Online Appendix

We begin by presenting two lemmas that are useful in our proof of Theorem 1. Fix any $(\alpha, E, E_s) \in (0.5, 1) \times \mathbb{R}^2$. Let $\boldsymbol{w}_{\alpha,E,E_s}$ denote the portfolio with minimum SCVaR at the $100\alpha\%$ confidence level among all portfolios with an expected return of E and a stressed expected return of E_s.

Lemma 1. *For any* $(\alpha, E, E_s) \in (0.5, 1) \times \mathbb{R}^2$, $\boldsymbol{w}_{\alpha,E,E_s}$ *solves:*

$$\min_{\boldsymbol{w} \in \mathbb{R}^N} \ \frac{1}{2} \boldsymbol{w}' \boldsymbol{\Sigma}_s \boldsymbol{w} \tag{8}$$

$$\boldsymbol{w}'\mathbf{1} = 1 \tag{9}$$

$$\boldsymbol{w}'\boldsymbol{\mu} = E \tag{10}$$

$$\boldsymbol{w}'\boldsymbol{\mu}_s = E_s. \tag{11}$$

Proof. Fix any $(\alpha, E, E_s) \in (0.5, 1) \times \mathbb{R}^2$. Suppose by way of a contradiction that $\boldsymbol{w}_{\alpha,E,E_s}$ does not solve minimization problem (8) subject to constraints (9)–(11). Then, there is a portfolio $\boldsymbol{w}^*$ with:

$$\sigma_s^2[r_{\boldsymbol{w}^*}] < \sigma_s^2[r_{\boldsymbol{w}_{\alpha,E,E_s}}], \tag{12}$$

$$E[r_{\boldsymbol{w}^*}] = E, \tag{13}$$

$$E_s[r_{\boldsymbol{w}^*}] = E_s. \tag{14}$$

Using Eqs. (4), (12), and (14), we have:

$$C_{s,\alpha}[r_{\boldsymbol{w}^*}] < C_{s,\alpha}[r_{\boldsymbol{w}_{\alpha,E,E_s}}]. \tag{15}$$

Eqs. (13)–(15) contradict the fact that $\boldsymbol{w}_{\alpha,E,E_s}$ has minimum SCVaR at the $100\alpha\%$ confidence level among all portfolios with an expected return of E and a stressed expected return of E_s. ■

Fix any $(E, E_s) \in \mathbb{R}^2$. Let $\boldsymbol{w}_{E,E_s}$ denote the portfolio that solves minimization problem (8) subject to constraints (9)–(11).

Lemma 2. *For any* $(E, E_s) \in \mathbb{R}^2$*, we have:*

$$\sigma_s^2[r_{\boldsymbol{w}_{E,E_s}}] = h_s + \frac{(E_s[r_{\boldsymbol{w}_{E,E_s}}] - i_s)^2}{g_s}, \tag{16}$$

where g_s *is defined in Section 2,* $h_s \equiv \frac{c_s E^2 - 2a_s E + b_s}{b_s c_s - a_s^2}$*, and* $i_s \equiv \frac{b_s d_s - a_s f_s + (c_s f_s - a_s d_s) E}{b_s c_s - a_s^2}$.[42] *Also,* $g_s \in \mathbb{R}_{++}$ *and* $h_s \in \mathbb{R}_{++}$*.*

Proof. Fix any $(E, E_s) \in \mathbb{R}^2$. First, we show that Eq. (16) holds. A first-order condition for $\boldsymbol{w}_{E,E_s}$ to solve minimization problem (8) subject to constraints (9)–(11) is:

$$\boldsymbol{\Sigma}_s \boldsymbol{w}_{E,E_s} - \lambda_1 \mathbf{1} - \lambda_2 \boldsymbol{\mu} - \lambda_3 \boldsymbol{\mu}_s = \mathbf{0}, \tag{17}$$

where λ_1, λ_2, and λ_3 are multipliers associated with such constraints. Since $rank(\boldsymbol{\Sigma}_s) = N$, Eq. (17) implies that:

$$\boldsymbol{w}_{E,E_s} = \lambda_1 \boldsymbol{\Sigma}_s^{-1} \mathbf{1} + \lambda_2 \boldsymbol{\Sigma}_s^{-1} \boldsymbol{\mu} + \lambda_3 \boldsymbol{\Sigma}_s^{-1} \boldsymbol{\mu}_s. \tag{18}$$

