and the implied risk aversion of the DMSS agent's aggregate portfolio	Endogenous interval	Affects the composition of the DMSS agent's optimal portfolios within all accounts
Exogenous	Unique	Unique	Affects the composition of the DMSS agent's optimal portfolio within only that account

Online Appendix C: proofs of theoretical results

Proof that portfolios on the MV frontier satisfy Eq. (10). Fix any $(d, p) \in \mathbb{R} \times \mathbb{R}_{++}$. Portfolio $\boldsymbol{q}_{d,p}$ solves:

$$\min_{\boldsymbol{q} \in \mathbb{R}^J} \frac{1}{2} \boldsymbol{q}' \boldsymbol{S} \boldsymbol{q} \tag{C.1}$$

$$\begin{aligned} s.t. \quad \boldsymbol{q}'\boldsymbol{d} &= d \quad &\text{(C.2)} \\ \boldsymbol{q}'\boldsymbol{p} &= p. \quad &\text{(C.3)} \end{aligned}$$

First-order conditions for $\boldsymbol{q}_{d,p}$ to solve problem (C.1) subject to constraints (C.2) and (C.3) are:

$$\begin{aligned} \boldsymbol{S}\boldsymbol{q}_{d,p} - \delta \boldsymbol{d} - \eta \boldsymbol{p} &= \boldsymbol{0}, \quad &\text{(C.4)} \\ \boldsymbol{q}'_{d,p}\boldsymbol{d} &= d, \quad &\text{(C.5)} \\ \boldsymbol{q}'_{d,p}\boldsymbol{p} &= p, \quad &\text{(C.6)} \end{aligned}$$

where δ and η are Lagrange multipliers associated with these constraints. Eq. (C.4) implies that:

$$\boldsymbol{q}_{d,p} = \delta(\boldsymbol{S}^{-1}\boldsymbol{d}) + \eta(\boldsymbol{S}^{-1}\boldsymbol{p}). \tag{C.7}$$

Premultiplying Eq. (C.7) by $\boldsymbol{d}'$ and using Eq. (C.5) along with the definitions of A and B, we have:

$$d = \delta B + \eta A. \tag{C.8}$$

Similarly, premultiplying Eq. (C.7) by $\boldsymbol{p}'$ and using Eq. (C.6) along with the definitions of A and C, we have:

$$p = \delta A + \eta C. \tag{C.9}$$

It follows from Eqs. (C.8) and (C.9) along with the definition of D that:

$$\delta = \frac{dC - pA}{D} \tag{C.10}$$

and:

$$\eta = \frac{pB - dA}{D}. \tag{C.11}$$

Eqs. (C.7), (C.10), and (C.11) imply that Eq. (10) holds with $\phi_{d,p} = A\delta$. ■

Proof that portfolios on the MV frontier satisfy Eq. (11). Eq. (C.7) implies that:

$$s_{\boldsymbol{q}}^2 = \delta^2 B + 2\delta\eta A + \eta^2 C. \tag{C.12}$$

Using Eqs. (C.10) and (C.11) in Eq. (C.12), we have:

$$s_{\boldsymbol{q}}^2 = \left(\frac{d_{\boldsymbol{q}}C - pA}{D}\right)^2 B + 2\left(\frac{d_{\boldsymbol{q}}C - pA}{D}\right)\left(\frac{pB - d_{\boldsymbol{q}}A}{D}\right) A + \left(\frac{pB - d_{\boldsymbol{q}}A}{D}\right)^2 C. \tag{C.13}$$

It follows from Eq. (C.13) and the definition of D that:

$$s_{\boldsymbol{q}}^2 = \frac{Cd_{\boldsymbol{q}}^2 - 2Ad_{\boldsymbol{q}}p + Bp^2}{D}. \tag{C.14}$$

Using Eq. (C.14) and the definition of D, we have Eq. (11). ■

Proof of Theorem 1. First-order conditions for $\boldsymbol{q}_0^*$ to solve maximization problem (1) subject to constraint (2) are:

$$\begin{aligned} \boldsymbol{d} - \gamma_0 \boldsymbol{S}\boldsymbol{q}_0^* - \lambda \boldsymbol{p} &= \boldsymbol{0} \quad &(C.15)\\ (\boldsymbol{q}_0^*)' \boldsymbol{p} &= \boldsymbol{q}_0' \boldsymbol{p}. \quad &(C.16) \end{aligned}$$

Using Eqs. (C.15) and (C.16), we have:

$$\boldsymbol{q}_0^* = \frac{\boldsymbol{S}^{-1}(\boldsymbol{d} - \lambda \boldsymbol{p})}{\gamma_0} \tag{C.17}$$

where:

$$\lambda = \frac{\boldsymbol{d}'\boldsymbol{S}^{-1}\boldsymbol{p} - \gamma_0 \boldsymbol{q}_0' \boldsymbol{p}}{\boldsymbol{p}'\boldsymbol{S}^{-1}\boldsymbol{p}}. \tag{C.18}$$

Eqs. (C.17) and (C.18) along with the definitions of A, $\boldsymbol{q}_{A/C,1}$, and $\boldsymbol{q}_{B/A,1}$ imply that Eq. (13) holds. ■

For any $0 < \alpha < \overline{\alpha}$ and $p > 0$, let $\boldsymbol{q}_{1-\alpha,p}$ denote the portfolio with minimum return VaR at confidence level $1-\alpha$ among the portfolios with a price of p. For brevity, let $V_{1-\alpha,p}$ denote its return VaR at the confidence level $1-\alpha$. For any $m \in \mathbb{M}$, let d_m^* and s_m^* denote, respectively the expected payoff and payoff standard deviation of the optimal portfolio within account m. The following two lemmas are useful in our proof of Theorem 2.

Lemma 1. *If $0 < \alpha < \overline{\alpha}$ and $p > 0$, then $V_{1-\alpha,p} = -H_\alpha$.*

Proof. Suppose that $0 < \alpha < \overline{\alpha}$ and $p > 0$. Using Eq. (6), portfolio $\boldsymbol{q}_{1-\alpha,p}$ is on the MV frontier. Using Eq. (12). that $r_{\boldsymbol{q}_{1-\alpha,p}}$ solves:

$$\min_{r \in \mathbb{R}} \quad z_\alpha \sqrt{1/C + \frac{(1 + r - A/C)^2}{D/C}} - r. \tag{C.19}$$

A first-order condition for $r_{\boldsymbol{q}_{1-\alpha,p}}$ to solve problem (C.19) is:

$$\frac{z_\alpha (1 + r_{\boldsymbol{q}_{1-\alpha,p}} - A/C)/(D/C)}{\sqrt{1/C + (1 + r_{\boldsymbol{q}_{1-\alpha,p}} - A/C)^2/(D/C)}} - 1 = 0. \tag{C.20}$$

It follows from Eq. (C.20) that:

$$r_{\boldsymbol{q}_{1-\alpha,p}} = \sqrt{\frac{D^2/C^3}{z_\alpha^2 - D/C}} + A/C - 1. \tag{C.21}$$

Using Eqs. (11) and (C.21), we have:

$$\sigma_{\boldsymbol{q}_{1-\alpha,p}} = \sqrt{\frac{z_\alpha^2/C}{z_\alpha^2 - D/C}}. \tag{C.22}$$

Eqs. (7), (15), (C.21), and (C.22) imply that $V_{1-\alpha,\boldsymbol{q}_{1-\alpha,p}} = -H_\alpha$. ■

Lemma 2. *Fix any account $m \in \mathbb{M}$. If $\alpha_m < \overline{\alpha}$ and $H_m \leq H_{\alpha_m}$, then the optimal portfolio within account m, $\boldsymbol{q}_m^*$, is on the MV frontier. Also, $d_{\boldsymbol{q}_m^*} > p_{\boldsymbol{q}_m}(A/C)$ and $v_{1-\alpha_m,\boldsymbol{q}_m^*} = -p_{\boldsymbol{q}_m}(1 + H_m)$.*

Proof. Fix any account $m \in \mathbb{M}$. Suppose that $\alpha_m < \overline{\alpha}$ and $H_m \leq H_{\alpha_m}$. First, we show that portfolio $\boldsymbol{q}_m^*$ is on the MV frontier. Assume by way of a contradiction that it is not. Then, there exists a portfolio $\widehat{\boldsymbol{q}}$ with $d_{\widehat{\boldsymbol{q}}} = d_{\boldsymbol{q}_m^*}$ and $s_{\widehat{\boldsymbol{q}}} < s_{\boldsymbol{q}_m^*}$. Let $\boldsymbol{q}_\zeta \equiv \zeta \widehat{\boldsymbol{q}} + (1-\zeta)\boldsymbol{q}$ where $\zeta > 0$ is arbitrarily small and $\widehat{\boldsymbol{q}}$ is a portfolio with $\widehat{d} \equiv d_{\widehat{\boldsymbol{q}}} > d_{\boldsymbol{q}}$ and $p_{\widehat{\boldsymbol{q}}} = p_{\boldsymbol{q}_m}$. Note that $p_{\boldsymbol{q}_\zeta} = p_{\boldsymbol{q}_m^*}$, $d_{\boldsymbol{q}_\zeta} > d_{\boldsymbol{q}_m^*}$, and $s_{\boldsymbol{q}_\zeta} < s_{\boldsymbol{q}_m^*}$. Since $p_{\boldsymbol{q}_\zeta} = p_{\boldsymbol{q}_m^*} > 0$, it follows that $r_{\boldsymbol{q}_\zeta} > r_{\boldsymbol{q}_m^*}$ and $\sigma_{\boldsymbol{q}_\zeta} < \sigma_{\boldsymbol{q}_m^*}$. Using Eq. (7) along with the facts that $r_{\boldsymbol{q}_\zeta} > r_{\boldsymbol{q}_m^*}$, $\sigma_{\boldsymbol{q}_\zeta} < \sigma_{\boldsymbol{q}_m^*}$, and $z_{\alpha_m} > 0$, we have $V_{1-\alpha_m,\boldsymbol{q}_\zeta} < V_{1-\alpha_m,\boldsymbol{q}_m^*}$. Equality $p_{\boldsymbol{q}_\zeta} = p_{\boldsymbol{q}_m^*}$ along with inequalities $d_{\boldsymbol{q}_\zeta} > d_{\boldsymbol{q}_m^*}$ and $V_{1-\alpha_m,\boldsymbol{q}_\zeta} < V_{1-\alpha_m,\boldsymbol{q}_m^*}$ contradict the fact that $\boldsymbol{q}_m^*$ is the optimal portfolio within account m. This completes the first part of our proof.

Second, we show that $d_{\boldsymbol{q}_m^*} > p_{\boldsymbol{q}_m}(A/C)$. For notational brevity, let $d \equiv d_{\boldsymbol{q}_m^*}$ and $p \equiv p_{\boldsymbol{q}_m}$. It follows from Eq. (7) that:

$$V_{1-\alpha_m,\boldsymbol{q}_m^*} = z_{\alpha_m}\left\{\frac{\sqrt{p^2(1/C) + [d - p(A/C)]^2/(D/C)}}{p}\right\} - \frac{d}{p} + 1. \tag{C.23}$$

Using Eq. (C.23), we have:

$$\frac{\partial V_{1-\alpha_m,\boldsymbol{q}_m^*}}{\partial d} = \left\{\frac{z_{\alpha_m}[d - p(A/C)]/(D/C)}{\sqrt{p^2(1/C) + [d - p(A/C)]^2/(D/C)}} - 1\right\}\frac{1}{p}. \tag{C.24}$$

Since $z_{\alpha_m} > 0$, Eq. (C.24) implies that if $d_{\boldsymbol{q}_m^*} \leq p_{\boldsymbol{q}_m}(A/C)$, then $\partial V_{1-\alpha_m,\boldsymbol{q}_m^*}/\partial d < 0$. Hence, we have $d_{\boldsymbol{q}_m^*} > p_{\boldsymbol{q}_m}(A/C)$. This completes the second part of our proof.

Third, we show that $v_{1-\alpha_m,\boldsymbol{q}_m^*} = -p_{\boldsymbol{q}_m}(1+H_m)$. Fix a portfolio $\boldsymbol{q}$ with $p_{\boldsymbol{q}} > 0$. Using the definition of $\widetilde{r}_{\boldsymbol{q}}$ in constraint (5), we have:

$$P[\widetilde{d}_{\boldsymbol{q}} \leq (1+H_m)\,p_{\boldsymbol{q}}] \leq \alpha_m. \tag{C.25}$$

It follows from Eq. (C.25) that a portfolio $\boldsymbol{q}$ with $p_{\boldsymbol{q}} > 0$ satisfies constraint (5) if and only if:

$$v_{1-\alpha,\boldsymbol{q}} \leq -(1+H_m)\,p_{\boldsymbol{q}}. \tag{C.26}$$

Eq. (C.26) implies that $v_{1-\alpha_m,\boldsymbol{q}_m^*} \leq -p_{\boldsymbol{q}_m}(1+H_m)$. Assume by way of a contradiction that $v_{1-\alpha_m,\boldsymbol{q}_m^*} < -p_{\boldsymbol{q}_m}(1+H_m)$. Let $\boldsymbol{q}_\xi \equiv \xi\widehat{\widehat{\boldsymbol{q}}} + (1-\xi)\boldsymbol{q}_m^*$ where $\xi > 0$ is arbitrarily small and $\widehat{\widehat{\boldsymbol{q}}}$ is a portfolio with $\widehat{\widehat{d}} \equiv d_{\widehat{\widehat{\boldsymbol{q}}}} > d_{\boldsymbol{q}_m^*}$ and $p_{\widehat{\widehat{\boldsymbol{q}}}} = p_{\boldsymbol{q}_m}$. Note that $p_{\boldsymbol{q}_\xi} = p_{\boldsymbol{q}_m}$, $d_{\boldsymbol{q}_\xi} > d_{\boldsymbol{q}_m^*}$, and $s_{\boldsymbol{q}_\xi} < s_{\boldsymbol{q}_m^*}$. Since $p_{\boldsymbol{q}_\xi} = p_{\boldsymbol{q}_m} > 0$, it follows that $r_{\boldsymbol{q}_\xi} > r_{\boldsymbol{q}_m^*}$ and $\sigma_{\boldsymbol{q}_\xi} < \sigma_{\boldsymbol{q}_m^*}$. Using Eq. (7) along with the facts that $r_{\boldsymbol{q}_\xi} > r_{\boldsymbol{q}_m^*}$, $\sigma_{\boldsymbol{q}_\xi} < \sigma_{\boldsymbol{q}_m^*}$, and $z_{\alpha_m} > 0$, we have $V_{1-\alpha_m,\boldsymbol{q}_\xi} < V_{1-\alpha_m,\boldsymbol{q}_m^*}$. Equality $p_{\boldsymbol{q}_\xi} = p_{\boldsymbol{q}_m^*}$ along with inequalities $d_{\boldsymbol{q}_\xi} > d_{\boldsymbol{q}_m^*}$ and $V_{1-\alpha_m,\boldsymbol{q}_\xi} < V_{1-\alpha_m,\boldsymbol{q}_m^*}$ contradict the fact that $\boldsymbol{q}_m^*$ is the optimal portfolio within account m. This completes the third part of our proof. ■

Proof of Theorem 2. Fix any account $m \in \mathbb{M}$. First, we show (i). Suppose that $\alpha_m \geq \overline{\alpha}$. Using the definition of z_{α_m} and Eq. (14), we have:

$$0 < z_{\alpha_m} \leq \sqrt{D/C}. \tag{C.27}$$

Fix any expected payoff $d \in \mathbb{R}$. Note that:

$$\frac{[d - p_{\boldsymbol{q}_m}(A/C)]/(D/C)}{\sqrt{p_{\boldsymbol{q}_m}^2(1/C) + [d - p_{\boldsymbol{q}_m}(A/C)]^2/(D/C)}} < \frac{1}{\sqrt{D/C}}. \tag{C.28}$$

Using Eqs. (C.24), (C.27), and (C.28), one can increase a portfolio's expected payoff and decrease a portfolio's return VaR at confidence level $1-\alpha_m$ by moving up along the MV frontier of portfolios with a price of $p_{\boldsymbol{q}_m}$. It follows that the optimal portfolio within account m does not exist.

Suppose now that $\alpha_m < \overline{\alpha}$ and $H_m > H_{\alpha_m}$. Recall that $V_{1-\alpha_m}$ denotes the return VaR at confidence level $1-\alpha_m$ of the portfolio with minimum return VaR at this confidence level among all portfolios with a price of $p_{\boldsymbol{q}_m}$. Using Lemma 1, we have $-H_m < -H_{\alpha_m} = V_{1-\alpha_m}$. Hence, there exists no portfolio $\boldsymbol{q}$ with $p_{\boldsymbol{q}} = p_{\boldsymbol{q}_m}$ that meets constraint (C.26). Therefore, the optimal portfolio within account m does not exist. This completes our proof of part (i).