Premultiplying Eq. (18) by $\mathbf{1}'$ and using the definitions of a_s, c_s, and d_s as well as Eq. (9), we have:

$$\lambda_1 c_s + \lambda_2 a_s + \lambda_3 d_s = 1. \tag{19}$$

Similarly, premultiplying Eq. (18) by $\boldsymbol{\mu}'$ and using the definitions of a_s, b_s, and f_s as well as Eq. (10), we have:

$$\lambda_1 a_s + \lambda_2 b_s + \lambda_3 f_s = E. \tag{20}$$

Also, premultiplying Eq. (18) by $\boldsymbol{\mu}_s'$ and using the definitions of d_s, e_s, and f_s as well as Eq. (11), we have:

$$\lambda_1 d_s + \lambda_2 f_s + \lambda_3 e_s = E_s. \tag{21}$$

Using Eqs. (19)–(21) and elementary algebra, we obtain:

$$\lambda_1 = \frac{\left(b_s e_s - f_s^2\right) + \left(d_s f_s - a_s e_s\right) E + \left(a_s f_s - b_s d_s\right) E_s}{2a_s d_s f_s - b_s d_s^2 - c_s f_s^2 + \left(b_s c_s - a_s^2\right) e_s}, \tag{22}$$

$$\lambda_2 = \frac{\left(d_s f_s - a_s e_s\right) + \left(c_s e_s - d_s^2\right) E + \left(a_s d_s - c_s f_s\right) E_s}{2a_s d_s f_s - b_s d_s^2 - c_s f_s^2 + \left(b_s c_s - a_s^2\right) e_s}, \tag{23}$$

$$\lambda_3 = \frac{\left(a_s f_s - b_s d_s\right) + \left(a_s d_s - c_s f_s\right) E + \left(b_s c_s - a_s^2\right) E_s}{2a_s d_s f_s - b_s d_s^2 - c_s f_s^2 + \left(b_s c_s - a_s^2\right) e_s}. \tag{24}$$

[42] While h_s depends on E, we write 'h_s' instead of '$h_{s,E}$' for brevity. A similar remark applies to i_s.

It follows from Eq. (18) that:

$$\sigma_s^2[r_{\boldsymbol{w}_{E,E_s}}] = \left(\lambda_1 \boldsymbol{\Sigma}_s^{-1}\mathbf{1} + \lambda_2 \boldsymbol{\Sigma}_s^{-1}\boldsymbol{\mu} + \lambda_3 \boldsymbol{\Sigma}_s^{-1}\boldsymbol{\mu}_s\right)' \left(\lambda_1 \mathbf{1} + \lambda_2 \boldsymbol{\mu} + \lambda_3 \boldsymbol{\mu}_s\right). \tag{25}$$

Using Eq. (25) as well as the definitions of a_s, b_s, c_s, d_s, e_s, and f_s, we obtain:

$$\sigma_s^2[r_{\boldsymbol{w}_{E,E_s}}] = \lambda_1^2 c_s + \lambda_2^2 b_s + \lambda_3^2 e_s + 2\lambda_1\lambda_2 a_s + 2\lambda_1\lambda_3 d_s + 2\lambda_2\lambda_3 f_s. \tag{26}$$

Eqs. (22)–(24) and (26) along with elementary algebra imply that:

$$\sigma_s^2[r_{\boldsymbol{w}_{E,E_s}}] = \frac{c_s E^2 - 2a_s E + b_s}{b_s c_s - a_s^2} + \frac{\left[E_s - \frac{b_s d_s - a_s f_s + (c_s f_s - a_s d_s)E}{b_s c_s - a_s^2}\right]^2}{\frac{2a_s d_s f_s - b_s d_s^2 - c_s f_s^2}{b_s c_s - a_s^2} + e_s}. \tag{27}$$

Eq. (16) follows from the definitions of g_s, h_s, and i_s along with Eq. (27).

Second, we show that $g_s > 0$. Let $\boldsymbol{\Theta}_2 \equiv \boldsymbol{\Psi}_2' \boldsymbol{\Sigma}_s^{-1} \boldsymbol{\Psi}_2$ and $\boldsymbol{\Theta}_3 \equiv \boldsymbol{\Psi}_3' \boldsymbol{\Sigma}_s^{-1} \boldsymbol{\Psi}_3$ where $\boldsymbol{\Psi}_2 \equiv [\mathbf{1} \quad \boldsymbol{\mu}]$ and $\boldsymbol{\Psi}_3 \equiv [\mathbf{1} \quad \boldsymbol{\mu} \quad \boldsymbol{\mu}_s]$. Observe that $det(\boldsymbol{\Theta}_2) = b_s c_s - a_s^2$ and $det(\boldsymbol{\Theta}_3) = 2a_s d_s f_s - b_s d_s^2 - c_s f_s^2 + \left(b_s c_s - a_s^2\right) e_s$, where $det(\cdot)$ denotes determinant. Noting that $g_s = \frac{det(\boldsymbol{\Theta}_3)}{det(\boldsymbol{\Theta}_2)}$, it suffices to show that $det(\boldsymbol{\Theta}_2) > 0$ and $det(\boldsymbol{\Theta}_3) > 0$. The fact that $\boldsymbol{\Sigma}_s$ is positive definite implies that $\boldsymbol{\Sigma}_s^{-1}$ is also positive definite. Since $rank(\boldsymbol{\Psi}_2) = 2$ and $\boldsymbol{\Sigma}_s^{-1}$ is positive definite, $\boldsymbol{\Theta}_2$ is also positive definite. Hence, $det(\boldsymbol{\Theta}_2) > 0$. Similarly, since $rank(\boldsymbol{\Psi}_3) = 3$ and $\boldsymbol{\Sigma}_s^{-1}$ is positive definite, $\boldsymbol{\Theta}_3$ is also positive definite. Hence, $det(\boldsymbol{\Theta}_3) > 0$.

Third, we show that $h_s > 0$. It suffices to show that the minimum value of h_s is positive. Note that $\frac{\partial h_s}{\partial E} = \frac{2(c_s E - a_s)}{b_s c_s - a_s^2}$. Since $\frac{\partial^2 h_s}{\partial E^2} = \frac{2c_s}{b_s c_s - a_s^2}$, $c_s > 0$, and $b_s c_s - a_s^2 > 0$, we have $\frac{\partial^2 h_s}{\partial E^2} > 0$. Hence, the minimum value of h_s occurs when $\frac{\partial h_s}{\partial E} = 0$ or $E = \frac{a_s}{c_s}$. It follows that the minimum value of h_s is $\frac{1}{c_s}$. Since $c_s > 0$, we have $\frac{1}{c_s} > 0$. ∎

Next, we provide proofs of Theorem 1 and Corollary 1.

Proof of Theorem 1. Fix any expected return $E \in \mathbb{R}$. First, suppose that $\alpha \leq \alpha_s$. Note that:

$$0 < y_\alpha \leq \sqrt{g_s}. \tag{28}$$

In order to prove that no portfolio is on the M-SCVaR frontier, it suffices to show that $\frac{\partial C_{s,\alpha}[r_{\boldsymbol{w}_{E,E_s}}]}{\partial E_s[r_{\boldsymbol{w}_{E,E_s}}]} < 0$. Using Eqs. (4) and (16), we obtain:

$$\frac{\partial C_{s,\alpha}[r_{\boldsymbol{w}_{E,E_s}}]}{\partial E_s[r_{\boldsymbol{w}_{E,E_s}}]} = y_\alpha \frac{\frac{E_s[r_{\boldsymbol{w}_{E,E_s}}]-i_s}{g_s}}{\sqrt{h_s + \frac{(E_s[r_{\boldsymbol{w}_{E,E_s}}]-i_s)^2}{g_s}}} - 1. \tag{29}$$

Since $h_s > 0$, Eqs. (28) and (29) imply that $\frac{\partial C_{s,\alpha}[r_{\boldsymbol{w}_{E,E_s}}]}{\partial E_s[r_{\boldsymbol{w}_{E,E_s}}]} < 0$. This completes the first part of our proof.

Second, suppose that $\alpha > \alpha_s$. Note that:

$$y_\alpha > \sqrt{g_s}. \tag{30}$$

Using Lemmas 1 and 2 along with Eq. (29), a first-order condition for E_s^{**} to solve $\min_{E_s \in \mathbb{R}} C_{s,\alpha}[r_{\boldsymbol{w}_{\alpha,E,E_s}}]$ is:

$$y_\alpha \frac{\frac{E_s^{**}-i_s}{g_s}}{\sqrt{h_s + \frac{(E_s^{**}-i_s)^2}{g_s}}} - 1 = 0. \tag{31}$$

It follows from Eq. (30) and elementary algebra that:

$$E_s^{**} = i_s + \sqrt{\frac{h_s}{y_\alpha^2 - g_s}} g_s. \tag{32}$$

Using Eq. (29) and elementary algebra, we have:

$$\frac{\partial^2 C_{s,\alpha}[r_{\boldsymbol{w}_{E,E_s}}]}{\partial (E_s[r_{\boldsymbol{w}_{E,E_s}}])^2} = \frac{\frac{y_\alpha h_s}{g_s}}{\left[h_s + \frac{(E_s[r_{\boldsymbol{w}_{E,E_s}}]-i_s)^2}{g_s}\right]^{3/2}}. \tag{33}$$