Second, we show part (ii). Suppose that $\alpha_m < \overline{\alpha}$ and $H_m \leq H_{\alpha_m}$. Lemma 2 and Eq. (11) imply that:

$$d_m^* = p_{\boldsymbol{q}_m}(A/C) + \sqrt{(D/C)[(s_m^*)^2 - p_{\boldsymbol{q}_m}^2(1/C)]}. \tag{C.29}$$

Using Eqs. (6) and (C.29) along with Lemma 2, we have:

$$z_{\alpha_m} s_{\boldsymbol{q}_m^*} - p_{\boldsymbol{q}_m}(A/C) - \sqrt{(D/C)\left[s_{\boldsymbol{q}_m^*}^2 - p_{\boldsymbol{q}_m}^2(1/C)\right]} = -(1+H_m)p_{\boldsymbol{q}_m}. \tag{C.30}$$

It follows from Eq. (C.30) that:

$$K_7(s_{\boldsymbol{q}_m^*}^2) + K_8(s_{\boldsymbol{q}_m^*}) + K_9 = 0, \tag{C.31}$$

where $K_7 \equiv z_{\alpha_m}^2 - D/C$, $K_8 \equiv -2z_{\alpha_m}(A/C - H_m - 1)\,p_{\boldsymbol{q}_m}$, and $K_9 \equiv [(A/C - H_m - 1)^2 + D/C^2]p_{\boldsymbol{q}_m}^2$. Using Eq. (C.31), we have:

$$s_{\boldsymbol{q}_m^*} = \left\{\frac{z_{\alpha_m}(A/C - H_m - 1) \pm \sqrt{(D/C)[(A/C - H_m - 1)^2 - (z_{\alpha_m}^2 - D/C)/C]}}{z_{\alpha_m}^2 - D/C}\right\} p_{\boldsymbol{q}_m}. \tag{C.32}$$

It follows from Eq. (15) that $H_{\alpha_m} < A/C - 1$. Noting that $H_m \leq H_{\alpha_m} < A/C - 1$, we have $A/C - H_m - 1 > 0$. Using the fact that $\alpha_m < \overline{\alpha}$ and Eq. (14), we obtain $z_{\alpha_m}^2 - D/C > 0$. Since $A/C - H_m - 1 > 0$, $z_{\alpha_m}^2 - D/C > 0$, and $\boldsymbol{q}_m^*$ solves maximization problem (3) subject to constraints (4) and (C.26), Eq. (C.32) implies that:

$$s_m^* = \left\{\frac{z_{\alpha_m}(A/C - H_m - 1) + \sqrt{(D/C)[(A/C - H_m - 1)^2 - (z_{\alpha_m}^2 - D/C)/C]}}{z_{\alpha_m}^2 - D/C}\right\} p_{\boldsymbol{q}_m}. \tag{C.33}$$

Using Eqs. (C.7), (C.11), and (C.10) along with the definition of $\boldsymbol{q}_{A/C,1}$ and $\boldsymbol{q}_{B/A,1}$, we have:

$$\boldsymbol{q}_m^* = \left(\frac{d_m^* AC - p_{\boldsymbol{q}_m} A^2}{D}\right)\boldsymbol{q}_{B/A,1} + \left(\frac{p_{\boldsymbol{q}_m} BC - d_m^* AC}{D}\right)\boldsymbol{q}_{A/C,1} \tag{C.34}$$

It follows from Eq. (C.34), the definition of D, and elementary algebra that Eq. (16) holds where:

$$\gamma_m^* = \frac{D/C}{d_m^* - p_{\boldsymbol{q}_m}(A/C)}. \tag{C.35}$$

The fact that γ_m^* is the implied risk aversion coefficient of the optimal portfolio within account m follows from Eqs. (13) and (16). This completes our proof of part (ii). ■

Proof of Theorem 3. Suppose that $\alpha_m < \overline{\alpha}$ and $H_m \leq H_{\alpha_m}$ for any account $m \in \mathbb{M}$. Using (ii) of Theorem 2, Eq. (16) holds for any account $m \in \mathbb{M}$. Summing Eq. (16) across all elements of $\mathbb{M}$, we have Eq. (17) where $\gamma_a^* = \left[\sum_{m\in\mathbb{M}}(1/\gamma_m^*)\right]^{-1}$. Using Eq. (**??**), we have $1/\gamma_m^* = \frac{d_m^* - p_{q_m}(A/C)}{D/C}$. It follows that γ_a^* is given by Eq. (18). The fact that γ_a^* is the implied risk aversion coefficient of the DMSS agent's aggregate portfolio follows from Eqs. (13) and (17). ■

Proof of Theorem 4. Fix any economy $[(\boldsymbol{d},\boldsymbol{S}),(\boldsymbol{q}_0,\gamma_0)]$ with $\gamma_0 \in \Gamma_0$. Also, fix any $\theta_0 \in \Theta_0$. Suppose that $\boldsymbol{p} = \boldsymbol{p}_{\theta_0,\gamma_0}$.[C.1] It follows from the definition of $\boldsymbol{q}_{B/A,1}$ that:

$$A\boldsymbol{q}_{B/A,1} = \boldsymbol{S}^{-1}\boldsymbol{d}. \tag{C.36}$$

Using Eqs. (13) and (C.36), we have:

$$\boldsymbol{q}_0^* = \left(p_{\mathbf{1}} - \frac{A}{\gamma_0}\right)\boldsymbol{q}_{A/C,1} + \frac{\boldsymbol{S}^{-1}\boldsymbol{d}}{\gamma_0} \tag{C.37}$$

The definitions of A, C, and $\boldsymbol{q}_{A/C,1}$ imply that:

$$A = \theta_0\left(B - \gamma_0 d_{\mathbf{1}}\right), \tag{C.38}$$

$$C = \theta_0^2\left(\gamma_0^2 s_{\mathbf{1}}^2 - 2\gamma_0 d_{\mathbf{1}} + B\right), \tag{C.39}$$

and:

$$\boldsymbol{q}_{A/C,1} = \frac{\boldsymbol{S}^{-1}\boldsymbol{d} \;-\; \gamma_0\mathbf{1}}{\theta_0\left(\gamma_0^2 s_{\mathbf{1}}^2 - 2\gamma_0 d_{\mathbf{1}} + B\right)}. \tag{C.40}$$

It follows from Eqs. (C.37)–(C.40) that:

$$\boldsymbol{q}_0^* = \mathbf{1}. \tag{C.41}$$

Given $\boldsymbol{p}$, $\mathbf{1}$ thus solves the MV agent's portfolio selection problem and asset markets clear. Hence, $(\boldsymbol{p}^*,\boldsymbol{q}_0^*) = (\boldsymbol{p}_{\theta_0,\gamma_0},\boldsymbol{q}^*_{0,\theta_0,\gamma_0})$ is an equilibrium for economy $[(\boldsymbol{d},\boldsymbol{S}),(\boldsymbol{q}_0,\gamma_0)]$. ■

The following lemma is useful in our proof of Theorem 5.

Lemma 3. *Fix any (H_1,α_1,γ_1) with $H_1 > -1$, $\alpha_1 > \underline{\alpha}$, and $\gamma_1 \in \Gamma_1$. If $\boldsymbol{p} = \boldsymbol{p}_{\gamma_1}$, then $\alpha_1 < \overline{\alpha}$.*

Proof. Fix any (H_1,α_1,γ_1) with $H_1 > -1$, $\alpha_1 > \underline{\alpha}$, and $\gamma_1 \in \Gamma_1$. Suppose that $\boldsymbol{p} = \boldsymbol{p}_{\gamma_1}$. The definitions of A, C, and $\boldsymbol{p}_{\gamma_1}$ imply that:

$$A = \frac{\left(d_{\mathbf{1}} - z_{\alpha_1}s_{\mathbf{1}}\right)\left(B - \gamma_1 d_{\mathbf{1}}\right)}{\left(H_1+1\right)\left(d_{\mathbf{1}} - \gamma_1 s_{\mathbf{1}}^2\right)} \tag{C.42}$$

and:

$$C = \frac{\left(d_{\mathbf{1}} - z_{\alpha_1}s_{\mathbf{1}}\right)^2\left(\gamma_1^2 s_{\mathbf{1}}^2 - 2\gamma_1 d_{\mathbf{1}} + B\right)}{\left(H_1+1\right)^2\left(d_{\mathbf{1}} - \gamma_1 s_{\mathbf{1}}^2\right)^2}. \tag{C.43}$$

Using the definition of D along with Eqs. (C.42) and (C.43), we have:

$$D/C = B - \frac{\left(B - \gamma_1 d_{\mathbf{1}}\right)^2}{\gamma_1^2 s_{\mathbf{1}}^2 - 2\gamma_1 d_{\mathbf{1}} + B}. \tag{C.44}$$

First, suppose that $\gamma_1 = \overline{\gamma}_{\alpha_1}$. Using Eqs. (C.42)–(C.44) along with elementary algebra, we have:

$$D/C = z_{\alpha_1}^2 - C\left(A/C - H_1 - 1\right)^2. \tag{C.45}$$

Since $C\left(A/C - H_1 - 1\right)^2 \geq 0$, we have $D/C > z_{\alpha_1}^2$ and thus $\alpha_1 < \overline{\alpha}$. This completes the first part of our proof.

Second, suppose that $0 < \gamma_1 < \overline{\gamma}_{\alpha_1}$. Let $f : \mathbb{R}_{++} \to \mathbb{R}$ be defined by:

$$f(\gamma) = B - \frac{\left(B - \gamma d_{\mathbf{1}}\right)^2}{\gamma^2 s_{\mathbf{1}}^2 - 2\gamma d_{\mathbf{1}} + B} \tag{C.46}$$

[C.1] Noting that (a) $\theta_0 \in \mathbb{R}_{++}$ and (b) $(\boldsymbol{d}-\gamma_0\boldsymbol{S}\mathbf{1}) \in \mathbb{R}^J_{++}$ (since $\gamma_0 \in \Gamma_0$), we have $\boldsymbol{p}_{\theta_0,\gamma_0} \in \mathbb{R}^J_{++}$.

for any $\gamma \in \mathbb{R}_{++}$. Eqs. (C.44) and (C.46) imply that $f(\gamma_1) = D/C$. Using Eq. (C.46), we have:

$$\frac{\partial f(\gamma)}{\partial \gamma} = \frac{2\left(B - \gamma d_{\mathbf{1}}\right)\left(B s_{\mathbf{1}}^2 - d_{\mathbf{1}}^2\right)\gamma}{(\gamma^2 s_{\mathbf{1}}^2 - 2\gamma d_{\mathbf{1}} + B)^2}. \tag{C.47}$$

Since $(B s_{\mathbf{1}}^2 - d_{\mathbf{1}}^2) > 0$, $\gamma > 0$, and $(\gamma^2 s_{\mathbf{1}}^2 - 2\gamma d_{\mathbf{1}} + B) > 0$, Eq. (C.47) implies that $\partial f(\gamma)/\partial \gamma > 0$ if $0 < \gamma < B/d_{\mathbf{1}}$.[C.2] Since $0.5 > \alpha_1 > \underline{\alpha}$, we have $d_{\mathbf{1}}/s_{\mathbf{1}} > z_{\alpha_1} > 0$. It follows that:

$$d_{\mathbf{1}}/s_{\mathbf{1}}^2 > z_{\alpha_1}/s_{\mathbf{1}}. \tag{C.48}$$

Since $B > d_{\mathbf{1}}^2/s_{\mathbf{1}}^2$, we have, we have:

$$B/d_{\mathbf{1}} > d_{\mathbf{1}}/s_{\mathbf{1}}^2. \tag{C.49}$$

Using Eqs. (C.48) and (C.49), we have:

$$B/d_{\mathbf{1}} > z_{\alpha_1}/s_{\mathbf{1}}. \tag{C.50}$$

Since $\gamma_1 < \overline{\gamma}_{\alpha_1} = z_{\alpha_1}/s_{\mathbf{1}}$, we have:

$$B/d_{\mathbf{1}} > \gamma_1. \tag{C.51}$$

It follows that $\partial f(\gamma_1)/\partial \gamma > 0$. Hence, $D/C > z_{\alpha_1}^2$ and thus $\alpha_1 < \overline{\alpha}$. This completes the second part of our proof. ■

Proof of Theorem 5. Fix any economy $[(\boldsymbol{d},\boldsymbol{S}),(\boldsymbol{q}_1,H_1,\alpha_1)]$ with $H_1 > -1$ and $\alpha_1 > \underline{\alpha}$. Suppose that $\gamma_1 \in \Gamma_1$. Let $\boldsymbol{p} = \boldsymbol{p}_{\gamma_1}$.[C.3] Lemma 3 implies that $\alpha_1 < \overline{\alpha}$. Since $\boldsymbol{q}_1 = \mathbf{1}$ and $\boldsymbol{p} = \boldsymbol{p}_{\gamma_1}$, we have:

$$p_{\boldsymbol{q}_1} = p_{\mathbf{1}} = \frac{d_{\mathbf{1}} - z_{\alpha_1} s_{\mathbf{1}}}{H_1 + 1}. \tag{C.52}$$

Using Eq. (C.52), the expected return and return standard deviation of market portfolio **1** are:

$$r_{\mathbf{1}} = \frac{(H_1+1)d_{\mathbf{1}}}{d_{\mathbf{1}} - z_{\alpha_1} s_{\mathbf{1}}} - 1 \tag{C.53}$$

and:

$$\sigma_{\mathbf{1}} = \frac{(H_1+1)s_{\mathbf{1}}}{d_{\mathbf{1}} - z_{\alpha_1} s_{\mathbf{1}}}, \tag{C.54}$$

respectively. Eqs. (7), (C.53), and (C.54) imply that its return VaR at confidence level $1 - \alpha_1$ is:

$$V_{1-\alpha_1,\mathbf{1}} = -H_1. \tag{C.55}$$

Using Lemma 1 with $\alpha = \alpha_1$, we have:

$$V_{1-\alpha_1,\mathbf{1}} \geq -H_{\alpha_1}. \tag{C.56}$$

Eqs. (C.55) and (C.56) imply that $H_1 \leq H_{\alpha_1}$.

Since $\alpha_1 < \overline{\alpha}$ and $H_1 \leq H_{\alpha_1}$, part (ii) of Theorem 2 holds with $m = 1$. It follows from Eqs. (C.42) and (C.43) that:

$$\frac{A}{C} = \left[\frac{(H_1+1)\left(d_{\mathbf{1}} - \gamma_1 s_{\mathbf{1}}^2\right)}{(d_{\mathbf{1}} - z_{\alpha_1} s_{\mathbf{1}})}\right]\left(\frac{B - \gamma_1 d_{\mathbf{1}}}{\gamma_1^2 s_{\mathbf{1}}^2 - 2\gamma_1 d_{\mathbf{1}} + B}\right). \tag{C.57}$$

[C.2] A proof that $(B s_{\mathbf{1}}^2 - d_{\mathbf{1}}^2) > 0$ is as follows. Letting $\boldsymbol{k} \equiv (s_{\mathbf{1}})\boldsymbol{d} - (d_{\mathbf{1}}/s_{\mathbf{1}})\boldsymbol{S}\mathbf{1}$, we have $\boldsymbol{k}'\boldsymbol{S}^{-1}\boldsymbol{k} = B s_{\mathbf{1}}^2 - d_{\mathbf{1}}^2$. Noting that $rank([\boldsymbol{d} \quad \boldsymbol{S}\mathbf{1}]) = 2$, we obtain $\boldsymbol{k} \neq \mathbf{0}$. Since $rank(\boldsymbol{S}^{-1}) = J$ and $\boldsymbol{k} \neq \mathbf{0}$, we have $\boldsymbol{k}'\boldsymbol{S}^{-1}\boldsymbol{k} > 0$. Similarly, a proof that $(\gamma^2 s_{\mathbf{1}}^2 - 2\gamma d_{\mathbf{1}} + B) > 0$ is as follows. Letting $\boldsymbol{l} \equiv \boldsymbol{d} - \gamma \boldsymbol{S}\mathbf{1}$, we have $\boldsymbol{l}'\boldsymbol{S}^{-1}\boldsymbol{l} = \gamma^2 s_{\mathbf{1}}^2 - 2\gamma d_{\mathbf{1}} + B$. Noting that $rank([\boldsymbol{d} \quad \boldsymbol{S}\mathbf{1}]) = 2$, we obtain $\boldsymbol{l} \neq \mathbf{0}$. Since $rank(\boldsymbol{S}^{-1}) = J$ and $\boldsymbol{l} \neq \mathbf{0}$, we have $\boldsymbol{l}'\boldsymbol{S}^{-1}\boldsymbol{l} > 0$.

[C.3] If $H_1 > -1$ and $\alpha_1 > \underline{\alpha}$, then $(H_1 + 1) > 0$ and $(d_{\mathbf{1}} - z_{\alpha_1} s_{\mathbf{1}}) > 0$. Also, if $\gamma_1 \in \Gamma_1$, then $\gamma_1 < \overline{\gamma}_0$, which implies that $(\boldsymbol{d} - \gamma_1 \boldsymbol{S}\mathbf{1}) \in \mathbb{R}_{++}^J$ and $(d_{\mathbf{1}} - \gamma_1 s_{\mathbf{1}}^2) > 0$. Hence, if $H_1 > -1$, $\alpha_1 > \underline{\alpha}$, and $\gamma_1 \in \Gamma_1$, then $\theta_{\gamma_1} \in \mathbb{R}_{++}$ and $\boldsymbol{p}_{\gamma_1} \in \mathbb{R}_{++}^J$.

Using Eqs. (C.43), (C.44), and (C.57) along with elementary algebra, we have:

$$\left(\frac{D}{C}\right)\left[\left(\frac{A}{C}-H_1-1\right)^2-\frac{z_{\alpha_1}^2-D/C}{C}\right]=\left[\frac{(z_{\alpha_1}^2-D/C)\,(H_1+1)\,s_{\mathbf{1}}}{d_{\mathbf{1}}-z_{\alpha_1}s_{\mathbf{1}}}-z_{\alpha_1}\left(\frac{A}{C}-H_1-1\right)\right]^2. \tag{C.58}$$

It follows from Eq. (C.33) with $m=1$ along with Eqs. (C.52) and (C.58) that:

$$s_1^*=s_{\mathbf{1}}. \tag{C.59}$$

Eqs. (C.29), (C.43), (C.44), (C.57), and (C.59) along with elementary algebra imply that:

$$d_1^*=d_{\mathbf{1}}. \tag{C.60}$$

Using Eqs. (C.44), (C.52), (C.57), and (C.60) along with elementary algebra, we obtain:

$$\frac{D/C}{d_1^*-(A/C)\,p_{\boldsymbol{q}_1}}=\gamma_1. \tag{C.61}$$

Using the definition of $\boldsymbol{q}_{A/C,1}$ and $\boldsymbol{p}^*_{\gamma_1}$ as well as Eq. (C.43), we have:

$$\boldsymbol{q}_{A/C,1}=\frac{(H_1+1)\left(d_{\mathbf{1}}-\gamma_1 s_{\mathbf{1}}^2\right)(\boldsymbol{S}^{-1}\boldsymbol{d}-\gamma_1\mathbf{1})}{(d_{\mathbf{1}}-z_{\alpha_1}s_{\mathbf{1}})\,(\gamma_1^2 s_{\mathbf{1}}^2-2\gamma_1 d_{\mathbf{1}}+B)}. \tag{C.62}$$

Eqs. (C.52) and (C.62) imply that:

$$p_{\boldsymbol{q}_1}\boldsymbol{q}_{A/C,1}=\frac{d_{\mathbf{1}}-\gamma_1 s_{\mathbf{1}}^2}{\gamma_1^2 s_{\mathbf{1}}^2+B-2\gamma_1 d_{\mathbf{1}}}(\boldsymbol{S}^{-1}\boldsymbol{d}-\gamma_1\mathbf{1}). \tag{C.63}$$

Also, Eqs. (C.42) and (C.62) imply that:

$$A\boldsymbol{q}_{A/C,1}=\frac{B-\gamma_1 d_{\mathbf{1}}}{\gamma_1^2 s_{\mathbf{1}}^2-2\gamma_1 d_{\mathbf{1}}+B}(\boldsymbol{S}^{-1}\boldsymbol{d}-\gamma_1\mathbf{1}). \tag{C.64}$$

Note that Eq. (C.36) holds. Using Eq. (16) with $m=1$ along with Eqs. (C.36), (C.61), (C.63), and (C.64), we have:

$$\boldsymbol{q}_1^*=\frac{d_{\mathbf{1}}-\gamma_1 s_{\mathbf{1}}^2}{\gamma_1^2 s_{\mathbf{1}}^2-2\gamma_1 d_{\mathbf{1}}+B}(\boldsymbol{S}^{-1}\boldsymbol{d}-\gamma_1\mathbf{1})+\frac{1}{\gamma_1}(\boldsymbol{S}^{-1}\boldsymbol{d})-\frac{B-\gamma_1 d_{\mathbf{1}}}{(\gamma_1^2 s_{\mathbf{1}}^2-2\gamma_1 d_{\mathbf{1}}+B)\gamma_1}(\boldsymbol{S}^{-1}\boldsymbol{d}-\gamma_1\mathbf{1}). \tag{C.65}$$

Eq. (C.65) implies that $\boldsymbol{q}_1^*=\mathbf{1}$. Given $\boldsymbol{p}$, $\mathbf{1}$ thus solves the DMSS agent's portfolio selection problem. Hence, $(\boldsymbol{p}^*,\boldsymbol{q}_1^*)=(\boldsymbol{p}_{\gamma_1},\boldsymbol{q}_{1,\gamma_1}^*)$ is an equilibrium for economy $[(\boldsymbol{d},\boldsymbol{S}),(\boldsymbol{q}_1,H_1,\alpha_1)]$. ■

For any $(m,\gamma_a,\theta_a)\in\mathbb{M}\times\Gamma_a\times\Theta_a$, let $g_m(\theta_a)$ denote the implied risk aversion coefficient of the DMSS agent's optimal portfolio within account m when $\boldsymbol{p}=\boldsymbol{p}^*_{\theta_a,\gamma_a}$. The following two lemmas are useful in our proof of Theorem 6.