Since $y_\alpha > 0$, $h_s > 0$, and $g_s > 0$, Eq. (33) implies that $\frac{\partial^2 C_{s,\alpha}[r_{\boldsymbol{w}_{E,E_s}}]}{\partial (E_s[r_{\boldsymbol{w}_{E,E_s}}])^2} > 0$. Hence, E_s^{**} solves $\min_{E_s \in \mathbb{R}} C_{s,\alpha}[r_{\boldsymbol{w}_{\alpha,E,E_s}}]$. Using Eqs. (22)–(24) with $E_s = E_s^{**}$ and elementary algebra, we have:

$$\begin{aligned}
\lambda_1 &= \frac{1}{b_s c_s - a_s^2}\left[(b_s - a_s E) + (a_s f_s - b_s d_s)\sqrt{\frac{h_s}{y_\alpha^2 - g_s}}\right], && (34)\\
\lambda_2 &= \frac{1}{b_s c_s - a_s^2}\left[(c_s E - a_s) + (a_s d_s - c_s f_s)\sqrt{\frac{h_s}{y_\alpha^2 - g_s}}\right], && (35)\\
\lambda_3 &= \sqrt{\frac{h_s}{y_\alpha^2 - g_s}}. && (36)
\end{aligned}$$

Let:

$$\theta_{0,\alpha,E} \equiv c_s \lambda_1, \tag{37}$$

$$\theta_{1,\alpha,E} \equiv a_s \lambda_2. \tag{38}$$

Using Eqs. (34)–(38), we obtain:

$$1-\theta_{0,\alpha,E}-\theta_{1,\alpha,E}=d_s\lambda_3. \tag{39}$$

Eq. (5) follows from Eqs. (18) and (34)–(39). This completes the second part of our proof. ■

Proof of Corollary 1. Fix any expected return $E \in \mathbb{R}$. Since $h_s > 0$, we have:

$$\sqrt{\frac{h_s}{y_\alpha^2-g_s}} \to 0 \text{ as } \alpha \to 1. \tag{40}$$

Let $\theta_{0,E} \equiv \left(\frac{b_s-a_sE}{b_sc_s-a_s^2}\right)c_s$ and $\theta_{1,E} \equiv \left(\frac{c_sE-a_s}{b_sc_s-a_s^2}\right)a_s$. Eqs. (34), (37), and (40) imply that $\theta_{0,\alpha,E} \to \theta_{0,E}$ as $\alpha \to 1$. Similarly, Eqs. (35), (38), and (40) imply that $\theta_{1,\alpha,E} \to \theta_{1,E}$ as $\alpha \to 1$. Also, Eqs. (36), (39), and (40) imply that $1-\theta_{0,\alpha,E}-\theta_{1,\alpha,E} \to 0$ as $\alpha \to 1$. ■

We now present two lemmas that are useful in our proof of Theorem 2. Fix any $(\alpha, E, E_s) \in (0.5,1)\times\mathbb{R}^2$. Let $\overline{\boldsymbol{w}}_{\alpha,E,E_s}$ denote the portfolio with minimum SCVaR at the 100α% confidence level among all portfolios with an expected return of E and a stressed expected return of E_s.

Lemma 3. *For any* $(\alpha, E, E_s) \in (0.5,1)\times\mathbb{R}^2$, $\underline{\boldsymbol{w}}_{\alpha,E,E_s}$ *solves:*

$$\min_{\boldsymbol{w}\in\mathbb{R}^N} \ \frac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}_s\boldsymbol{w} \tag{41}$$

$$\boldsymbol{w}'\left(\boldsymbol{\mu}-\mathbf{1}r_f\right) = E-r_f \tag{42}$$

$$\boldsymbol{w}'\left(\boldsymbol{\mu}_s-\mathbf{1}r_{f,s}\right) = E_s-r_{f,s}. \tag{43}$$

Proof. Fix any $(\alpha, E, E_s) \in (0.5,1)\times\mathbb{R}^2$. Suppose by way of a contradiction that $\underline{\boldsymbol{w}}_{\alpha,E,E_s}$ does not solve minimization problem (41) subject to constraints (42) and (43). Then, there is a portfolio $\overline{\boldsymbol{w}}^*$ with:

$$\sigma_s^2[r_{\overline{\boldsymbol{w}}^*}] < \sigma_s^2[r_{\overline{\boldsymbol{w}}_{\alpha,E,E_s}}], \tag{44}$$

$$E[r_{\overline{\boldsymbol{w}}^*}] = E, \tag{45}$$

$$E_s[r_{\overline{\boldsymbol{w}}^*}] = E_s. \tag{46}$$

Using Eqs. (4), (44), and (46), we have:

$$C_{s,\alpha}[r_{\overline{\boldsymbol{w}}^*}] < C_{s,\alpha}[r_{\overline{\boldsymbol{w}}_{\alpha,E,E_s}}]. \tag{47}$$