Lemma 4. *Suppose that $H_m>-1$ and $\alpha_m>\underline{\alpha}$ for any account $m\in\mathbb{M}$. For any $(m,\gamma_a,\theta_a)\in\mathbb{M}\times\Gamma_a\times\Theta_a$, we have $\partial g_m(\theta_a)/\partial\theta_a>0$.*

Proof. Suppose that $H_m>-1$ and $\alpha_m>\underline{\alpha}$ for any account $m\in\mathbb{M}$. Fix any $(m,\gamma_a,\theta_a)\in\mathbb{M}\times\Gamma_a\times\Theta_a$. Let $\boldsymbol{p}=\boldsymbol{p}_{\theta_a,\gamma_a}$. The definitions of A, C, D, and $\boldsymbol{p}_{\theta_a,\gamma_a}$ imply that:

$$\frac{A}{C}=\frac{B-\gamma_a d_{\mathbf{1}}}{\theta_a(\gamma_a^2 s_{\mathbf{1}}^2-2\gamma_a d_{\mathbf{1}}+B)} \tag{C.66}$$

and:

$$\frac{D}{C} = B - \frac{(B - \gamma_a d_{\mathbf{1}})^2}{\gamma_a^2 s_{\mathbf{1}}^2 - 2\gamma_a d_{\mathbf{1}} + B}. \tag{C.67}$$

Using arguments similar to those in the proof of Lemma 3, we obtain $\alpha_1 < \overline{\alpha}$. Since $\alpha_m \leq \alpha_1$ (by assumption) and $\alpha_1 < \overline{\alpha}$, we have $\alpha_m < \overline{\alpha}$. Using Eq. (15) with $\alpha = \alpha_m$ as well as Eqs. (24), (C.66), and (C.67), we have:

$$(1 + H_m) = (1 + H_{\alpha_m})\,(\theta_a/\overline{\theta}_m). \tag{C.68}$$

Noting that $0 < \theta_a < \overline{\theta}_a \leq \overline{\theta}_m$, we have $0 < \theta_a/\overline{\theta}_m < 1$.[C.4] The facts that $(1 + H_m) > 0$ and $0 < \theta_a/\overline{\theta}_m < 1$ along with Eq. (C.68) imply that $H_m \leq H_{\alpha_m}$. Since $\alpha_m < \overline{\alpha}$ and $H_m \leq H_{\alpha_m}$, the DMSS agent's optimal portfolio within account m exists (see Theorem 2).

Using Eqs. (**??**)–(C.33), its implied risk aversion coefficient is:

$$g_m(\theta_a) \equiv \frac{\sqrt{D/C}}{(\boldsymbol{p}'_{1,\gamma_a}\boldsymbol{q}_m)\sqrt{h_m^2(\theta_a) - 1/K_3}} \tag{C.69}$$

where $\boldsymbol{p}_{1,\gamma_a} = \boldsymbol{d} - \gamma_a \boldsymbol{S1}$ (using the definition of $\boldsymbol{p}_{\theta_a,\gamma_a}$ with $\theta_a = 1$) and:

$$h_m(\theta_a) \equiv \frac{z_{\alpha_m}\left[K_1 - \theta_a\,(H_m+1)\right] + \sqrt{(D/C)\left\{\left[K_1 - \theta_a\,(H_m+1)\right]^2 - (z_{\alpha_m}^2 - D/C)/K_3\right\}}}{z_{\alpha_m}^2 - D/C} \tag{C.70}$$

(K_1 and K_3 are defined in Section 3.3). Eq. (C.69) implies that:

$$\frac{\partial g_m(\theta_a)}{\partial \theta_a} = \frac{\partial g_m(h_m(\theta_a))}{\partial h_m}\frac{\partial h_m(\theta_a)}{\partial \theta_a} \tag{C.71}$$

and:

$$\frac{\partial g_m(h_m(\theta_a))}{\partial h_m} = -\left(\frac{\sqrt{D/C}}{\boldsymbol{p}'_{1,\gamma_a}\boldsymbol{q}_m}\right)\left\{\frac{h_m(\theta_a)}{[h_m^2(\theta_a) - 1/K_3]^{3/2}}\right\}. \tag{C.72}$$

Noting that $\boldsymbol{p}_{1,\gamma_a} \in \mathbb{R}^J_{++}$ (since $\gamma_a \in \Gamma_a$) and $\boldsymbol{q}_m \in \mathbb{R}^J_{++}$, we have $\boldsymbol{p}'_{1,\gamma_a}\boldsymbol{q}_m > 0$. Since $D/C > 0$, $\boldsymbol{p}'_{1,\gamma_a}\boldsymbol{q}_m > 0$, $h_m(\theta_a) > 0$, and $h_m^2(\theta_a) > 1/K_3$, Eq. (C.72) implies that:

$$\frac{\partial g_m(h_m(\theta_a))}{\partial h_m} < 0. \tag{C.73}$$

It follows from Eq. (C.70) and elementary algebra that:

$$\frac{\partial h_m(\theta_a)}{\partial \theta_a} = -\left(\frac{H_m+1}{z_{\alpha_m}^2 - D/C}\right)\left(z_{\alpha_m} + \frac{(D/C)\left[K_1 - \theta_a\,(H_m+1)\right]}{\sqrt{(D/C)\left\{\left[K_1 - \theta_a\,(H_m+1)\right]^2 - (z_{\alpha_m}^2 - D/C)/K_3\right\}}}\right). \tag{C.74}$$

Using the facts that $(H_m + 1) > 0$, $K_2 = z_{\alpha_m}^2 - D/C > 0$, $K_3 > 0$, and $0 < \theta_a < \overline{\theta}_a \leq \overline{\theta}_m$ along with Eq. (24), we have:

$$-\left(\frac{H_m+1}{z_{\alpha_m}^2 - D/C}\right) < 0 \tag{C.75}$$

[C.4] A proof that $\overline{\theta}_m > 0$ is as follows. Since $\gamma_a \in \Gamma_a$, we have $(B - \gamma_a d_{\mathbf{1}}) > 0$. Arguments similar to those in the second part of footnote C.2 imply that $K_3 > 0$. Since $(B - \gamma_a d_{\mathbf{1}}) > 0$ and $K_3 > 0$, we have $K_1 > 0$. Noting that $\alpha_m < \overline{\alpha}$, we have $K_2 > 0$. Since $K_1 > 0$, $K_2 > 0$, $K_3 > 0$, and $(1 + H_m) > 0$, it suffices to show that $K_1^2 - K_2/K_3 > 0$. Using the definitions of K_1, K_2, and K_3, we have $K_1^2 - K_2/K_3 = \frac{B - z_{\alpha m}^2}{K_3}$. Noting that $0.5 > \alpha_m > \underline{\alpha}$, we have $d_{\mathbf{1}}/s_{\mathbf{1}} > z_{\alpha_m} > 0$ and thus $d_{\mathbf{1}}^2/s_{\mathbf{1}}^2 > z_{\alpha_m}^2$. Since $B > d_{\mathbf{1}}^2/s_{\mathbf{1}}^2$, we have $B > z_{\alpha m}^2$ and thus $(B - z_{\alpha_m}^2) > 0$. Using the facts that $K_3 > 0$ and $(B - z_{\alpha_m}^2) > 0$, we have $K_1^2 - K_2/K_3 > 0$ and thus $\overline{\theta}_m > 0$.

and:

$$K_1 - \theta_a\,(H_m+1) > K_1 - \overline{\theta}_m\,(H_m+1) = \sqrt{\frac{K_2}{K_3}} = \sqrt{\frac{z_{\alpha_m}^2 - D/C}{K_3}} > 0. \tag{C.76}$$

It follows from Eq. (C.76) that:

$$[K_1 - \theta_a\,(H_m+1)]^2 > \frac{z_{\alpha_m}^2 - D/C}{K_3}. \tag{C.77}$$

Since $z_{\alpha_m} > 0$ and $D/C > 0$, Eqs. (C.76) and (C.77) imply that:

$$z_{\alpha_m} + \frac{(D/C)\,[K_1 - \theta_a\,(H_m+1)]}{\sqrt{(D/C)\,\{[K_1 - \theta_a\,(H_m+1)]^2 - (z_{\alpha_m}^2 - D/C)/K_3\}}} > 0. \tag{C.78}$$

It follows from Eqs. (C.75) and (C.78) that:

$$\frac{\partial h_m(\theta_a)}{\partial \theta_a} < 0. \tag{C.79}$$

Using Eqs. (C.71), (C.73), and (C.79), we have $\partial g_m(\theta_a)/\partial\theta_a > 0$. ■

Lemma 5. *Suppose that $H_m > -1$ and $\alpha_m > \underline{\alpha}$ for any account $m \in \mathbb{M}$. For any $(\gamma_a, \theta_a) \in \Gamma_a \times \Theta_a$, we have $\partial g_a(\theta_a)/\partial\theta_a > 0$.*

Proof. Suppose that $H_m > -1$ and $\alpha_m > \underline{\alpha}$ for any account $m \in \mathbb{M}$. Fix any $(\gamma_a, \theta_a) \in \Gamma_a \times \Theta_a$. Note that:

$$g_a(\theta_a) = \left[\sum_{m\in\mathbb{M}} 1/g_m(\theta_a)\right]^{-1}. \tag{C.80}$$

Using Lemma 4 and Eq. (C.80), we have $\partial g_a(\theta_a)/\partial\theta_a > 0$. ■

Proof of Theorem 6. Fix any economy $[(\boldsymbol{d},\boldsymbol{S}), \{(\boldsymbol{q}_m, H_m, \alpha_m)\}_{m\in\mathbb{M}}]$ with $H_m > -1$ and $\alpha_m > \underline{\alpha}$ for any account $m \in \mathbb{M}$. Also, fix any $\gamma_a \in \Gamma_a$ with $\underline{g}_a < \gamma_a < \overline{g}_a$. Using Lemma 5, we have $\partial g_a(\theta_a)/\partial\theta_a > 0$ for any $\theta_a \in \Theta_a$. Hence, there exists $\theta_a^* \in \Theta_a$ such that $g_a(\theta_a^*) = \gamma_a$. Let $\boldsymbol{p} = \boldsymbol{p}_{\theta_a^*,\gamma_a}$.

Fix any $m \in \mathbb{M}$. By definition, given $\boldsymbol{p} = \boldsymbol{p}_{\theta_a^*,\gamma_a}$, $\boldsymbol{q}^*_{m,\theta_a^*,\gamma_a}$ solves the DMSS agent's portfolio selection problem within account m. Since Eq. (C.36) holds, it follows from Eq. (17) that the DMSS agent's aggregate portfolio is:

$$\boldsymbol{q}^*_{a,\theta_a^*,\gamma_a} = \left(p_{\boldsymbol{1}} - \frac{A}{\gamma_a}\right)\boldsymbol{q}_{A/C,1} + \frac{\boldsymbol{S}^{-1}\boldsymbol{d}}{\gamma_a} \tag{C.81}$$

The definitions of A, C, and $\boldsymbol{q}_{A/C,1}$ imply that:

$$A = \theta_a^*\,(B - \gamma_a d_{\boldsymbol{1}}), \tag{C.82}$$

$$C = (\theta_a^*)^2(\gamma_a^2 s_{\boldsymbol{1}}^2 - 2\gamma_a d_{\boldsymbol{1}} + B), \tag{C.83}$$

and:

$$\boldsymbol{q}_{A/C,1} = \frac{\boldsymbol{S}^{-1}\boldsymbol{d} \;-\; \gamma_a\boldsymbol{1}}{\theta_a^*(\gamma_a^2 s_{\boldsymbol{1}}^2 - 2\gamma_a d_{\boldsymbol{1}} + B)}. \tag{C.84}$$

It follows from Eqs. (C.81)–(C.84) that $\boldsymbol{q}_a^* = \boldsymbol{1}$ and thus asset markets clear. Hence, $(\boldsymbol{p}^*, \{\boldsymbol{q}_m^*\}_{m\in\mathbb{M}}) = (\boldsymbol{p}^*_{\theta_a}, \{\boldsymbol{q}^*_{m,\theta_a^*,\gamma_a}\}_{m\in\mathbb{M}})$ is an equilibrium for economy $[(\boldsymbol{d},\boldsymbol{S}), \{(\boldsymbol{q}_m, H_m, \alpha_m)\}_{m\in\mathbb{M}}]$ and $\gamma_a^* = \gamma_a$ is the implied risk aversion coefficient of the DMSS agent's aggregate portfolio. ■

For any $(m, \gamma_{0,a}, \theta_{0,a}) \in \mathbb{M} \times \Gamma_{0,a} \times \Theta_{0,a}$, let $g_m(\theta_{0,a})$ denote the implied risk aversion coefficient of the DMSS agent's optimal portfolio within account m when $\boldsymbol{p} = \boldsymbol{p}^*_{\theta_a}$. The following two lemmas are useful in our proof of Theorem 7.

Lemma 6. *For any* $(m, \gamma_{0,a}, \theta_{0,a}) \in \mathbb{M} \times \Gamma_{0,a} \times \Theta_{0,a}$*, we have* $\partial g_{0,m}(\theta_{0,a})/\partial\theta_{0,a} > 0$.

Proof. Similar to the proof of Lemma 4. ■

Lemma 7. *For any* $(\gamma_{0,a}, \theta_{0,a}) \in \Gamma_{0,a} \times \Theta_{0,a}$*, we have* $\partial g_{0,a}(\theta_{0,a})/\partial\theta_{0,a} > 0$.

Proof. Fix any $(\gamma_{0,a}, \theta_{0,a}) \in \Gamma_{0,a} \times \Theta_{0,a}$. Note that:

$$g_{0,a}(\theta_{0,a}) = \left[\sum_{m\in\mathbb{M}} 1/g_{0,m}(\theta_{0,a})\right]^{-1}. \tag{C.85}$$

Using Lemma 6 and Eq. (C.85), we have $\partial g_{0,a}(\theta_{0,a})/\partial\theta_{0,a} > 0$. ■

Proof of Theorem 7. Fix any economy $[(\boldsymbol{d}, \boldsymbol{S}), (\boldsymbol{q}_0, \gamma_0), \{(\boldsymbol{q}_m, H_m, \alpha_m)\}_{m\in\mathbb{M}}]$ with $\gamma_0 > 0$ as well as $H_m > -1$ and $\alpha_m > \underline{\alpha}$ for all $m \in \mathbb{M}$. Also, fix any $\gamma_{0,a} \in \Gamma_{0,a}$ with $\underline{g}_a < \gamma_{0,a} < \overline{g}_a$. Using Lemma 7, we have $\partial g_{0,a}(\theta_{0,a})/\partial\theta > 0$ for any $\theta_{0,a} \in \Theta_{0,a}$. Hence, there exists $\theta^*_{0,a} \in \Theta_{0,a}$ such that $g_{0,a}(\theta^*_{0,a}) = \gamma_{0,a}$. Let $\boldsymbol{p} = \boldsymbol{p}_{\theta^*_{0,a},\varphi_{0,a}}$.

By definition, given $\boldsymbol{p} = \boldsymbol{p}_{\theta^*_{0,a},\varphi_{0,a}}$, $\boldsymbol{q}^*_{0,\theta^*_{0,a},\varphi_{0,a}}$ solves the MV agent's portfolio selection problem. Note that Eq. (C.36) holds. Using Eqs. (17) and (C.36), the MV agent's optimal portfolio is:

$$\boldsymbol{q}^*_{0,\theta^*_{0,a},\varphi_{0,a}} = \left(p_{\boldsymbol{q}_0} - \frac{A}{\gamma_0}\right) \boldsymbol{q}_{A/C,1} + \frac{\boldsymbol{S}^{-1}\boldsymbol{d}}{\gamma_0}. \tag{C.86}$$

Similarly, fix any $m \in \mathbb{M}$. By definition, given $\boldsymbol{p} = \boldsymbol{p}_{\theta^*_{0,a},\varphi_{0,a}}$, $\boldsymbol{q}^*_{0,\theta^*_{0,a},\varphi_{0,a}}$ solves the DMSS agent's portfolio selection problem within account m.[C.5] Using Eqs. (17) and (C.36), the DMSS agent's aggregate portfolio is:

$$\boldsymbol{q}^*_{a,\theta^*_{0,a},\varphi_{0,a}} = \left(p_{\boldsymbol{q}_a} - \frac{A}{\gamma_{0,a}}\right) \boldsymbol{q}_{A/C,1} + \frac{\boldsymbol{S}^{-1}\boldsymbol{d}}{\gamma_{0,a}}. \tag{C.87}$$

The definitions of A, C, and $\boldsymbol{q}_{A/C,1}$ imply that:

$$A = \theta^*_{0,a}\left(B - \varphi_{0,a} d_{\mathbf{1}}\right), \tag{C.88}$$
$$C = (\theta^*_{0,a})^2\left(\varphi^2_{0,a} s^2_{\mathbf{1}} - 2\varphi_{0,a} d_{\mathbf{1}} + B\right), \tag{C.89}$$

and:

$$\boldsymbol{q}_{A/C,1} = \frac{\boldsymbol{S}^{-1}\boldsymbol{d} \; - \; \varphi_{0,a}\mathbf{1}}{\theta^*_{0,a}(\varphi^2_{0,a} s^2_{\mathbf{1}} - 2\varphi_{0,a} d_{\mathbf{1}} + B)}. \tag{C.90}$$

Using Eqs. (C.86)–(C.90), we have $\boldsymbol{q}^*_{0,\theta_{0,a}} + \boldsymbol{q}^*_{a,\theta_{0,a}} = \mathbf{1}$ and thus asset markets clear. It follows that $[\boldsymbol{p}^*, (\boldsymbol{q}^*_0, \{\boldsymbol{q}^*_m\}_{m\in\mathbb{M}})] = [\boldsymbol{p}_{\theta^*_{0,a},\varphi_{0,a}}, (\boldsymbol{q}^*_{0,\theta^*_{0,a},\varphi_{0,a}} \{\boldsymbol{q}^*_{m,\theta^*_{0,a},\varphi_{0,a}}\}_{m\in\mathbb{M}})]$ is an equilibrium for economy $[(\boldsymbol{d}, \boldsymbol{S}), (\boldsymbol{q}_0, \gamma_0), \{(\boldsymbol{q}_m, H_m, \alpha_m)\}_{m\in\mathbb{M}}]$ and $\gamma^*_a = \gamma_{0,a}$ is the implied risk aversion coefficient of the DMSS agent's aggregate portfolio. ■

Proof of Theorem 8. Using Eq. (13), the MV agent's optimal portfolio is on the MV frontier. Premultiplying Eq. (13) by $\boldsymbol{d}'$ and using the definition of D, the expected payoff of this portfolio is $d^*_0 = p_{\boldsymbol{q}_0}(A/C) + \frac{D/C}{\gamma_0}$. It follows that $d^*_0 > p^*_{\boldsymbol{q}_0}(A/C)$. Similarly, using Eqs. (16) and (C.29), the DMSS agent's optimal portfolio within account m is on the MV frontier and $d^*_m > p^*_{\boldsymbol{q}_m}(A/C)$

[C.5] For any $m \in \mathbb{M}$, if $(\theta_{0,a}, \gamma_{0,a}) \in \Theta_{0,a} \times \Gamma_{0,a}$, then $\alpha_m < \overline{\alpha}$ and $H_m \leq H_{\alpha_m}$. Hence, $\boldsymbol{q}^*_{m,\theta_{0,a},\varphi_{0,a}}$ exists; see Theorem 2.

for any $m \in \mathbb{M}$. Eq. (10) and asset market clearing imply that in equilibrium market portfolio $\mathbf{1}$ is on the MV frontier and $d'\mathbf{1} > p^*_{\mathbf{1}}(A/C)$.

Let $\overline{d}$, $\overline{\overline{d}}$, and p be three constants with $\overline{d} \neq \overline{\overline{d}}$ and $p > 0$. Using Eqs. (C.7), (C.10) and (C.11), along elementary algebra, the covariance between the returns on portfolios $\boldsymbol{q}_{p,\overline{d}}$ and $\boldsymbol{q}_{p,\overline{\overline{d}}}$ is:

$$\frac{\boldsymbol{q}'_{p,\overline{d}}\boldsymbol{S}\boldsymbol{q}_{p,\overline{\overline{d}}}}{p^2} = \frac{\frac{C}{D}\left[\overline{d} - p\,(A/C)\right]\left[\overline{\overline{d}} - p\,(A/C)\right]}{p^2} + \frac{1}{C}. \tag{C.91}$$

For brevity, let $p^* = (\boldsymbol{p}^*)'\mathbf{1}$ and $d = \boldsymbol{d}'\mathbf{1}$. Using Eq. (C.91), the definition of portfolio $\boldsymbol{q}_{zc}$, and elementary algebra, we have:[C.6]

$$\boldsymbol{q}'_{zc}\boldsymbol{d} = (p^*)\,(A/C) - (p^*)^2\left[\frac{D/C^2}{d - p^*\,(A/C)}\right]. \tag{C.92}$$

Since $\boldsymbol{q}'_{zc}\boldsymbol{p} = p^*$, Eq. (C.92) implies that:

$$r^*_{\boldsymbol{q}_{zc}} = (A/C - 1) - \frac{D/C^2}{r^*_{\mathbf{1}} - (A/C - 1)}. \tag{C.93}$$

Using the fact that the market portfolio is on the MV frontier and Eq. (10), we have:

$$\mathbf{1} = p^*\boldsymbol{q}_{A/C,1} + \left(\frac{dAC - p^*A^2}{D}\right)(\boldsymbol{q}_{B/A,1} - \boldsymbol{q}_{A/C,1}). \tag{C.94}$$

Fix any portfolio $\boldsymbol{q}$ with $p^*_{\boldsymbol{q}} \equiv \boldsymbol{q}'\boldsymbol{p}^* > 0$. Note that:

$$\sigma^*_{\boldsymbol{q},\mathbf{1}} = \frac{\boldsymbol{q}'\boldsymbol{S}\mathbf{1}}{p^*_{\boldsymbol{q}}p^*}. \tag{C.95}$$

Using Eqs. (C.94) and (C.95), the definitions of $\boldsymbol{q}_{A/C,1}$ and $\boldsymbol{q}_{B/A,1}$, and elementary algebra, we have:

$$\sigma^*_{\boldsymbol{q},\mathbf{1}} = 1/C + \frac{[r^*_{\boldsymbol{q}} - (A/C - 1)]\,[r^*_{\mathbf{1}} - (A/C - 1)]}{D/C}. \tag{C.96}$$

Fix any $j \in \mathbb{J}$. Let $\boldsymbol{\iota}_j$ denote the $J \times 1$ vector with the j^{th} entry being one and the other $J - 1$ entries being zero. Using Eq. (C.96) with $\boldsymbol{q} = \boldsymbol{\iota}_j$, we have:

$$\sigma^*_{j,\mathbf{1}} = 1/C + \frac{[r^*_j - (A/C - 1)]\,[r^*_{\mathbf{1}} - (A/C - 1)]}{D/C}. \tag{C.97}$$

Similarly, using Eq. (C.96) with $\boldsymbol{q} = \mathbf{1}$, we have:

$$\sigma^*_{\mathbf{1},\mathbf{1}} = 1/C + \frac{[r^*_{\mathbf{1}} - (A/C - 1)]^2}{D/C}. \tag{C.98}$$

Eq. (C.97) implies that:

$$r^*_j = \left[(A/C - 1) - \frac{D/C^2}{r^*_{\mathbf{1}} - (A/C - 1)}\right] + \frac{D/C}{r^*_{\mathbf{1}} - (A/C - 1)}\sigma^*_{j,\mathbf{1}}. \tag{C.99}$$

Since $\sigma^*_{j,\mathbf{1}} = \sigma^*_{\mathbf{1},\mathbf{1}}\beta^*_j$, we have:

$$r^*_j = \left[(A/C - 1) - \frac{D/C^2}{r^*_{\mathbf{1}} - (A/C - 1)}\right] + \frac{D/C}{r^*_{\mathbf{1}} - (A/C - 1)}\sigma^*_{\mathbf{1},\mathbf{1}}\beta^*_j. \tag{C.100}$$

[C.6] Note that $\boldsymbol{q}_{zc}$ can be found by using Eq. (10) with $d = \left(A/C - \frac{D/C^2}{d_{\mathbf{1}}/p^*_{\mathbf{1}} - A/C}\right)p^*_{\mathbf{1}}$, $p = p^* = p^*_{\mathbf{1}}$, and $\boldsymbol{p} = \boldsymbol{p}^*$.

Using Eq. (C.97) in Eq. (C.100) along with elementary algebra, we obtain:

$$r_j^* = \left[(A/C - 1) - \frac{D/C^2}{r_{\mathbf{1}}^* - (A/C - 1)} \right] + \left[r_{\mathbf{1}}^* + \frac{D/C^2}{r_{\mathbf{1}}^* - (A/C - 1)} - (A/C - 1) \right] \beta_j^*. \tag{C.101}$$

Eq. (30) follows from Eqs. (C.93) and (C.101). ■

Online Appendix D: adding a risk-free asset

In this appendix, we examine the case where a risk-free asset is present. Accordingly, in addition to J risky assets, suppose that a risk-free asset ($j = J+1$) is available. This asset is in zero net supply and has a payoff of one. Portfolios (e.g., $\boldsymbol{q}$), asset endowments (e.g., $\boldsymbol{q}_0$), and asset prices (e.g., $\boldsymbol{p}$) are now $(J+1)\times 1$ vectors. Let $\widehat{\boldsymbol{q}}$ denote the $J\times 1$ vector with the first J components of $\boldsymbol{q}$. Similarly, let $\widehat{\boldsymbol{p}}$ denote the $J\times 1$ vector with the first J components of $\boldsymbol{p}$.

D.1. MV frontier

Since the price of the risk-free asset is $p_{J+1} > 0$, the risk-free return is $r_f \equiv 1/p_{J+1} - 1$. Let $F \equiv A - C(1+r_f)$ and $G \equiv B - A(1+r_f) - F(1+r_f)$. Suppose that $F \neq 0$ and $rank([\boldsymbol{d} \quad \widehat{\boldsymbol{p}}]) = 2$. Since $rank(\boldsymbol{S}) = J$ and $rank([\boldsymbol{d} \quad \widehat{\boldsymbol{p}}]) = 2$, we have $G > 0$.

For any given expected payoff $d \in \mathbb{R}$ and any given price $p \in \mathbb{R}_{++}$, the corresponding portfolio on the MV frontier is:

$$\boldsymbol{q}_{d,p} = p(\boldsymbol{q}_{f,1}) + \phi_{d,p}(\boldsymbol{q}_{t,1} - \boldsymbol{q}_{f,1}) \tag{D.1}$$

where $\boldsymbol{q}_{t,1} \equiv \left[\frac{(\boldsymbol{d}-\widehat{\boldsymbol{p}}/p_{J+1})'\boldsymbol{S}^{-1}}{F} \quad 0\right]'$, $\boldsymbol{q}_{f,1} \equiv [\boldsymbol{0}' \quad 1/p_{J+1}]'$, and $\phi_{d,p} \equiv (d - p/p_{J+1})\frac{F}{G}$.[D.1] Portfolios on it with price p are represented in $(d_{\boldsymbol{q}}, s_{\boldsymbol{q}})$ space by:

$$s_{\boldsymbol{q}} = \sqrt{\frac{(d_{\boldsymbol{q}} - p/p_{J+1})^2}{G}}. \tag{D.2}$$

Since $s_{\boldsymbol{q}} = p\sigma_{\boldsymbol{q}}$ and $d_{\boldsymbol{q}} = p(1 + r_{\boldsymbol{q}})$, such portfolios are represented in ($r_{\boldsymbol{q}}$,$\sigma_{\boldsymbol{q}}$) space by:

$$\sigma_{\boldsymbol{q}} = \sqrt{\frac{(r_{\boldsymbol{q}} - r_f)^2}{G}}. \tag{D.3}$$

Hence, their location in this space depends on r_f and G.

D.2. Optimal portfolios

This section characterizes the agents' optimal portfolios.

D.2.1. MV agent

Next, we examine the MV agent's optimal portfolio.

Theorem D.1. *The MV agent's optimal portfolio is:*

$$\boldsymbol{q}_0^* = (p_{\boldsymbol{q}_0})(\boldsymbol{q}_{f,1}) + (F/\gamma_0)(\boldsymbol{q}_{t,1} - \boldsymbol{q}_{f,1}). \tag{D.4}$$

[D.1] Note that $\boldsymbol{q}_{f,1}$ and $\boldsymbol{q}_{t,1}$ are, respectively, the risk-free and tangency portfolios with a price of one.

Theorem D.1 differs from Theorem 1 in that $\boldsymbol{q}_{f,1}$, $\boldsymbol{q}_{t,1}$, and F are used instead of, respectively, $\boldsymbol{q}_{A/C,1}$, $\boldsymbol{q}_{B/A,1}$, and A.

D.2.2. DMSS agent

Let:

$$\overline{\overline{\alpha}} \equiv \Phi(-\sqrt{G}). \tag{D.5}$$

Since $G > 0$, Eq. (D.5) implies that $\overline{\overline{\alpha}} \in (0, 0.5)$.

We now examine the DMSS agent's optimal portfolios within accounts.

Theorem D.2. *Fix any account $m \in \mathbb{M}$. (i) If either (a) $\alpha_m \geq \overline{\overline{\alpha}}$ or (b) $\alpha_m < \overline{\overline{\alpha}}$ and $H_m > r_f$, then the DMSS agent's optimal portfolio within account m does not exist. (ii) If $\alpha_m < \overline{\overline{\alpha}}$ and $H_m \leq r_f$, then it exists and is:*

$$\boldsymbol{q}_m^* = (p_{\boldsymbol{q}_m})(\boldsymbol{q}_{f,1}) + (F/\gamma_m^*)(\boldsymbol{q}_{t,1} - \boldsymbol{q}_{f,1}) \tag{D.6}$$

where its implied risk aversion coefficient is:

$$\gamma_m^* = \begin{cases} \frac{z_{\alpha_m}\sqrt{G} - G}{(r_f - H_m)p_{\boldsymbol{q}_m}} & if \quad H_m < r_f \\ \infty & if \quad H_m = r_f \end{cases}. \tag{D.7}$$

Theorem D.2 differs from Theorem 2 in two respects. First, $\boldsymbol{q}_{f,1}$, $\boldsymbol{q}_{t,1}$, and F are used instead of, respectively, $\boldsymbol{q}_{A/C,1}$, $\boldsymbol{q}_{B/A,1}$, and A. Second, the formula for γ_m^* is somewhat less complex.

Next, we examine the DMSS agent's aggregate portfolio.

Theorem D.3. *Suppose that $\alpha_m < \overline{\overline{\alpha}}$ and $H_m \leq r_f$ for any account $m \in \mathbb{M}$. The DMSS agent's aggregate portfolio is:*

$$\boldsymbol{q}_a^* = (p_{\boldsymbol{q}_a})(\boldsymbol{q}_{f,1}) + (F/\gamma_a^*)(\boldsymbol{q}_{t,1} - \boldsymbol{q}_{f,1}) \tag{D.8}$$

where:

$$\gamma_a^* = \left[\sum_{m \in \mathbb{M}} (1/\gamma_m^*)\right]^{-1} \tag{D.9}$$

is its implied risk aversion coefficient and γ_m^* is given by Eq. (D.7).

Theorem D.3 differs from Theorem 3 in that $\boldsymbol{q}_{f,1}$, $\boldsymbol{q}_{t,1}$, and F are used instead of, respectively, $\boldsymbol{q}_{A/C,1}$, $\boldsymbol{q}_{B/A,1}$, and A.

D.3. Equilibrium

In this section, we characterize equilibria in four types of economies: (1) a single-agent economy with an MV agent; (2) a single-agent economy with a DMSS agent and a single account; (3) a single-agent economy with a DMSS agent and multiple accounts; and (4) a two-agent economy with an MV agent and a DMSS agent with multiple accounts.

D.3.1. Single-agent economy with an MV agent

Consider a single-agent economy with an MV agent. For any $\theta_0 \in \Theta_0$ and $\gamma_0 \in \Gamma_0$, let $\boldsymbol{p}_{\theta_0,\gamma_0} \equiv \theta_0[(\boldsymbol{d}-\gamma_0\boldsymbol{S1})' \quad 1]'$.

The following result characterizes equilibria.

Theorem D.4. *Fix any economy* $[(\boldsymbol{d},\boldsymbol{S}),(\boldsymbol{q}_0,\gamma_0)]$ *where* $\gamma_0 \in \Gamma_0$. *For any* $\theta_0 \in \Theta_0$, $(\boldsymbol{p}^*,\boldsymbol{q}_0^*) = (\boldsymbol{p}_{\theta_0,\gamma_0},\boldsymbol{q}_{0,\gamma_0}^*)$ *is an equilibrium for it.*

Theorem D.4 is similar to Theorem 4 except that θ_0 is now the risk-free asset price.

D.3.2. Single-agent economy with a DMSS agent and a single account

Consider a single-agent economy with a DMSS agent and a single account. For any $\gamma_1 \in \Gamma_1$, let $\boldsymbol{p}_{\gamma_1} \equiv \theta_{\gamma_1}[(\boldsymbol{d}-\gamma_1\boldsymbol{S1})' \quad 1]'$ where θ_{γ_1} is defined in Section 3.2.

The following result characterizes equilibria.

Theorem D.5. *Fix any economy* $[(\boldsymbol{d},\boldsymbol{S}),(\boldsymbol{q}_1,H_1,\alpha_1)]$ *where* $H_1 > -1$ *and* $\alpha_1 > \underline{\alpha}$. *For any* $\gamma_1 \in \Gamma_1$, $(\boldsymbol{p}^*,\boldsymbol{q}_1^*) = (\boldsymbol{p}_{\gamma_1},\boldsymbol{q}_{1,\gamma_1}^*)$ *is an equilibrium for it and* $\gamma_1^* = \gamma_1$ *is the implied risk aversion coefficient of the DMSS agent's optimal portfolio within account 1.*

Theorem D.5 is similar to Theorem 5 except that θ_{γ_1} is now the risk-free asset price.

D.3.3. Single-agent economy with a DMSS agent and multiple accounts

Consider a single-agent economy with a DMSS agent and multiple accounts. Fix any $\gamma_a \in \Gamma_a$. Let $\boldsymbol{p}_{\gamma_a} \equiv \theta_{\gamma_a}[(\boldsymbol{d}-\gamma_a\boldsymbol{S1})' \quad 1]'$ where:

$$\theta_{\gamma_a} \equiv \frac{\left(\sum_{m\in\mathbb{M}} \kappa_{m,\gamma_a}\right) - 1}{\sum_{m\in\mathbb{M}} \left[(H_m+1)\,\kappa_{m,\gamma_a}\right]} \tag{D.10}$$

and:

$$\kappa_{m,\gamma_a} \equiv \frac{(\boldsymbol{d} - \gamma_a \boldsymbol{S}\boldsymbol{1})'\widehat{\boldsymbol{q}}_m + q_{m,J+1}}{z_{\alpha_m} s_{\boldsymbol{1}} - \gamma_a s_{\boldsymbol{1}}^2} \tag{D.11}$$

for any $m \in \mathbb{M}$. Let $\overline{\Gamma}_a$ be the subset of Γ_a where $1/\theta_{\gamma_a} - 1 - H_m > 0$ and $(\boldsymbol{d} - \gamma_a \boldsymbol{S}\boldsymbol{1})'\widehat{\boldsymbol{q}}_m + q_{m,J+1} > 0$ for any $m \in \mathbb{M}$. Let $\boldsymbol{q}^*_{m,\gamma_a}$ denote the DMSS agent's optimal portfolio within account m when the asset price vector is $\boldsymbol{p}_{\gamma_a}$.

The following result characterizes equilibria.

Theorem D.6. *Fix any economy* $[(\boldsymbol{d}, \boldsymbol{S}), \{(\boldsymbol{q}_m, H_m, \alpha_m)\}_{m\in\mathbb{M}}]$ *where* $H_m > -1$ *and* $\alpha_m > \underline{\alpha}$ *for any account* $m \in \mathbb{M}$. *For any* $\gamma_a \in \overline{\Gamma}_a$, $(\boldsymbol{p}^*, \{\boldsymbol{q}^*_m\}_{m\in\mathbb{M}}) = (\boldsymbol{p}_{\gamma_a}, \{\boldsymbol{q}^*_{m,\gamma_a}\}_{m\in\mathbb{M}})$ *is an equilibrium for it and* $\gamma^*_a = \gamma_a$ *is the implied risk aversion coefficient of the DMSS agent's aggregate portfolio.*

Theorem D.6 is similar to Theorem 6 except that θ_{γ_a} is now the risk-free asset price and has a closed-form expression.

D.3.4. Two-agent economy with an MV agent and a DMSS agent with multiple accounts

Consider a two-agent economy with an MV agent and a DMSS agent with multiple accounts. Fix any $\gamma_{0,a} \in \Gamma_{0,a}$. Let $\boldsymbol{p}_{\gamma_{0,a}} \equiv \theta_{\gamma_{0,a}}[(\boldsymbol{d} - \varphi_{0,a}\boldsymbol{S}\boldsymbol{1})' \quad 1]'$ where:

$$\theta_{\gamma_{0,a}} \equiv \frac{\left(\sum_{m\in\mathbb{M}} \kappa_{m,\gamma_{0,a}}\right) - 1 + \varphi_{0,a}/\gamma_0}{\sum_{m\in\mathbb{M}} \left[(H_m + 1)\,\kappa_{m,\gamma_a}\right]} \tag{D.12}$$

and:

$$\kappa_{m,\gamma_{0,a}} \equiv \frac{(\boldsymbol{d} - \varphi_{0,a}\boldsymbol{S}\boldsymbol{1})'\widehat{\boldsymbol{q}}_m + q_{m,J+1}}{z_{\alpha_m} s_{\boldsymbol{1}} - \varphi_{0,a} s_{\boldsymbol{1}}^2} \tag{D.13}$$

for any $m \in \mathbb{M}$, and $\varphi_{0,a} = \frac{1}{1/\gamma_0 + 1/\gamma_{0,a}}$ (as before). Let $\overline{\Gamma}_{0,a}$ be the subset of $\Gamma_{0,a}$ where $1/\theta_{\gamma_a} - 1 - H_m > 0$ and $(\boldsymbol{d} - \varphi_{0,a}\boldsymbol{S}\boldsymbol{1})'\widehat{\boldsymbol{q}}_m + q_{m,J+1} > 0$ for any $m \in \mathbb{M}$. Let $\boldsymbol{q}^*_{0,\gamma_{0,a}}$ denote the MV agent's optimal portfolio when the asset price vector is $\boldsymbol{p}_{\gamma_{0,a}}$. Similarly, let $\boldsymbol{q}^*_{m,\gamma_{0,a}}$ denote the DMSS agent's optimal portfolio within account m when the asset price vector is $\boldsymbol{p}_{\gamma_{0,a}}$.

The following result characterizes equilibria.