Eqs. (45)–(47) contradict the fact that $\overline{\boldsymbol{w}}_{\alpha,E,E_s}$ has minimum SCVaR at the $100\alpha\%$ confidence level among all portfolios with an expected return of E and a stressed expected return of E_s. ■

Fix any $(E, E_s) \in \mathbb{R}^2$. Let $\overline{\boldsymbol{w}}_{E,E_s}$ denote the portfolio that solves minimization problem (41) subject to constraints (42) and (43).

Lemma 4. *For any* $(E, E_s) \in \mathbb{R}^2$*, we have:*

$$\sigma_s^2[r_{\overline{\boldsymbol{w}}_{E,E_s}}] = \overline{h}_s + \frac{\left(E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}] - \overline{i}_s\right)^2}{\overline{g}_s}, \tag{48}$$

where $\overline{g}_s$ *is defined in Section 3,* $\overline{h}_s \equiv \frac{1}{j_s}(E - r_f)^2$*, and* $\overline{i}_s \equiv r_{f,s} + \frac{k_s}{j_s}(E - r_f)$.[43] *Also,* $\overline{g}_s \in \mathbb{R}_{++}$ *and* $\overline{h}_s \in \mathbb{R}_+$.

Proof. Fix any $(E, E_s) \in \mathbb{R}^2$. First, we show that Eq. (48) holds. A first-order condition for $\underline{\boldsymbol{w}}_{E,E_s}$ to solve minimization problem (41) subject to constraints (42) and (43) is:

$$\boldsymbol{\Sigma}_s \underline{\boldsymbol{w}}_{E,E_s} - \lambda_4 (\boldsymbol{\mu} - \mathbf{1} r_f) - \lambda_5 (\boldsymbol{\mu}_s - \mathbf{1} r_{f,s}) = \mathbf{0}, \tag{49}$$

where λ_4 and λ_5 are multipliers associated with such constraints. Since $rank(\boldsymbol{\Sigma}_s) = N$, Eq. (49) implies that:

$$\underline{\boldsymbol{w}}_{E,E_s} = \lambda_4 \boldsymbol{\Sigma}_s^{-1} (\boldsymbol{\mu} - \mathbf{1} r_f) + \lambda_5 \boldsymbol{\Sigma}_s^{-1} (\boldsymbol{\mu}_s - \mathbf{1} r_{f,s}). \tag{50}$$

Premultiplying Eq. (50) by $(\boldsymbol{\mu} - \mathbf{1} r_f)'$ and using the definitions of j_s and k_s as well as Eq. (42), we have:

$$\lambda_4 j_s + \lambda_5 k_s = E - r_f. \tag{51}$$

Similarly, premultiplying Eq. (50) by $(\boldsymbol{\mu}_s - \mathbf{1} r_{f,s})'$ and using the definitions of k_s and l_s as well as Eq. (43), we have:

$$\lambda_4 k_s + \lambda_5 l_s = E_s - r_{f,s}. \tag{52}$$

[43] While $\overline{h}_s$ depends on E, we write '$\overline{h}_s$' instead of '$\overline{h}_{s,E}$' for brevity. A similar remark applies to $\overline{i}_s$.

Using Eqs. (51) and (52) along with elementary algebra, we obtain:

$$\lambda_4 = \frac{l_s\left(E - r_f\right) - k_s\left(E_s - r_{f,s}\right)}{j_s l_s - k_s^2}, \tag{53}$$

$$\lambda_5 = \frac{j_s\left(E_s - r_{f,s}\right) - k_s\left(E - r_f\right)}{j_s l_s - k_s^2}. \tag{54}$$

It follows from Eq. (50) that:

$$\sigma_s^2[r_{\overline{\boldsymbol{w}}_{E,E_s}}] = \left[\lambda_4 \boldsymbol{\Sigma}_s^{-1}\left(\boldsymbol{\mu} - \mathbf{1}r_f\right) + \lambda_5 \boldsymbol{\Sigma}_s^{-1}\left(\boldsymbol{\mu}_s - \mathbf{1}r_{f,s}\right)\right]' \left[\lambda_4\left(\boldsymbol{\mu} - \mathbf{1}r_f\right) + \lambda_5\left(\boldsymbol{\mu}_s - \mathbf{1}r_{f,s}\right)\right]. \tag{55}$$