Theorem D.7. *Fix any economy* $[(\boldsymbol{d}, \boldsymbol{S}), (\boldsymbol{q}_0, \gamma_0), \{(\boldsymbol{q}_m, H_m, \alpha_m)\}_{m\in\mathbb{M}}]$ *where* $\gamma_0 > 0$ *as well as* $H_m > -1$ *and* $\alpha_m > \underline{\alpha}$ *for any account* $m \in \mathbb{M}$. *For any* $\gamma_{0,a} \in \overline{\Gamma}_{0,a}$, $(\boldsymbol{p}^*, \{\boldsymbol{q}^*_m\}_{m\in\mathbb{M}}) = (\boldsymbol{p}_{\gamma_{0,a}}, \{\boldsymbol{q}^*_{m,\gamma_{0,a}}\}_{m\in\mathbb{M}})$ *is an equilibrium for it and* $\gamma^*_a = \gamma_{0,a}$ *is the implied risk aversion coefficient* *of the DMSS agent's aggregate portfolio.*

Theorem D.7 is similar to Theorem 7 except that $\theta_{\gamma_{0,a}}$ is now the risk-free asset price and has a closed-form solution.

D.3.5. Equilibrium asset prices

As before, r_j^* and β_j^* denote, respectively, the equilibrium expected return and beta of asset j, whereas $r_{\mathbf{1}}^*$ denotes the the equilibrium expected return of market portfolio $\mathbf{1}$. The equilibrium risk-free return is denoted by r_f^*.

The following result characterizes expected asset returns in equilibrium.

Theorem D.8. *For any asset $j \in \mathbb{J}$, its equilibrium expected return is:*

$$r_j^* = r_f^* + \beta_j^*(r_{\mathbf{1}}^* - r_f^*). \tag{D.14}$$

Theorem D.8 differs from Theorem 8 in that the CAPM of Sharpe (1964) now holds instead of the zero-beta CAPM of Black (1972).

D.4. Example

In this section, we add a risk-free asset to the example of Section 4.

D.4.1. Single-agent economy with an MV agent

Consider a single-agent economy with an MV agent where his or her endowments are: (a) one for each risky asset; and (b) zero for the risk-free asset; see Panel A1 of Table D.1. Assume that $\gamma_0 = 1$ and $\theta_0 = 0.95$; see Theorem D.4. Panels A, B, and C of Fig. D.1 report the equilibrium prices, expected returns, and betas of the risky assets. The results are identical to those in Fig. 4 where a risk-free is absent. Note that the price, expected return, and beta of the risk-free asset (not reported in the panels of Fig. D.1) are, respectively, 0.95, 5.26% [= 1/0.95 − 1], and zero.

D.4.2. Single-agent economy with a DMSS agent and one account

Consider a single-agent economy with a DMSS agent and one account where his or her endowment of each risky asset is one and his or her endowment of the risk-free asset is zero; see Panel A2 of Table D.1. Panels A, B, and C of Fig. D.2 report the equilibrium prices, expected returns, and betas of the risky assets. The results are identical to those in Fig. 5 where a risk-free is absent. Note that the price, expected return, and beta of the risk-free asset (not reported in the panels of Fig. D.2) are very close to those of asset 1. This result follows from: (1) the risk-free asset's payoff (one) equals asset 1's expected payoff (also one); and (2) the risk-free asset's beta (zero) is very close to asset 1's beta (near zero).

D.4.3. Single-agent economy with a DMSS agent and three accounts

Consider a single-agent economy with a DMSS agent and three accounts. The agent's endowments of each risky asset in accounts 1, 2, and 3 are, respectively, 0.2, 0.2, and 0.6; see rows $j = 1$, 2, and 3 of Panel A3 of Table D.1. The agent's endowments of the risk-free asset in accounts 1, 2, and 3 are, respectively, -0.2, -0.2, and 0.4; see row $j = 4$. Panels A, B, and C of Fig. D.3 report the equilibrium prices, expected returns, and betas of the risky assets. The results are similar to those in Fig. 6 where a risk-free is absent. Also, the size of the thresholds affects asset prices. For example, fixing the value of γ_a^*, the prices of the risky assets in panel D of Fig. D.3 where $\alpha_1 = 15\%$ exceed those in panel A where $\alpha_1 = 10\%$. Note that the price, expected return, and beta of the risk-free asset (not reported in the panels of Fig. D.3) are very close to those of asset 1. This result follows from: (1) the risk-free asset's payoff (one) equals asset 1's expected payoff (also one); and (2) the risk-free asset's beta (zero) is very close to asset 1's beta (near zero).

In panel A of Fig. D.4, the solid and dotted lines report the optimal holdings of, respectively, each risky asset (the same holding for assets 1, 2, and 3) and the risk-free asset (i.e., asset 4) in account 1. The results differ from those in panel A of Fig. 7 where a risk-free is absent in one respect. While in Fig. D.4 the agent optimally holds the same quantity of each risky asset within account 1 (given some value of γ_a^*), that is not true in Fig. 7. This difference in the results also holds within accounts 2 and 3; see panels B and C of Fig. D.4.

D.4.4. Two-agent economy with an MV agent and a DMSS agent with three accounts

Consider a two-agent economy with an MV agent and a DMSS agent with three accounts. For each risky asset, the MV agent's endowment of 0.5 equals the DMSS agent's aggregate endowment of 0.5 [$= 0.1 + 0.1 + 0.3$]; see rows $j = 1$, 2, and 3 of panel A4 of Table D.1. Similarly, for the risk-free asset, the MV agent's endowment of 0 equals the DMSS agent's aggregate endowment of 0 [$= (-0.1) + (-0.1) + (0.2)$]; see row $j = 4$. Panels A, B, and C of Fig. D.5 report the equilibrium prices, expected returns, and betas of the risky assets. The results are similar to those in Fig. 8 where a risk-free is absent. Also, the size of the thresholds affects asset prices. For example, fixing the value of γ_a^*, the prices of the risky assets in panel G of Fig. D.5 where $\alpha_1 = 15\%$ exceed those in panel A where $\alpha_1 = 10\%$. Note that the price, expected return, and beta of the risk-free asset (not reported in the panels of Fig. D.5) are very close to those of asset 1. This result follows from: (1) the risk-free asset's payoff (one) equals asset 1's expected payoff (also one); and (2) the risk-free asset's beta (zero) is very close to asset 1's beta (near zero).

In panel A of Fig. D.6, the solid and dotted lines report the MV agent's optimal holdings of, respectively, each risky asset (the same holding for assets 1, 2, and 3) and the risk-free asset (i.e., asset 4) in account 1. The results differ from those in panel A of Fig. 9 where a risk-free is absent in one respect. While in Fig. D.6 the agent optimally holds the same quantity of each risky asset (given some value of γ_a^*), that is not true in Fig. 9. This difference in the results also holds for the DMSS agent's optimal portfolios within accounts; see panels B–D.

D.5. Proofs of theoretical results in Appendix D

Proof that portfolios on the MV frontier satisfy Eq. (D.1). Fix any $(d,p) \in \mathbb{R} \times \mathbb{R}_{++}$. Note that portfolio $\boldsymbol{q}_{d,p}$ solves:

$$\min_{\boldsymbol{q} \in \mathbb{R}^{J+1}} \quad \frac{1}{2}\widehat{\boldsymbol{q}}'\boldsymbol{S}\widehat{\boldsymbol{q}} \tag{D.15}$$

$$\begin{aligned} s.t. \qquad \widehat{\boldsymbol{q}}'\boldsymbol{d} + q_{J+1} &= d & \text{(D.16)} \\ \widehat{\boldsymbol{q}}'\widehat{\boldsymbol{p}} + q_{J+1} \times p_{J+1} &= p. & \text{(D.17)} \end{aligned}$$

It follows from Eq. (D.17) that:

$$q_{J+1} = \frac{p - \widehat{\boldsymbol{q}}'\widehat{\boldsymbol{p}}}{p_{J+1}}. \tag{D.18}$$

Using Eqs. (D.16) and (D.18), we have:

$$\widehat{\boldsymbol{q}}'(\boldsymbol{d} - \widehat{\boldsymbol{p}}/p_{J+1}) = d - p/p_{J+1}. \tag{D.19}$$

Hence, $\widehat{\boldsymbol{q}}_{d,p}$ solves problem (D.15) subject to constraint (D.19). First-order conditions for $\widehat{\boldsymbol{q}}_{d,p}$ to solve this problem are:

$$\begin{aligned} \boldsymbol{S}\widehat{\boldsymbol{q}}_{d,p} - \delta(\boldsymbol{d} - \widehat{\boldsymbol{p}}/p_{J+1}) &= \boldsymbol{0} & \text{(D.20)} \\ \widehat{\boldsymbol{q}}'_{d,p}(\boldsymbol{d} - \widehat{\boldsymbol{p}}/p_{J+1}) &= d - p/p_{J+1} & \text{(D.21)} \end{aligned}$$

where δ is a Lagrange multiplier associated with such a constraint. Eq. (D.20) implies that:

$$\widehat{\boldsymbol{q}}_{d,p} = \delta \boldsymbol{S}^{-1}(\boldsymbol{d} - \widehat{\boldsymbol{p}}/p_{J+1}). \tag{D.22}$$

Premultiplying Eq. (D.22) by $(\boldsymbol{d} - \widehat{\boldsymbol{p}}/p_{J+1})'$ and using Eq. (D.21) along with the definition of G, we have:

$$d - p/p_{J+1} = \delta G. \tag{D.23}$$

It follows from Eq. (D.23) that:

$$\delta = \frac{d - p/p_{J+1}}{G}. \tag{D.24}$$

The definitions of $\boldsymbol{q}_{f,1}$, $\boldsymbol{q}_{t,1}$, and $\phi_{d,p}$ along with Eqs. (D.18), (D.22), and (D.24) imply that Eq. (D.1) holds. ■

Proof that portfolios on the MV frontier satisfy Eq. (D.3). Using Eqs. (D.22) and (D.24), we have:

$$\widehat{\boldsymbol{q}}_{d,p} = \left(\frac{d - p/p_{J+1}}{G}\right)\boldsymbol{S}^{-1}(\boldsymbol{d} - \widehat{\boldsymbol{p}}/p_{J+1}). \tag{D.25}$$

Since $G = (\boldsymbol{d} - \widehat{\boldsymbol{p}}/p_{J+1})'\boldsymbol{S}^{-1}(\boldsymbol{d} - \widehat{\boldsymbol{p}}/p_{J+1})$, Eq. (D.25) implies that:

$$\widehat{\boldsymbol{q}}'_{d,p}\boldsymbol{S}\widehat{\boldsymbol{q}}_{d,p} = \frac{(d - p/p_{J+1})^2}{G}. \tag{D.26}$$

The desired result follows from Eq. (D.26). ■

Proof of Theorem D.1. The MV agent's budget constraint is:

$$\widehat{\boldsymbol{q}}'\widehat{\boldsymbol{p}} + q_{J+1}p_{J+1} = \widehat{\boldsymbol{q}}_0'\widehat{\boldsymbol{p}} + q_{0,J+1}p_{J+1}. \tag{D.27}$$

Using Eq. (D.27), we have:

$$q_{J+1} = (\widehat{\boldsymbol{q}}_0 - \widehat{\boldsymbol{q}})'\widehat{\boldsymbol{p}}/p_{J+1} + q_{0,J+1}. \tag{D.28}$$

Hence, $\widehat{\boldsymbol{q}}_0^*$ solves:

$$\max_{\widehat{\boldsymbol{q}} \in \mathbb{R}^J} \ \widehat{\boldsymbol{q}}'(\boldsymbol{d} - \widehat{\boldsymbol{p}}/p_{J+1}) + ({\boldsymbol{q}_0}'\boldsymbol{p}/p_{J+1} + q_{0,J+1}) - \frac{\gamma_0}{2}(\widehat{\boldsymbol{q}}'\boldsymbol{S}\widehat{\boldsymbol{q}}). \tag{D.29}$$

A first-order condition for $\widehat{\boldsymbol{q}}_0^*$ to solve (D.29) is:

$$(\boldsymbol{d} - \widehat{\boldsymbol{p}}/p_{J+1}) - \gamma_0 \boldsymbol{S}\widehat{\boldsymbol{q}}_0^* = \boldsymbol{0}. \tag{D.30}$$

It follows from Eq. (D.30) that:

$$\widehat{\boldsymbol{q}}_0^* = \frac{\boldsymbol{S}^{-1}(\boldsymbol{d} - \widehat{\boldsymbol{p}}/p_{J+1})}{\gamma_0}. \tag{D.31}$$

Using Eq. (D.31) and the definition of $\boldsymbol{q}_{t,1}$, we have:

$$\widehat{\boldsymbol{q}}_0^* = (F/\gamma_0)\widehat{\boldsymbol{q}}_{t,1}. \tag{D.32}$$

Eq. (D.27) implies that:

$$q_{0,J+1}^* = (\widehat{\boldsymbol{q}}_0 - \widehat{\boldsymbol{q}}_0^*)'\widehat{\boldsymbol{p}}/p_{J+1} + q_{0,J+1}. \tag{D.33}$$

Using the right-hand side of Eq. (D.32) in the right-hand side of Eq. (D.33), we have:

$$q_{0,J+1}^* = \widehat{\boldsymbol{q}}_0'\widehat{\boldsymbol{p}}/p_{J+1} - (F/\gamma_0)/p_{J+1} + q_{0,J+1}. \tag{D.34}$$

It follows from Eq. (D.34) that:

$$q_{0,J+1}^* = (p_{\boldsymbol{q}_0} - F/\gamma_0)/p_{J+1}. \tag{D.35}$$

Eq. (D.4) follows from the definition of $\boldsymbol{q}_{f,1}$ along with Eqs. (D.32) and (D.35). ■

Proof of Theorem D.2. Fix any account $m \in \mathbb{M}$. First, we show (i). Suppose that $\alpha_m \geq \overline{\overline{\alpha}}$. Fix any portfolio $\boldsymbol{q}$ on the MV frontier with a payoff of $d > p/p_{J+1}$ and a price of $p = p_{\boldsymbol{q}_m}$. Since $d > p/p_{J+1}$, we have $r_{\boldsymbol{q}} > r_f$. Using Eq. (D.3), we obtain:

$$r_{\boldsymbol{q}} = r_f + \sqrt{G}\sigma_{\boldsymbol{q}}. \tag{D.36}$$

It follows from Eqs. (7) and (D.36) that:

$$V_{1-\alpha_m,\boldsymbol{q}} = (z_{\alpha_m} - \sqrt{G})\sigma_{\boldsymbol{q}} - r_f. \tag{D.37}$$

Assume that $\alpha_m > \overline{\overline{\alpha}}$. Then, the definition of z_{α_m} and Eq. (D.5) imply that:

$$z_{\alpha_m} < \sqrt{G}. \tag{D.38}$$

Using Eqs. (D.37) and (D.38), one can increase a portfolio's expected return and decrease a portfolio's return VaR at confidence level $1 - \alpha_m$ by moving up along the MV frontier of portfolios with a price of $p_{\boldsymbol{q}_m}$. It follows that the optimal portfolio within account m does not exist.

Assume that $\alpha_m = \overline{\overline{\alpha}}$. Then, the definition of z_{α_m} and Eq. (D.5) imply that:

$$z_{\alpha_m} = \sqrt{G}. \tag{D.39}$$

Eqs. (D.37) and (D.39) imply that:

$$V_{1-\alpha_m,\boldsymbol{q}} = -r_f. \tag{D.40}$$

Note that the portfolio with minimum return VaR at confidence level $1-\alpha_m$ among portfolios with a positive price has a return VaR at this confidence level of $-r_f$. If $H_m > r_f$, then there is no portfolio with a return VaR at confidence level $1-\alpha_m$ less than $-H_m < -r_f$. Hence, the optimal portfolio within account m does not exist. If $H_m \leq r_f$, then one can increase a portfolio's expected return while keeping the portfolio's return VaR at confidence level $1-\alpha_m$ fixed at $-r_f$ by moving up along the MV frontier of portfolios with a price of $p_{\boldsymbol{q}_m}$. Hence, the optimal portfolio within account m does not exist.

Suppose now that $\alpha_m < \overline{\overline{\alpha}}$ and $H_m > r_f$. Using the definition of z_{α_m} and Eq. (D.5), we have:

$$z_{\alpha_m} > \sqrt{G}. \tag{D.41}$$

Again, the portfolio with minimum return VaR at confidence level $1-\alpha_m$ among portfolios with a positive price has a return VaR at this confidence level of $-r_f$. It follows that there is no portfolio with a return VaR at confidence level $1-\alpha_m$ less than $-H_m < -r_f$. Hence, the optimal portfolio within account m does not exist. This completes the first part of our proof.

Second, we show (ii). Suppose that $\alpha_m < \overline{\overline{\alpha}}$ and $H_m \leq r_f$. The optimal portfolio within account m lies in $(r_{\boldsymbol{q}}, \sigma_{\boldsymbol{q}})$ space at the point where the line corresponding to the constraint:

$$V_{1-\alpha_m,\boldsymbol{q}} \leq -H_m. \tag{D.42}$$

intersects the top half-line representing the MV frontier. It follows that:

$$r_{\boldsymbol{q}_m^*} = r_f + \sqrt{G}\sigma_{\boldsymbol{q}_m^*}. \tag{D.43}$$

and:

$$r_{\boldsymbol{q}_m^*} = H_m + z_{\alpha_m}\sigma_{\boldsymbol{q}_m^*}. \tag{D.44}$$

Using Eqs. (D.43) and (D.44), we have:

$$\sigma_{\boldsymbol{q}_m^*} = \frac{r_f - H_m}{z_{\alpha_m} - \sqrt{G}} \tag{D.45}$$

and:

$$r_{\boldsymbol{q}_m^*} = r_f + \sqrt{G}\left(\frac{r_f - H_m}{z_{\alpha_m} - \sqrt{G}}\right). \tag{D.46}$$

Assume that $H_m = r_f$. Then, Eq. (D.46) implies that $\sigma_{\boldsymbol{q}_m^*} = 0$. Hence, Eq. (D.6) holds with $\gamma_m^* = \infty$.

Assume now that $H_m < r_f$. Using the definition of $\phi_{d,p}$ with $d = p_{\boldsymbol{q}_m}(1 + r_{\boldsymbol{q}_m^*})$ and $p = p_{\boldsymbol{q}_m}$, we have:

$$\phi_{d,p} = \frac{F}{(z_{\alpha_m}\sqrt{G} - G)/[(r_f - H_m)p_{\boldsymbol{q}_m}]}. \tag{D.47}$$

It follows from Eqs. (D.1) and (D.47) that Eq. (D.6) holds with $\gamma_m^* = \frac{z_{\alpha_m}\sqrt{G}-G}{(r_f-H_m)p_{\boldsymbol{q}_m}}$. This completes the second part of our proof. ■

Proof of Theorem D.3. Suppose that $\alpha_m < \overline{\overline{\alpha}}$ and $H_m \leq r_f$ for any account $m \in \mathbb{M}$. Using (ii) of Theorem D.2, Eq. (D.6) holds for any account $m \in \mathbb{M}$. Summing Eq. (D.6) across all elements of $\mathbb{M}$, we have Eq. (D.8) where γ_a^* is given by Eq. (D.9). ■

Proof of Theorem D.4. Fix any economy $[(\boldsymbol{d},\boldsymbol{S}),(\boldsymbol{q}_0,\gamma_0)]$ where $\gamma_0 \in \Gamma_0$. Also, fix any $\theta_0 \in \Theta_0$. Suppose that $\boldsymbol{p} = \boldsymbol{p}_{\theta_0,\gamma_0}$. Using the definitions of A, C, and $\boldsymbol{p}_{\theta_0,\gamma_0}$, we have:

$$A = \theta_0(B - \gamma_0 d_{\mathbf{1}}) \tag{D.48}$$

and:

$$C = \theta_0^2(\gamma_0^2 s_{\mathbf{1}}^2 - 2\gamma_0 d_{\mathbf{1}} + B). \tag{D.49}$$

Using Eqs. (D.48) and (D.49) as well as the definition of F, we have:

$$F = \theta_0\gamma_0(d_{\mathbf{1}} - \gamma_0 s_{\mathbf{1}}^2). \tag{D.50}$$

The definitions of $\boldsymbol{p}$ and $p_{\boldsymbol{q}_0}$ as well as the fact that $\boldsymbol{q}_0 = [\mathbf{1}' \quad 0]'$ imply that:

$$p_{\boldsymbol{q}_0} = \theta_0(d_{\mathbf{1}} - \gamma_0 s_{\mathbf{1}}^2). \tag{D.51}$$

Using Eqs. (D.50) and (D.51), we have:

$$p_{\boldsymbol{q}_0} = F/\gamma_0. \tag{D.52}$$

It follows from the definitions of $\boldsymbol{q}_{t,1}$ and $p_{\boldsymbol{q}_0}$ that:

$$\widehat{\boldsymbol{q}}_{t,1} = (\gamma_0/F)\,\mathbf{1}. \tag{D.53}$$

Using Eqs. (D.4), (D.52), and (D.53), we obtain:

$$\boldsymbol{q}_0^* = [\mathbf{1}' \quad 0]'. \tag{D.54}$$

Hence, $(\boldsymbol{p}^*,\boldsymbol{q}_0^*) = (\boldsymbol{p}_{\gamma_0},\boldsymbol{q}_{\theta_0,\gamma_0}^*)$ is an equilibrium for $[(\boldsymbol{d},\boldsymbol{S}),(\boldsymbol{q}_0,\gamma_0)]$. ■

Proof of Theorem D.5. Fix any economy $[(\boldsymbol{d},\boldsymbol{S}),(\boldsymbol{q}_1,H_1,\alpha_1)]$ where $H_1 > -1$ and $\alpha_1 > \underline{\alpha}$. Suppose that $\gamma_1 \in \Gamma_1$. Let $\boldsymbol{p} = \boldsymbol{p}_{\gamma_1}$. Using the definitions of A, C, and $\boldsymbol{p}_{\gamma_1}$, we have:

$$A = \theta_{\gamma_1}(B - \gamma_1 d_{\mathbf{1}}) \tag{D.55}$$

and:

$$C = \theta_{\gamma_1}^2(\gamma_1^2 s_{\mathbf{1}}^2 - 2\gamma_1 d_{\mathbf{1}} + B). \tag{D.56}$$

It follows from Eqs. (D.55) and (D.56) as well as the definition of F that:

$$F = \theta_{\gamma_1}\gamma_1(d_{\mathbf{1}} - \gamma_1 s_{\mathbf{1}}^2). \tag{D.57}$$

Eqs. (D.55) and (D.57) as well as the definition of G imply that:

$$G = \gamma_1^2 s_{\mathbf{1}}^2. \tag{D.58}$$

Since $\gamma_1 \in \Gamma_1$, we obtain $\gamma_1 < z_{\alpha_1}/s_{\mathbf{1}}$. It follows that:

$$z_{\alpha_1} > \gamma_1 s_{\mathbf{1}}. \tag{D.59}$$

It follows from Eqs. (D.58) and (D.59) that:

$$z_{\alpha_1} > \sqrt{G}. \tag{D.60}$$

The definition of z_{α_1} along with Eqs. (D.5) and (D.60) imply that $\alpha_1 < \overline{\overline{\alpha}}$.

Since $\theta_{\gamma_1} = \frac{d_{\mathbf{1}} - z_{\alpha_1} s_{\mathbf{1}}}{(H_1+1)(d_{\mathbf{1}} - \gamma_1 s_{\mathbf{1}}^2)}$, it follows from the definition of $\boldsymbol{p}_{\gamma_1}$ that:

$$r_f = \frac{(H_1 + 1)\,(d_{\mathbf{1}} - \gamma_1 s_{\mathbf{1}}^2)}{d_{\mathbf{1}} - z_{\alpha_1} s_{\mathbf{1}}} - 1. \tag{D.61}$$

Using Eq. (D.59), we obtain $z_{\alpha_1} s_{\mathbf{1}} > \gamma_1 s_{\mathbf{1}}^2$. It follows that:

$$\frac{d_{\mathbf{1}} - \gamma_1 s_{\mathbf{1}}^2}{d_{\mathbf{1}} - z_{\alpha_1} s_{\mathbf{1}}} > 1. \tag{D.62}$$

It follows from Eqs. (D.61) and (D.62) that $H_1 < r_f$.

Since $\alpha_1 < \overline{\overline{\alpha}}$ and $H_1 < r_f$, part (ii) of Theorem D.2 holds with $m = 1$. Using Eq. (D.7) with $m = 1$, we have:

$$\gamma_1^* = \frac{z_{\alpha_1}\sqrt{G} - G}{(r_f - H_1)p_{\boldsymbol{q}_1}}. \tag{D.63}$$

Eq. (D.58) implies that:

$$z_{\alpha_1}\sqrt{G} - G = \gamma_1(z_{\alpha_1} s_{\mathbf{1}} - \gamma_1 s_{\mathbf{1}}^2). \tag{D.64}$$

Using the definitions of $\boldsymbol{p}_{\gamma_1}$ and $p_{\boldsymbol{q}_1}$ as well as the fact that $\boldsymbol{q}_1 = [\mathbf{1}' \quad 0]'$, we obtain:

$$p_{\boldsymbol{q}_1} = \theta_{\gamma_1}(d_{\mathbf{1}} - \gamma_1 s_{\mathbf{1}}^2). \tag{D.65}$$

Using Eqs. (D.61) and (D.65), the definition of θ_{γ_1}, and elementary algebra, we have:

$$(r_f - H_1)p_{\boldsymbol{q}_1} = z_{\alpha_1} s_{\mathbf{1}} - \gamma_1 s_{\mathbf{1}}^2. \tag{D.66}$$

It follows from Eqs. (D.63), (D.64), and (D.66) that:

$$\gamma_1^* = \gamma_1. \tag{D.67}$$

Eqs. (D.57), (D.65), and (D.67) imply that:

$$p_{\boldsymbol{q}_1} = F/\gamma_1^*. \tag{D.68}$$

Using the definitions of $\boldsymbol{q}_{t,1}$ and $\boldsymbol{p}_{\gamma_1}$, we have:

$$\boldsymbol{q}_{t,1} = (\gamma_1/F)\,[\mathbf{1}' \quad 0]'. \tag{D.69}$$

It follows from Eqs. (D.67) and (D.69) that:

$$(F/\gamma_1^*)\,\boldsymbol{q}_{t,1} = [\mathbf{1}' \quad 0]'. \tag{D.70}$$

Using Eq. (D.6) with $m = 1$ as well as Eqs. (D.68) and (D.70), we have:

$$\boldsymbol{q}_1^* = [\mathbf{1}' \quad 0]'. \tag{D.71}$$

Hence, $(\boldsymbol{p}^*, \boldsymbol{q}_1^*) = (\boldsymbol{p}_{\gamma_1}, \boldsymbol{q}_{1,\gamma_1}^*)$ is an equilibrium for economy $[(\boldsymbol{d}, \boldsymbol{S}), (\boldsymbol{q}_1, H_1, \alpha_1)]$. ■

Proof of Theorem D.6. Fix any economy $[(\boldsymbol{d}, \boldsymbol{S}), \{(\boldsymbol{q}_m, H_m, \alpha_m)\}_{m\in\mathbb{M}}]$ with $H_m > -1$ and $\alpha_m > \underline{\alpha}$ for any account $m \in \mathbb{M}$. Also, fix any $\gamma_a \in \overline{\Gamma}_a$. Let $\boldsymbol{p} = \boldsymbol{p}_{\gamma_a}$. Using the definitions of A, C, and $\boldsymbol{p}_{\gamma_a}$, we have:

$$A = \theta_{\gamma_a}(B - \gamma_a d_{\mathbf{1}}). \tag{D.72}$$

and:

$$C = \theta_{\gamma_a}^2(\gamma_a^2 s_{\mathbf{1}}^2 - 2\gamma_a d_{\mathbf{1}} + B). \tag{D.73}$$

It follows from Eqs. (D.72) and (D.73) as well as the definition of F that:

$$F = \theta_{\gamma_a}\gamma_a(d_{\mathbf{1}} - \gamma_a s_{\mathbf{1}}^2). \tag{D.74}$$

Eqs. (D.72) and (D.74) as well as the definition of G imply that:

$$G = \gamma_a^2 s_{\mathbf{1}}^2. \tag{D.75}$$

Using the arguments in the proof of Theorem D.5, we have $\alpha_m < \overline{\overline{\alpha}}$ for any account $m \in \mathbb{M}$. Since $\gamma_a \in \overline{\Gamma}_a$, we have $H_m < r_f$ for any account $m \in \mathbb{M}$. Hence, Theorem D.3 holds. It follows from Eqs. (D.7) and (D.9) that:

$$\gamma_a^* = \left[\sum_{m\in\mathbb{M}} \frac{(r_f - H_m)p_{\boldsymbol{q}_m}}{z_{\alpha_m}\sqrt{G} - G}\right]^{-1}. \tag{D.76}$$

Using (D.10) and the definition of $\boldsymbol{p}_{\gamma_a}$, we have:

$$r_f = \frac{\sum_{m\in\mathbb{M}} \;\; [(H_m + 1)\,\kappa_{m,\gamma_a}]}{\left(\sum_{m\in\mathbb{M}} \;\; \kappa_{m,\gamma_a}\right) - 1} - 1. \tag{D.77}$$

Fix any account $m \in \mathbb{M}$. Using the definitions of $p_{\boldsymbol{q}_m}$ and κ_{m,γ_a}, we have:

$$p_{\boldsymbol{q}_m} = \kappa_{m,\gamma_a}(z_{\alpha_m}s_{\mathbf{1}} - \gamma_a s_{\mathbf{1}}^2)\left\{\frac{\left(\sum_{m\in\mathbb{M}} \;\; \kappa_{m,\gamma_a}\right) - 1}{\sum_{m\in\mathbb{M}} \;\; [(H_m + 1)\,\kappa_{m,\gamma_a}]}\right\}. \tag{D.78}$$

Eqs. (D.77) and (D.78) imply that:

$$(r_f - H_m)p_{\boldsymbol{q}_m} = \left(1 - (H_m + 1)\left\{\frac{\left(\sum_{m\in\mathbb{M}} \;\; \kappa_{m,\gamma_a}\right) - 1}{\sum_{m\in\mathbb{M}} \;\; [(H_m + 1)\,\kappa_{m,\gamma_a}]}\right\}\right)\kappa_{m,\gamma_a}(z_{\alpha_m}s_{\mathbf{1}} - \gamma_a s_{\mathbf{1}}^2). \tag{D.79}$$

Eq. (D.75) implies that:

$$z_{\alpha_m}\sqrt{G} - G = \gamma_a(z_{\alpha_m}s_{\mathbf{1}} - \gamma_a s_{\mathbf{1}}^2). \tag{D.80}$$

It follows from Eqs. (D.79) and (D.80) that:

$$\frac{(r_f - H_m)p_{\boldsymbol{q}_m}}{z_{\alpha_m}\sqrt{G} - G} = \frac{1}{\gamma_a}\left(1 - (H_m + 1)\left\{\frac{\left(\sum_{m\in\mathbb{M}} \;\; \kappa_{m,\gamma_a}\right) - 1}{\sum_{m\in\mathbb{M}} \;\; [(H_m + 1)\,\kappa_{m,\gamma_a}]}\right\}\right)\kappa_{m,\gamma_a}. \tag{D.81}$$

Using Eqs. (D.76) and (D.81), we have:

$$\gamma_a^* = \gamma_a. \tag{D.82}$$

Using the definitions of $\boldsymbol{p}_{\gamma_a}$ and $p_{\boldsymbol{q}_a}$ as well as the fact that $\boldsymbol{q}_a = [\mathbf{1}' \quad 0]'$, we obtain:

$$p_{\boldsymbol{q}_a} = \theta_{\gamma_a}(d_{\mathbf{1}} - \gamma_a s_{\mathbf{1}}^2). \tag{D.83}$$

Eqs. (D.74), (D.82), and (D.83) imply that:

$$p_{\boldsymbol{q}_a} = F/\gamma_a^*. \tag{D.84}$$

Using the definitions of $\boldsymbol{q}_{t,1}$ and $\boldsymbol{p}_{\gamma_a}$, we have:

$$\boldsymbol{q}_{t,1} = (\gamma_a/F)\,[\mathbf{1}' \quad 0]'. \tag{D.85}$$

It follows from Eqs. (D.82) and (D.85) that:

$$(F/\gamma_a^*)\boldsymbol{q}_{t,1} = [\mathbf{1}' \quad 0]'. \tag{D.86}$$

Using Eqs. (D.8), (D.84), (D.86), we have:

$$\boldsymbol{q}_a^* = [\mathbf{1}' \quad 0]'. \tag{D.87}$$

Hence, $(\boldsymbol{p}^*, \{\boldsymbol{q}_m^*\}_{m\in\mathbb{M}}) = (\boldsymbol{p}_{\gamma_a}, \{\boldsymbol{q}_{m,\gamma_a}^*\}_{m\in\mathbb{M}})$ is an equilibrium for economy $[(\boldsymbol{d},\boldsymbol{S}), \{(\boldsymbol{q}_m, H_m, \alpha_m)\}_{m\in\mathbb{M}}]$ and $\gamma_a^* = \gamma_a$ is the implied risk aversion coefficient of the DMSS agent's aggregate portfolio. ■

Proof of Theorem D.7. Fix any economy $[(\boldsymbol{d},\boldsymbol{S}), (\boldsymbol{q}_0, \gamma_0), \{(\boldsymbol{q}_m, H_m, \alpha_m)\}_{m\in\mathbb{M}}]$ with $\gamma_0 > 0$ as well as $H_m > -1$ and $\alpha_m > \underline{\alpha}$ for all $m \in \mathbb{M}$. Also, fix any $\gamma_{0,a} \in \overline{\Gamma}_{0,a}$. Let $\boldsymbol{p} = \boldsymbol{p}_{\gamma_{0,a}}$. Using the definitions of A, C, and $\boldsymbol{p}_{\gamma_{0,a}}$, we have:

$$A = \theta_{\gamma_{0,a}}(B - \varphi_{0,a} d_{\mathbf{1}}) \tag{D.88}$$

and:

$$C = \theta_{\gamma_{0,a}}^2(\varphi_{0,a}^2 s_{\mathbf{1}}^2 - 2\varphi_{0,a} d_{\mathbf{1}} + B). \tag{D.89}$$

It follows from Eqs. (D.88) and (D.89) as well as the definition of F that:

$$F = \theta_{\gamma_{0,a}} \varphi_{0,a}(d_{\mathbf{1}} - \varphi_{0,a} s_{\mathbf{1}}^2). \tag{D.90}$$

Eqs. (D.88) and (D.90) as well as the definition of G imply that:

$$G = \varphi_{0,a}^2 s_{\mathbf{1}}^2. \tag{D.91}$$

Using the arguments in the proof of Theorem D.5, we have $\alpha_m < \overline{\overline{\alpha}}$ for any account $m \in \mathbb{M}$. Since $\gamma_a \in \overline{\Gamma}_{0,a}$, we have $H_m < r_f$ for any account $m \in \mathbb{M}$. Hence, Theorem D.3 holds. It follows from Eqs. (D.7) and (D.9) that:

$$\gamma_a^* = \left[\sum_{m\in\mathbb{M}} \frac{(r_f - H_m)p_{\boldsymbol{q}_m}}{z_{\alpha_m}\sqrt{G} - G}\right]^{-1}. \tag{D.92}$$

Using Eq. (D.12) and the definition of $\boldsymbol{p}_{\gamma_{0,a}}$, we have:

$$r_f = \frac{\sum_{m\in\mathbb{M}} [(H_m + 1)\,\kappa_{m,\gamma_a}]}{\left(\sum_{m\in\mathbb{M}} \kappa_{m,\gamma_{0,a}}\right) - 1 + \varphi_{0,a}/\gamma_0} - 1. \tag{D.93}$$

Fix any given account $m \in \mathbb{M}$. Using the definitions of $\boldsymbol{p}_{\gamma_{0,a}}$ and $\kappa_{m,\gamma_{0,a}}$, we have:

$$p_{\boldsymbol{q}_m} = \kappa_{m,\gamma_{0,a}}(z_{\alpha_m} s_{\mathbf{1}} - \varphi_{0,a} s_{\mathbf{1}}^2) \left\{ \frac{\left(\sum_{m\in\mathbb{M}} \kappa_{m,\gamma_{0,a}}\right) - 1 + \varphi_{0,a}/\gamma_0}{\sum_{m\in\mathbb{M}} [(H_m + 1)\,\kappa_{m,\gamma_{0,a}}]} \right\}. \tag{D.94}$$

Eqs. (D.93) and (D.94) imply that:

$$(r_f - H_m)p_{\boldsymbol{q}_m} = \left(1 - (H_m + 1)\left\{ \frac{\left(\sum_{m\in\mathbb{M}} \kappa_{m,\gamma_{0,a}}\right) - 1 + \varphi_{0,a}/\gamma_0}{\sum_{m\in\mathbb{M}} [(H_m + 1)\,\kappa_{m,\gamma_{0,a}}]} \right\}\right) \kappa_{m,\gamma_{0,a}}(z_{\alpha_m} s_{\mathbf{1}} - \varphi_{0,a} s_{\mathbf{1}}^2). \tag{D.95}$$

Eq. (D.91) implies that:

$$z_{\alpha_m}\sqrt{G} - G = \varphi_{0,a}(z_{\alpha_m} s_{\mathbf{1}} - \varphi_{0,a} s_{\mathbf{1}}^2). \tag{D.96}$$

It follows from Eqs. (D.95) and (D.96) that:

$$\frac{(r_f - H_m)p_{\boldsymbol{q}_m}}{z_{\alpha_m}\sqrt{G} - G} = \frac{1}{\varphi_{0,a}}\left(1 - (H_m + 1)\left\{\frac{\left(\sum_{m\in\mathbb{M}} \quad \kappa_{m,\gamma_{0,a}}\right) - 1 + \varphi_{0,a}/\gamma_0}{\sum_{m\in\mathbb{M}} \quad [(H_m + 1)\,\kappa_{m,\gamma_{0,a}}]}\right\}\right)\kappa_{m,\gamma_{0,a}}. \tag{D.97}$$

Using Eqs. (D.92) and (D.97), we have:

$$\gamma_a^* = \gamma_{0,a}. \tag{D.98}$$

Using the definitions of $\boldsymbol{p}_{\gamma_{0,a}}$, $p_{\boldsymbol{q}_0}$, and $p_{\boldsymbol{q}_a}$ as well as the fact that $\boldsymbol{q}_0 + \boldsymbol{q}_a = [\mathbf{1}' \quad 0]'$, we obtain:

$$p_{\boldsymbol{q}_0} + p_{\boldsymbol{q}_a} = \theta_{\gamma_{0,a}}(d_{\mathbf{1}} - \varphi_{0,a}s_{\mathbf{1}}^2). \tag{D.99}$$