Using Eq. (55) as well as the definitions of j_s, k_s, and l_s, we obtain:

$$\sigma_s^2[r_{\overline{\boldsymbol{w}}_{E,E_s}}] = \lambda_4^2 j_s + \lambda_5^2 l_s + 2\lambda_4\lambda_5 k_s. \tag{56}$$

Eqs. (53), (54), and (56) along with elementary algebra imply that:

$$\sigma_s^2[r_{\overline{\boldsymbol{w}}_{E,E_s}}] = \frac{1}{j_s}\left(E - r_f\right)^2 + \frac{\left(E_s[r_{\boldsymbol{w}_{E,E_s}}] - \left[r_{f,s} + \frac{k_s}{j_s}\left(E - r_f\right)\right]\right)^2}{l_s - k_s^2/j_s}. \tag{57}$$

Eq. (48) follows from the definitions of $\overline{g}_s$, $\overline{h}_s$, and $\overline{i}_s$ along with Eq. (57).

Second, we show that $\overline{g}_s > 0$. Let $\overline{\boldsymbol{\Theta}}_2 \equiv \overline{\boldsymbol{\Psi}}_2' \boldsymbol{\Sigma}_s^{-1} \overline{\boldsymbol{\Psi}}_2$ where $\overline{\boldsymbol{\Psi}}_2 \equiv [\boldsymbol{\mu}_s - \mathbf{1}r_{f,s} \quad \boldsymbol{\mu} - \mathbf{1r_f}]$. Observe that $det(\overline{\boldsymbol{\Theta}}_2) = l_s j_s - k_s^2$. Noting that $\overline{g}_s = \frac{det(\overline{\boldsymbol{\Theta}}_2)}{j_s}$, it suffices to show that $det(\boldsymbol{\Theta}_2) > 0$ and $j_s > 0$. Since $rank([\mathbf{1} \quad \boldsymbol{\mu} \quad \boldsymbol{\mu}_s])= 3$, we have $rank(\overline{\boldsymbol{\Psi}}_2) = 2$. Since $rank(\overline{\boldsymbol{\Psi}}_2) = 2$ and $\boldsymbol{\Sigma}_s^{-1}$ is positive definite, $\overline{\boldsymbol{\Theta}}_2$ is also positive definite. Hence, $det(\overline{\boldsymbol{\Theta}}_2) > 0$. Observe that $j_s = \left(\boldsymbol{\mu} - \mathbf{1}r_f\right)' \boldsymbol{\Sigma}_s^{-1}\left(\boldsymbol{\mu} - \mathbf{1}r_f\right)$. Noting that $rank([\mathbf{1} \quad \boldsymbol{\mu}])= 2$, we have $\boldsymbol{\mu} - \mathbf{1}r_f \neq \mathbf{0}$. Since $\boldsymbol{\Sigma}_s^{-1}$ is positive definite and $\boldsymbol{\mu} - \mathbf{1}r_f \neq \mathbf{0}$, we have $j_s > 0$.

Third, we show that $\overline{h}_s \geq 0$. Since $\overline{h}_s = \frac{1}{j_s}\left(E - r_f\right)^2$ and $j_s > 0$, we have $\overline{h}_s \geq 0$. ■

Next, we provide proofs of Theorem 2 and Corollary 2.

Proof of Theorem 2. Fix any expected return $E \in \mathbb{R}$. First, suppose that $\alpha < \overline{\alpha}_s$. Note that:

$$0 < y_\alpha < \sqrt{\overline{g}_s}. \tag{58}$$

In order to prove that no portfolio is on the M-SCVaR frontier at the $100\alpha\%$ confidence level, it suffices to show that $\frac{\partial C_{s,\alpha}[r_{\overline{\boldsymbol{w}}_{E,E_s}}]}{\partial E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}]} < 0$. Using Eqs. (4) and (48), we obtain:

$$\frac{\partial C_{s,\alpha}[r_{\overline{\boldsymbol{w}}_{E,E_s}}]}{\partial E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}]} = y_\alpha \frac{\frac{E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}]-\overline{i}_s}{\overline{g}_s}}{\sqrt{\overline{h}_s + \frac{(E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}]-\overline{i}_s)^2}{\overline{g}_s}}} - 1. \tag{59}$$

Since $\overline{h}_s \geq 0$, Eqs. (58) and (59) imply that $\frac{\partial C_{s,\alpha}[r_{\overline{\boldsymbol{w}}_{E,E_s}}]}{\partial E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}]} < 0$. This completes the first part of our proof.