Eqs. (D.90) and (D.99) imply that:

$$p_{\boldsymbol{q}_0} + p_{\boldsymbol{q}_a} = F/\varphi_{0,a}. \tag{D.100}$$

Using the definitions of $\boldsymbol{q}_{t,1}$ and $\boldsymbol{p}_{\gamma_{0,a}}$, we have:

$$\boldsymbol{q}_{t,1} = (\varphi_{0,a}/F)[\mathbf{1}' \quad 0]'. \tag{D.101}$$

Eqs. (D.4), (D.8), and (D.98) imply that:

$$\boldsymbol{q}_0^* = (p_{\boldsymbol{q}_0})(\boldsymbol{q}_{f,1}) + (F/\gamma_0)(\boldsymbol{q}_{t,1} - \boldsymbol{q}_{f,1}) \tag{D.102}$$

and:

$$\boldsymbol{q}_a^* = (p_{\boldsymbol{q}_a})(\boldsymbol{q}_{f,1}) + (F/\gamma_{0,a})(\boldsymbol{q}_{t,1} - \boldsymbol{q}_{f,1}). \tag{D.103}$$

It follows from Eq. (D.100) that:

$$[p_{\boldsymbol{q}_0} - (F/\gamma_0)] + [p_{\boldsymbol{q}_a} - (F/\gamma_{0,a})] = F/\varphi_{0,a} - F/\gamma_0 - F/\gamma_{0,a}. \tag{D.104}$$

Since $1/\varphi_{0,a} = 1/\gamma_0 + 1/\gamma_{0,a}$, Eq. (D.104) implies that:

$$[p_{\boldsymbol{q}_0} - (F/\gamma_0)] + [p_{\boldsymbol{q}_a} - (F/\gamma_{0,a})] = 0. \tag{D.105}$$

It follows from Eq. (D.101) that:

$$(F/\varphi_{0,a})\boldsymbol{q}_{t,1} = [\mathbf{1}' \quad 0]'. \tag{D.106}$$

Since $1/\varphi_{0,a} = 1/\gamma_0 + 1/\gamma_{0,a}$, we have:

$$F/\gamma_0 + F/\gamma_{0,a} = F/\varphi_{0,a}. \tag{D.107}$$

It follows from Eqs. (D.102), (D.103), and (D.107) that:

$$\boldsymbol{q}_0^* + \boldsymbol{q}_a^* = \{[p_{\boldsymbol{q}_0} - (F/\gamma_0)] + [p_{\boldsymbol{q}_a} - (F/\gamma_{0,a})]\}\boldsymbol{q}_{f,1} + (F/\varphi_{0,a})\boldsymbol{q}_{t,1}. \tag{D.108}$$

Using Eqs. (D.105), (D.106), and (D.108), we have:

$$\boldsymbol{q}_0^* + \boldsymbol{q}_a^* = [\mathbf{1}' \quad 0]'. \tag{D.109}$$

It follows that $[\boldsymbol{p}^*, (\boldsymbol{q}_0^*, \{\boldsymbol{q}_m^*\}_{m\in\mathbb{M}})] = [\boldsymbol{p}_{\gamma_{0,a}}, (\boldsymbol{q}_{0,\gamma_{0,a}}^* \{\boldsymbol{q}_{m,\gamma_{0,a}}^*\}_{m\in\mathbb{M}})]$ is an equilibrium for economy $[(\boldsymbol{d}, \boldsymbol{S}), (\boldsymbol{q}_0, \gamma_0), \{(\boldsymbol{q}_m, H_m, \alpha_m)\}_{m\in\mathbb{M}}]$ and $\gamma_a^* = \gamma_{0,a}$ is the implied risk aversion coefficient of the DMSS agent's aggregate portfolio. ■

Proof of Theorem D.8. Asset market clearing implies that in equilibrium market portfolio **1** is on the MV frontier. Hence, using Eq. (D.1) and the definition of $\widehat{\boldsymbol{q}}_{t,1}$, we obtain:

$$p_{\mathbf{1}}^{*}\left[\frac{\boldsymbol{S}^{-1}(\boldsymbol{d}-\widehat{\boldsymbol{p}}^{*}/p_{J+1})}{F}\right]=\mathbf{1}. \tag{D.110}$$

Premultiplying Eq. (D.110) by $\boldsymbol{d}'$ and using the definitions of A and B, we have:

$$d_{\mathbf{1}}=p_{\mathbf{1}}^{*}\left[\frac{B-A(1+r_f^{*})}{F}\right]. \tag{D.111}$$

It follows from Eq. (D.111) that:

$$r_{\mathbf{1}}^{*}-r_f^{*}=\frac{B-A(1+r_f^{*})-F(1+r_f^{*})}{F}. \tag{D.112}$$

Eq. (D.112) and the definition of G imply that:

$$r_{\mathbf{1}}^{*}-r_f^{*}=\frac{G}{F}. \tag{D.113}$$

Using Eq. (D.110), the definition of $\sigma_{j,\mathbf{1}}^{*}$, and elementary algebra, we have:

$$\sigma_{j,\mathbf{1}}^{*}=\frac{r_j^{*}-r_f^{*}}{F}. \tag{D.114}$$

Eq. (D.3) implies that:

$$(\sigma_{\mathbf{1}}^{*})^2=\frac{(r_{\mathbf{1}}^{*}-r_f^{*})^2}{G}. \tag{D.115}$$

It follows from Eqs. (D.114) and (D.115) that:

$$\beta_j^{*}=\frac{\frac{r_j^{*}-r_f^{*}}{F}}{\frac{(r_{\mathbf{1}}^{*}-r_f^{*})^2}{G}}. \tag{D.116}$$

Using Eqs. (D.113) and (D.116), we have $\beta_j^{*}=\frac{r_j^{*}-r_f^{*}}{r_{\mathbf{1}}^{*}-r_f^{*}}$, which implies Eq. (D.14). ■

Table D.1. Asset endowments when a risk-free asset is present

In the economies examined in Appendix D, there are three risky assets ($j = 1, 2, 3$) with each having one share outstanding and a risk-free asset ($j = 4$) in zero net supply. Panel A of Table 1 shows the expected payoff vector and the variance-covariance matrix for risky asset payoffs. The risk-free asset has a payoff of one. Panel A of Table D.1 provides the asset endowments in four cases. Case A1 shows the endowments in a single-agent economy with an MV agent ($q_{0,j}$, $j = 1, 2, 3, 4$). Case A2 shows the endowments a single-agent economy with a DMSS agent and one account ($q_{1,j}$, $j = 1, 2, 3, 4$). Case A3 shows the endowments in a single-agent economy with a DMSS agent and three accounts ($q_{m,j}$, $m = 1, 2, 3$, $j = 1, 2, 3, 4$). Case A4 shows the endowments in a two-agent economy with an MV agent and a DMSS agent with three accounts.

Panel A. Asset endowments

A1. Single-agent economy with an MV agent

j	$q_{0,j}$
1	1.0
2	1.0
3	1.0
4	0.0

A2. Single-agent economy with a DMSS agent and one account

j	$q_{1,j}$
1	1.0
2	1.0
3	1.0
4	0.0

A3. Single-agent economy with a DMSS agent and three accounts

j	$q_{1,j}$	$q_{2,j}$	$q_{3,j}$
1	0.2	0.2	0.6
2	0.2	0.2	0.6
3	0.2	0.2	0.6
4	−0.2	−0.2	0.4

A4. Two-agent economy with an MV agent and a DMSS agent with three accounts

j	$q_{0,j}$	$q_{1,j}$	$q_{2,j}$	$q_{3,j}$
1	0.5	0.1	0.1	0.3
2	0.5	0.1	0.1	0.3
3	0.5	0.1	0.1	0.3
4	0.0	−0.1	−0.1	0.2

Fig. D.1. Asset prices, expected returns, and betas in a single-agent economy with an MV agent when a risk-free asset is present

Panel A plots equilibrium asset prices ($p_j^*, j = 1, 2, 3$) as a function of the MV agent's risk aversion coefficient (γ_0) while setting the risk-free equilibrium asset price θ_0 to 0.95 (see Theorem D.4). Panels B and C plot, respectively, the corresponding expected asset returns ($r_j^*, j = 1, 2, 3$) and asset betas ($\beta_j^*, j = 1, 2, 3$). In all panels, the solid, dashed, and dotted lines refer to, respectively, assets 1, 2, and 3. In each panel, economy parameters other than γ_0 take the values in panel A of Table 1 and panel A1 of Table D.1. The price, expected return, and beta of the risk-free asset (not reported in the panels of Fig. D.1) are, respectively, 0.95, 5.26% [= 1/0.95 − 1], and zero. Expected returns are reported in percentage points.

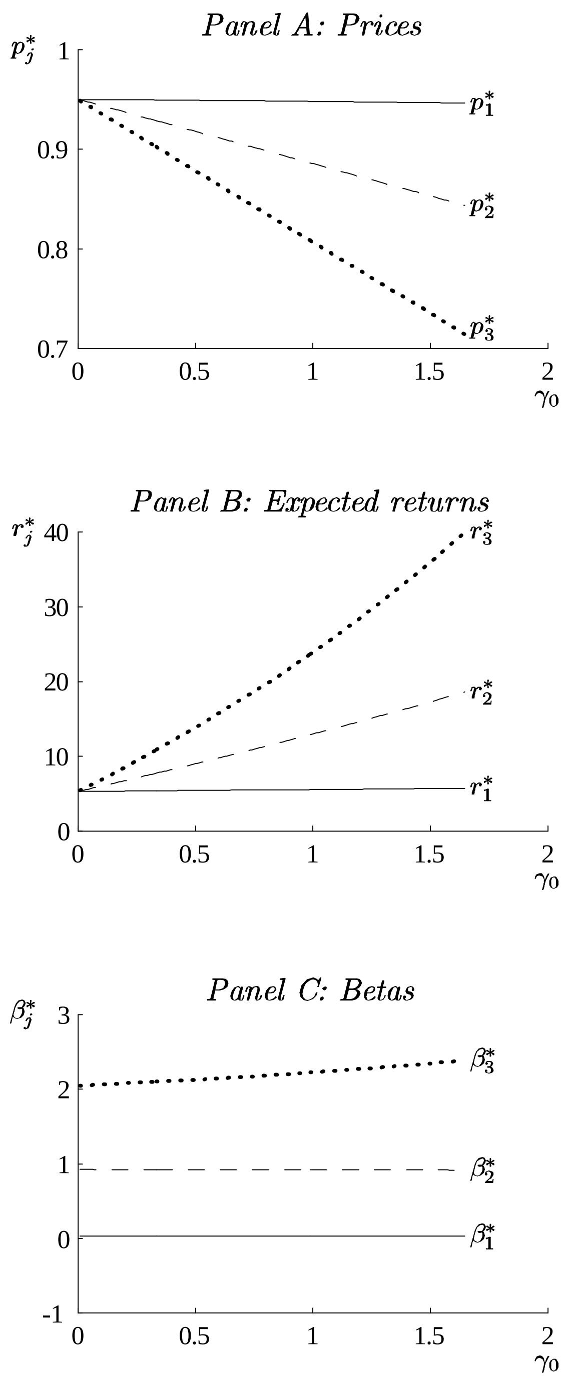


Fig. D.2. Asset prices, expected returns, and betas in a single-agent economy with a DMSS agent and one account when a risk-free asset is present

Given a threshold probability (α_1) and a threshold return (H_1) for account 1, panels A, D, G, and J plot equilibrium asset prices (p_j^*, $j = 1, 2, 3$) as a function of the implied risk aversion coefficient of the DMSS agent's optimal portfolio within account 1 (γ_1^*). Panels B, E, H, and K plot the corresponding expected asset returns (r_j^*, $j = 1, 2, 3$). Panels C, F, I, and L plot the corresponding asset betas (β_j^*, $j = 1, 2, 3$). In all panels, the solid, dashed, and dotted, lines refer to, respectively, assets 1, 2, and 3. In each panel, parameters other than α_1 and H_1 (shown in the title of the corresponding part of the figure) take the values in panel A of Table 1 and panel A2 of Table D.1. The price, expected return, and beta of the risk-free asset (not reported in the panels of Fig. D.2) are very close to those of asset 1. The beta of the risk-free asset is zero. Expected returns are reported in percentage points.

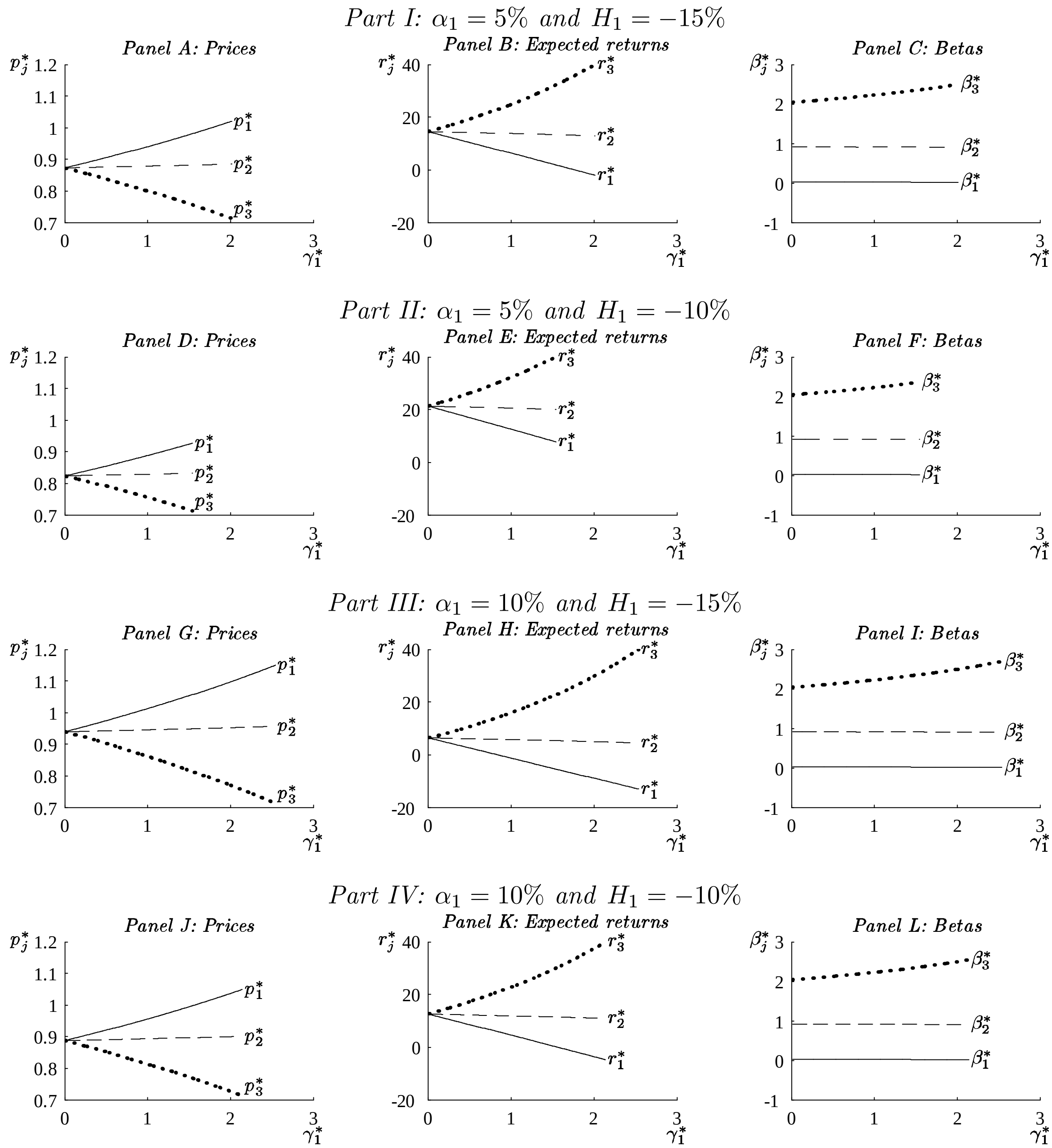
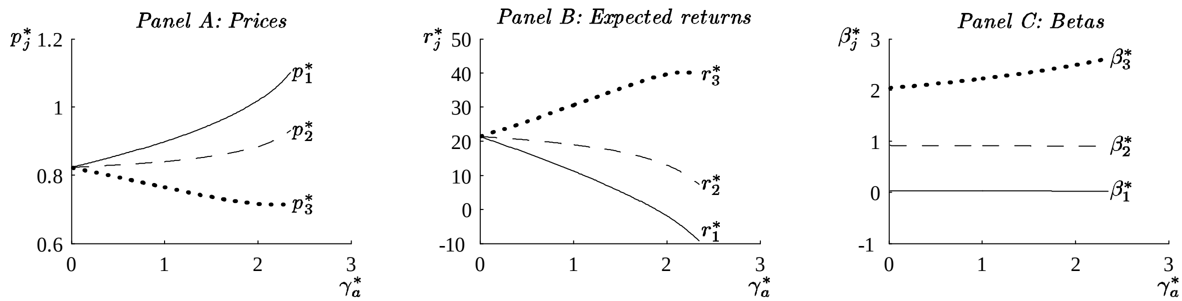


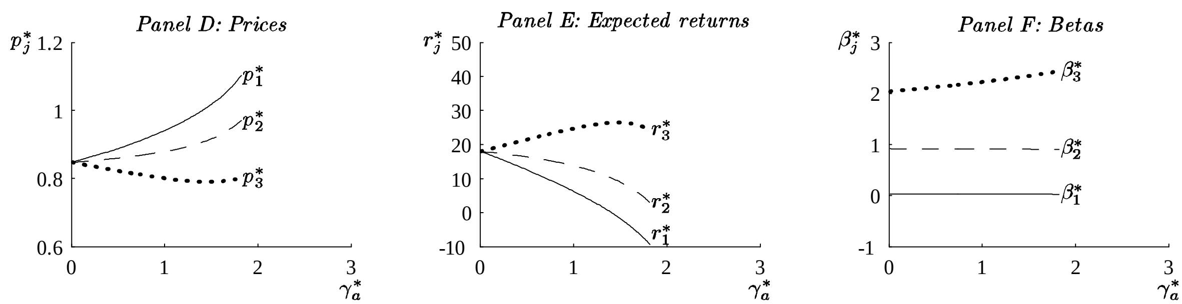
Fig. D.3. Asset prices, expected returns, and betas in a single-agent economy with a DMSS agent and three accounts when a risk-free asset is present

Given the threshold probabilities ($\alpha_m, m = 1,2,3$) and threshold returns ($H_m, m = 1,2,3$) for the accounts, panels A, D, and G plot equilibrium asset prices (p_j^*, $j = 1,2,3$) as a function of the implied risk aversion coefficient of the DMSS agent's aggregate portfolio (γ_a^*). Panels B, E, and H plot the corresponding expected asset returns (r_j^*, $j = 1,2,3$). Panels C, F, and I plot the corresponding asset betas (β_j^*, $j = 1,2,3$). In all panels, the solid, dashed, and dotted lines refer to, respectively, assets 1, 2, and 3. In each panel, parameters other than thresholds (shown in the title of the corresponding part of the figure) take the values in panel A of Table 1 and panel A3 of Table D.1. The price, expected return, and beta of the risk-free asset (not reported in the panels of Fig. D.3) are close to those of asset 1. The beta of the risk-free asset is zero. Expected returns are reported in percentage points.

Part I: $(\alpha_1, \alpha_2, \alpha_3) = (10\%, 5\%, 1\%)$ *and* $(H_1, H_2, H_3) = (-30\%, -25\%, -10\%)$



Part II: $(\alpha_1, \alpha_2, \alpha_3) = (15\%, 5\%, 1\%)$ *and* $(H_1, H_2, H_3) = (-30\%, -25\%, -10\%)$



Part III: $(\alpha_1, \alpha_2, \alpha_3) = (10\%, 5\%, 1\%)$ *and* $(H_1, H_2, H_3) = (-25\%, -25\%, -10\%)$

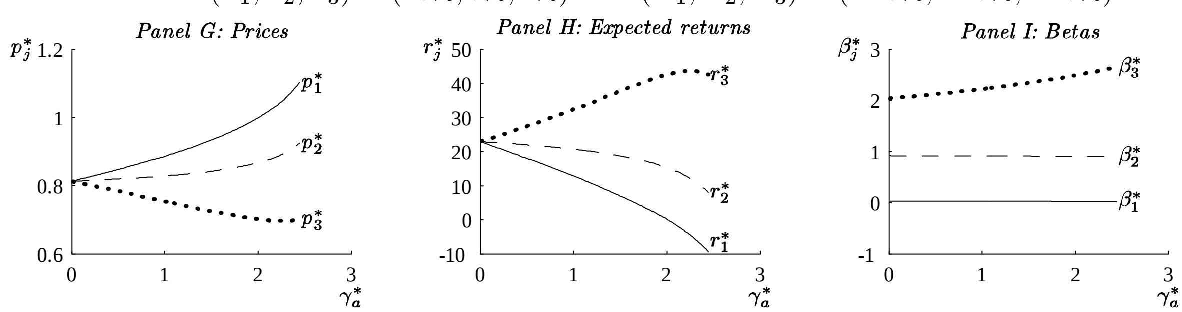
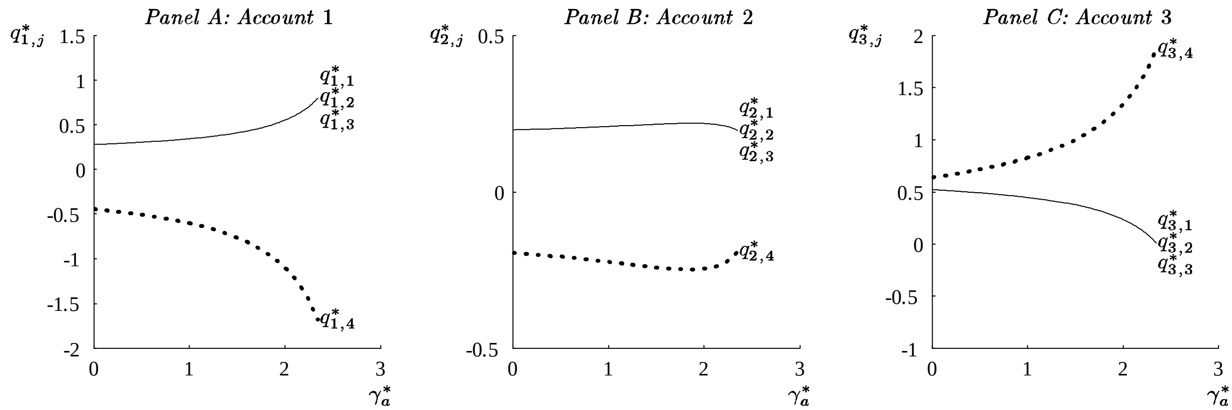


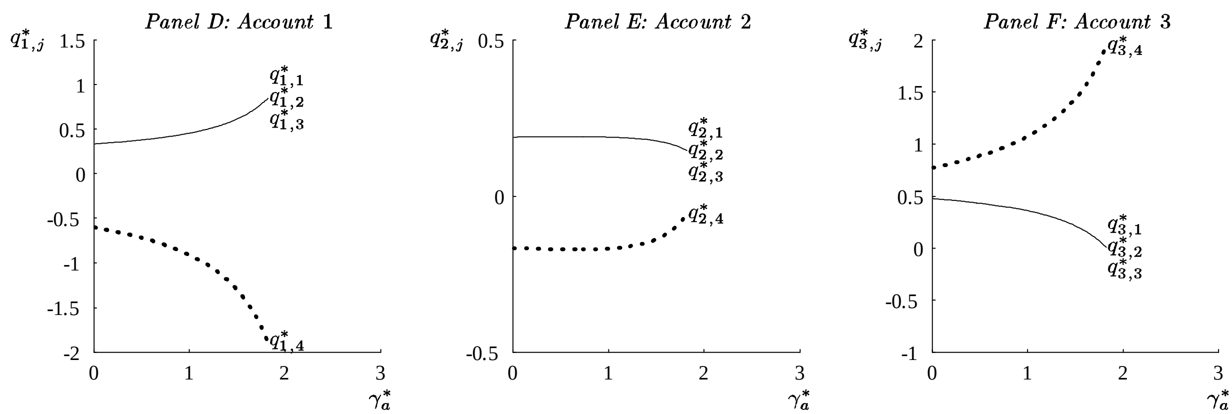
Fig. D.4. Optimal portfolios in a single-agent economy with a DMSS agent and three accounts when a risk-free asset is present

Given the threshold probabilities ($\alpha_m, m = 1,2,3$) and threshold returns ($H_m, m = 1,2,3$) for the accounts, each panel plots the DMSS agent's optimal portfolio within a given account ($q^*_{m,j}$, $m = 1,2,3$, $j = 1,2,3,4$) as a function of the implied risk aversion coefficient of his or her aggregate portfolio (γ^*_a). Panels A, D, and G consider account 1. While panels B, E, and H consider account 2, panels C, F, and I consider account 3. In all panels, the solid and dotted lines report the optimal holding of each risky asset (the same holding for assets 1, 2, and 3) and that of the risk-free asset (i.e., asset 4). In each panel, parameters other than thresholds (shown in the title of the corresponding part of the figure) take the values in panel A of Table 1 and panel A3 of Table D.1.

Part I: $(\alpha_1, \alpha_2, \alpha_3) = (10\%, 5\%, 1\%)$ *and* $(H_1, H_2, H_3) = (-30\%, -25\%, -10\%)$



Part II: $(\alpha_1, \alpha_2, \alpha_3) = (15\%, 5\%, 1\%)$ *and* $(H_1, H_2, H_3) = (-30\%, -25\%, -10\%)$



Part III: $(\alpha_1, \alpha_2, \alpha_3) = (10\%, 5\%, 1\%)$ *and* $(H_1, H_2, H_3) = (-25\%, -25\%, -10\%)$

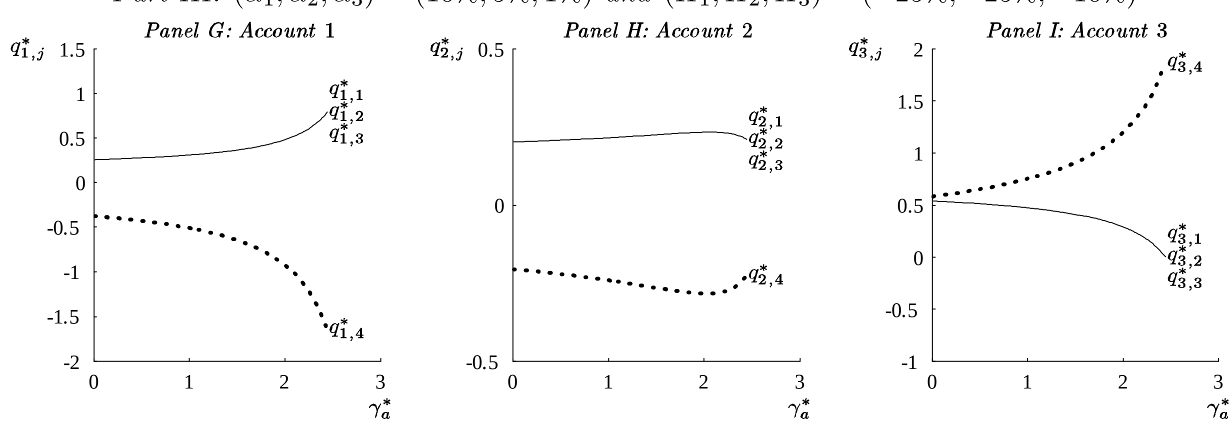
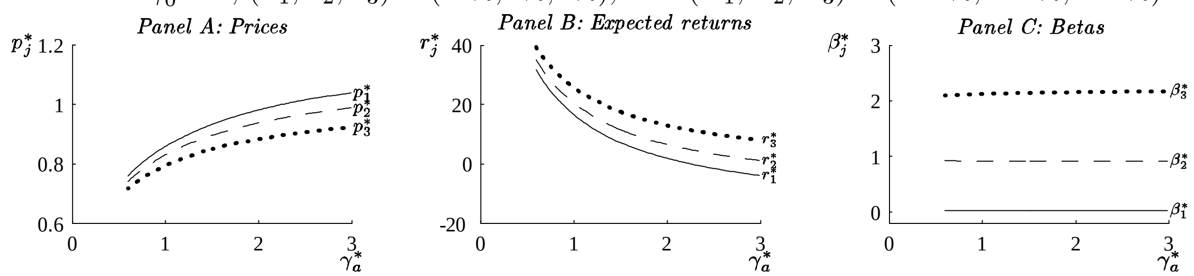
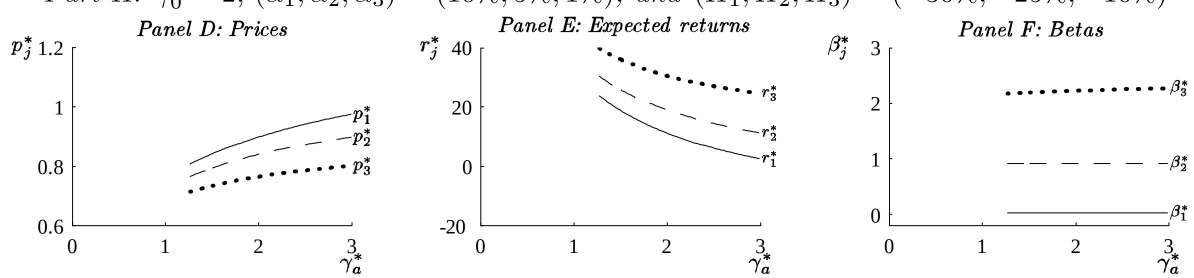


Fig. D.5. Asset prices, expected returns, and betas in a two-agent economy with an MV agent and a DMSS agent with three accounts when a risk-free asset is present
Given the MV agent's risk aversion coefficient (γ_0) as well as the threshold probabilities (α_m, $m = 1, 2, 3$) and threshold returns (H_m, $m = 1, 2, 3$) for the DMSS agent's accounts, panels A, D, G, and J plot equilibrium asset prices (p_j^*, $j = 1, 2, 3$) as a function of the implied risk aversion coefficient of the DMSS agent's aggregate portfolio (γ_a^*). Panels B, E, H, and K plot the corresponding expected asset returns (r_j^*, $j = 1, 2, 3$). Panels C, F, I, and L plot the corresponding asset betas (β_j^*, $j = 1, 2, 3$). In all panels, the solid, dashed, and dotted lines refer to, respectively, assets 1, 2, and 3. In each panel, parameters other than the MV agent's risk aversion coefficient and thresholds (shown in the title of the corresponding part of the figure) take the values in panel A of Table 1 and panel A4 of Table D.1. The price, expected return, and beta of the risk-free asset (not reported in the panels of Fig. D.5) are close to those of asset 1. The beta of the risk-free asset is zero. Expected returns are reported in percentage points.

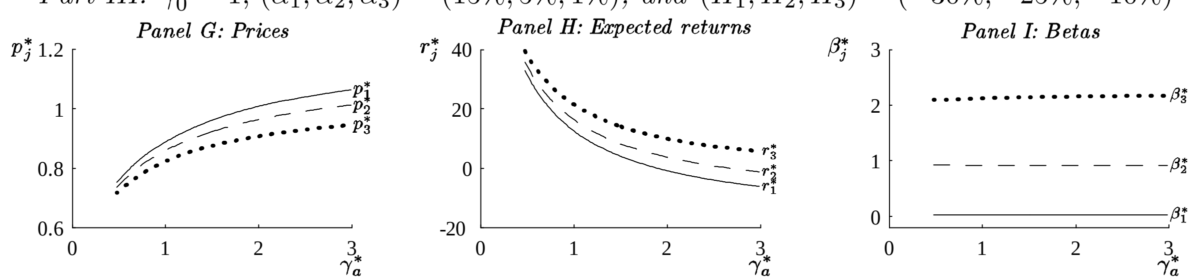
Part I: $\gamma_0 = 1$, $(\alpha_1, \alpha_2, \alpha_3) = (10\%, 5\%, 1\%)$, *and* $(H_1, H_2, H_3) = (-30\%, -25\%, -10\%)$



Part II: $\gamma_0 = 2$, $(\alpha_1, \alpha_2, \alpha_3) = (10\%, 5\%, 1\%)$, *and* $(H_1, H_2, H_3) = (-30\%, -25\%, -10\%)$



Part III: $\gamma_0 = 1$, $(\alpha_1, \alpha_2, \alpha_3) = (15\%, 5\%, 1\%)$, *and* $(H_1, H_2, H_3) = (-30\%, -25\%, -10\%)$



Part IV: $\gamma_0 = 1$, $(\alpha_1, \alpha_2, \alpha_3) = (10\%, 5\%, 1\%)$ *and* $(H_1, H_2, H_3) = (-25\%, -25\%, -10\%)$

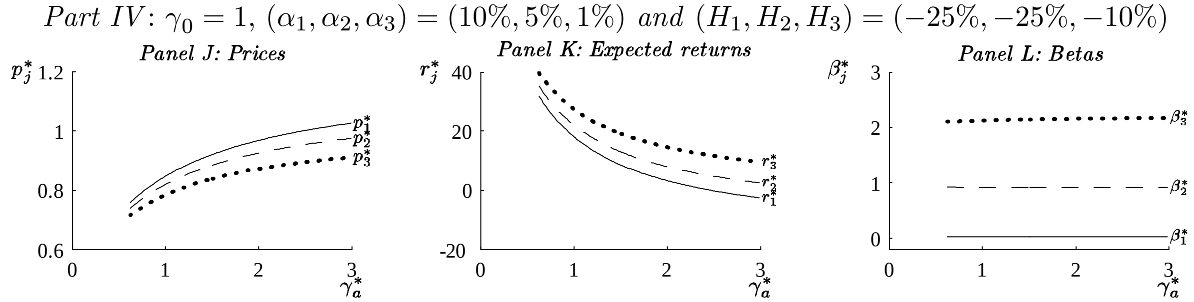
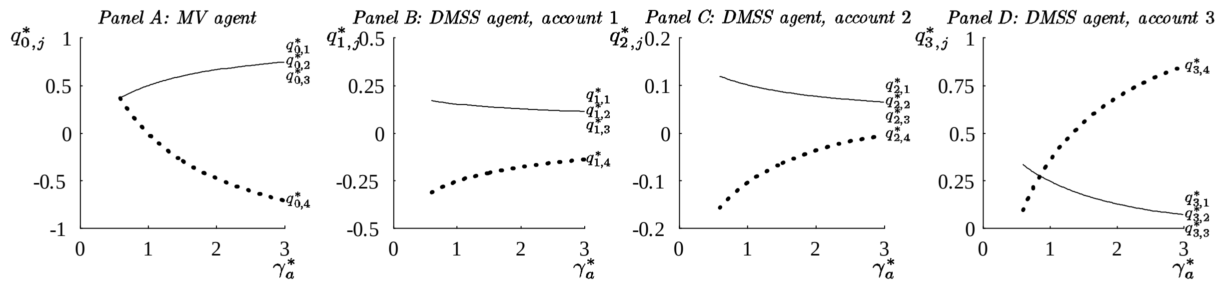


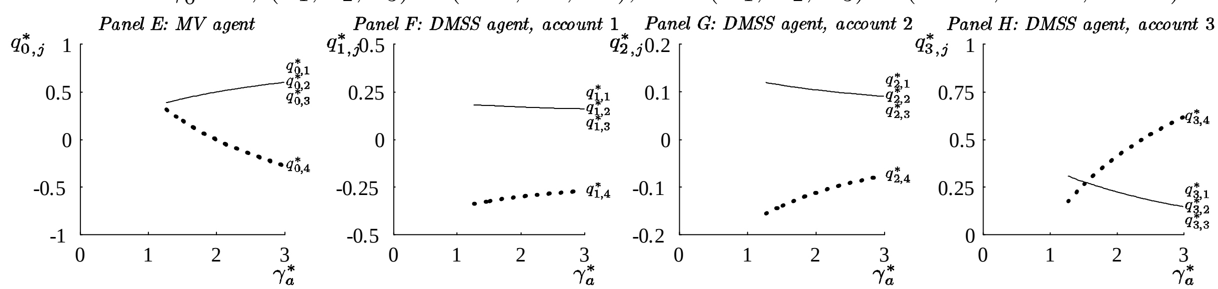
Fig. D.6. Optimal portfolios in a two-agent economy with an MV agent and a DMSS agent with three accounts when a risk-free asset is present

Given the MV agent's risk aversion coefficient (γ_0) as well as the threshold probabilities (α_m, $m = 1, 2, 3$) and threshold returns (H_m, $m = 1, 2, 3$) for the DMSS agent's accounts, panels A, E, I, and M show the MV agent's optimal portfolio ($q^*_{0,j}, j = 1, 2, 3, 4$) as a function of the implied risk aversion coefficient of the DMSS agent's aggregate portfolio (γ^*_a). Similarly, panels B–D, F–H, J–L, and N–P show the DMSS agent's optimal portfolios within accounts ($q^*_{m,j}$, $m = 1, 2, 3, j = 1, 2, 3, 4$) as a function of γ^*_a. In all panels, the solid and dotted lines report the optimal holding of each risky asset (the same holding for assets 1, 2, and 3) and that of the risk-free asset (i.e., asset 4). In each panel, parameters other than the MV agent's risk aversion coefficient and thresholds (shown in the title of the corresponding part of the figure) take the values in panel A of Table 1 and panel A4 of Table D.1.

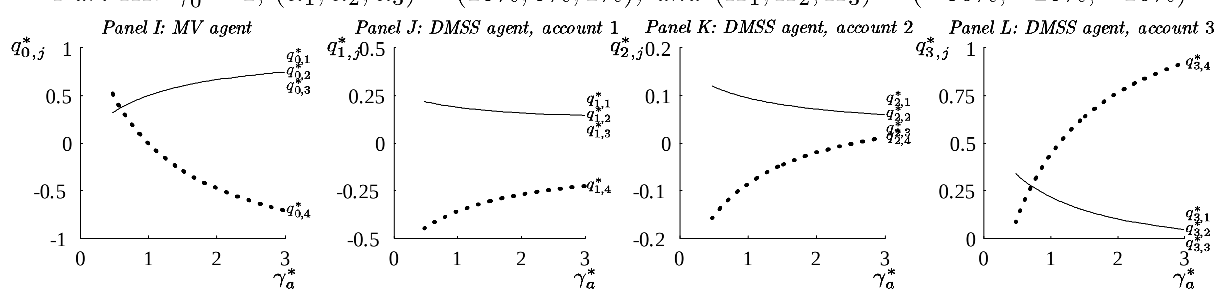
Part I: $\gamma_0 = 1$, $(\alpha_1, \alpha_2, \alpha_3) = (10\%, 5\%, 1\%)$, *and* $(H_1, H_2, H_3) = (-30\%, -25\%, -10\%)$



Part II: $\gamma_0 = 2$, $(\alpha_1, \alpha_2, \alpha_3) = (10\%, 5\%, 1\%)$, *and* $(H_1, H_2, H_3) = (-30\%, -25\%, -10\%)$



Part III: $\gamma_0 = 1$, $(\alpha_1, \alpha_2, \alpha_3) = (15\%, 5\%, 1\%)$, *and* $(H_1, H_2, H_3) = (-30\%, -25\%, -10\%)$



Part IV: $\gamma_0 = 1$, $(\alpha_1, \alpha_2, \alpha_3) = (10\%, 5\%, 1\%)$ *and* $(H_1, H_2, H_3) = (-25\%, -25\%, -10\%)$