Second, suppose that $\alpha = \overline{\alpha}_s$ and $E \neq r_f$. Note that:

$$y_\alpha = \sqrt{\overline{g}_s}. \tag{60}$$

and:

$$\overline{h}_s > 0. \tag{61}$$

In order to prove that no portfolio is on the M-SCVaR frontier at the $100\alpha\%$ confidence level, it suffices to show that $\frac{\partial C_{s,\alpha}[r_{\overline{\boldsymbol{w}}_{E,E_s}}]}{\partial E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}]} < 0$. Eqs. (59)–(61) imply that $\frac{\partial C_{s,\alpha}[r_{\overline{\boldsymbol{w}}_{E,E_s}}]}{\partial E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}]} < 0$. This completes the second part of our proof.

Third, suppose that $\alpha = \overline{\alpha}_s$ and $E = r_f$. Note that Eq. (60) holds and:

$$\overline{h}_s = 0. \tag{62}$$

Using Eqs. (48), (60), and (62), we have:

$$\sigma_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}] = \begin{cases} \frac{E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}]-r_{f,s}}{\sqrt{\overline{g}_s}} & \Leftarrow \quad E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}] > r_{f,s} \\ -\frac{E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}]-r_{f,s}}{\sqrt{\overline{g}_s}} & \Leftarrow \quad E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}] < r_{f,s} \end{cases}. \tag{63}$$

It follows from Eqs. (4), (60), and (63) that:

$$\frac{\partial C_{s,\alpha}[r_{\overline{\boldsymbol{w}}_{E,E_s}}]}{\partial E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}]} = \begin{cases} 0 & \Leftarrow \quad E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}] > r_{f,s} \\ -2 & \Leftarrow \quad E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}] < r_{f,s} \end{cases}. \tag{64}$$

Lemmas 3 and 4 along with Eq. (64) imply that for any $E_s \geq r_{f,s}$ portfolio $\overline{\boldsymbol{w}}_{E,E_s}$ is on the M-SCVaR frontier at the $100\alpha\%$ confidence level. Note that $\overline{\boldsymbol{w}}_{E,E_s} = \overline{\boldsymbol{w}}_0$ if $E = r_f$ and $E_s = r_{f,s}$. This completes the third part of our proof.

Fourth, suppose that $\alpha > \overline{\alpha}_s$. Note that:

$$y_\alpha > \sqrt{\overline{g}_s}. \tag{65}$$

Assume that $E \neq r_f$. Then, Eq. (61) holds. Using Lemmas 3 and 4 along with Eq. (59), a first-order condition for $\overline{E}_s^{**}$ to solve $\min_{E_s \in \mathbb{R}} C_{s,\alpha}[r_{\overline{\boldsymbol{w}}_{E,E_s}}]$ is:

$$y_\alpha \frac{\frac{\overline{E}_s^{**} - \overline{i}_s}{\overline{g}_s}}{\sqrt{\overline{h}_s + \frac{(\overline{E}_s^{**} - \overline{i}_s)^2}{\overline{g}_s}}} - 1 = 0. \tag{66}$$

It follows from Eq. (66) and elementary algebra that:

$$E_s^{**} = \overline{i}_s + \sqrt{\frac{\overline{h}_s}{y_\alpha^2 - \overline{g}_s}} \overline{g}_s. \tag{67}$$

Using Eq. (59) and elementary algebra, we have:

$$\frac{\partial^2 C_{s,\alpha}[r_{\overline{\boldsymbol{w}}_{E,E_s}}]}{\partial (E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}])^2} = \frac{\frac{y_\alpha \overline{h}_s}{\overline{g}_s}}{\left[\overline{h}_s + \frac{(E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}] - \overline{i}_s)^2}{\overline{g}_s}\right]^{3/2}}. \tag{68}$$

Since $y_\alpha > 0$, $\overline{g}_s > 0$, and $\overline{h}_s > 0$, Eq. (68) implies that $\frac{\partial^2 C_{s,\alpha}[r_{\overline{\boldsymbol{w}}_{E,E_s}}]}{\partial (E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}])^2} > 0$. Hence, $\overline{E}_s^{**}$ solves $\min_{E_s \in \mathbb{R}} C_{s,\alpha}[r_{\overline{\boldsymbol{w}}_{E,E_s}}]$. Using Eqs. (53) and (54) with $E_s = \overline{E}_s^{**}$, the definitions of $\overline{g}_s$, $\overline{h}_s$, and $\overline{i}_s$, and elementary algebra, we have:

$$\lambda_4 = \frac{1}{j_s}\left[(E - r_f) - k_s \sqrt{\frac{\frac{1}{j_s}(E - r_f)^2}{y_\alpha^2 - \overline{g}_s}}\right], \tag{69}$$

$$\lambda_5 = \sqrt{\frac{\frac{1}{j_s}(E - r_f)^2}{y_\alpha^2 - \overline{g}_s}}. \tag{70}$$

Let:

$$\overline{\theta}_{0,\alpha,E} \equiv 1 - (a_s - c_s r_f)\lambda_4 - (d_s - c_s r_{f,s})\lambda_5, \tag{71}$$

$$\overline{\theta}_{1,\alpha,E} \equiv (a_s - c_s r_f)\lambda_4. \tag{72}$$

Using Eqs. (69)–(72), we obtain:

$$1 - \overline{\theta}_{0,\alpha,E} - \overline{\theta}_{1,\alpha,E} = (d_s - c_s r_{f,s})\lambda_5. \tag{73}$$

Eq. (7) follows from Eqs. (50) and (69)–(73).[44]

Assume that $E = r_f$. Then, Eq. (63) holds. Using Eqs. (63) and (65), we have $\frac{\partial C_{s,\alpha}[r_{\overline{\boldsymbol{w}}_{E,E_s}}]}{\partial E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}]} < 0$ if $E_s < r_{f,s}$ and $\frac{\partial C_{s,\alpha}[r_{\overline{\boldsymbol{w}}_{E,E_s}}]}{\partial E_s[r_{\overline{\boldsymbol{w}}_{E,E_s}}]} > 0$ if $E_s > r_{f,s}$. Hence, $\overline{E}_s^{**} = r_{f,s}$ solves $\min_{E_s \in \mathbb{R}} C_{s,\alpha}[r_{\overline{\boldsymbol{w}}_{E,E_s}}]$. It follows that Eq. (7) holds with $\overline{\theta}_{0,\alpha,E} = 1$ and $\overline{\theta}_{1,\alpha,E} = 0$. This completes the fourth part of our proof. ■

Proof of Corollary 2. Fix any expected return $E \in \mathbb{R}$. Let $\overline{\theta}_{0,E} \equiv 1 - \frac{1}{j_s}(a_s - c_s r_f)(E - r_f)$ and $\overline{\theta}_{1,E} \equiv \frac{1}{j_s}(a_s - c_s r_f)(E - r_f)$. First, suppose that $E = r_f$. It follows from the proof of Theorem 2 that $\overline{\theta}_{0,\alpha,E} = 1$, $\overline{\theta}_{1,\alpha,E} = 0$, and $1 - \overline{\theta}_{0,\alpha,E} - \overline{\theta}_{1,\alpha,E} = 0$. The desired claims follow from the fact that $\overline{\theta}_{0,E} = 1$ and $\overline{\theta}_{1,E} \equiv 0$.

Second, suppose that $E \neq r_f$. Note that:

$$\sqrt{\frac{\frac{1}{j_s}(E - r_f)^2}{y_\alpha^2 - \overline{g}_s}} \to 0 \text{ as } \alpha \to 1. \tag{74}$$

Eqs. (69), (70), and (74) imply that $\overline{\theta}_{0,\alpha,E} \to \overline{\theta}_{0,E}$ as $\alpha \to 1$. Similarly, Eqs. (69), (72), and (74) imply that $\overline{\theta}_{1,\alpha,E} \to \overline{\theta}_{1,E}$ as $\alpha \to 1$. Also, Eqs. (70), (73), and (74) imply that $1 - \overline{\theta}_{0,\alpha,E} - \overline{\theta}_{1,\alpha,E} \to 0$ as $\alpha \to 1$. ■

[44] If $\alpha > \overline{\alpha}_s$, an alternative characterization of the composition of portfolios on the M-SCVaR frontier at the $100\alpha\%$ confidence level is as follows. Suppose that $E < r_f$. Then, $\overline{\boldsymbol{w}}_{\alpha,E} = \overline{\theta}_{0,\alpha,E}\overline{\boldsymbol{w}}_0 + (1 - \overline{\theta}_{0,\alpha,E})\overline{\boldsymbol{w}}_{3,s}$ where $\overline{\boldsymbol{w}}_{3,s} \equiv \psi \overline{\boldsymbol{w}}_{1,s} + (1 - \psi)\overline{\boldsymbol{w}}_{2,s}$, $\psi \equiv \frac{(1+k_s m_s)(a_s - c_s r_f)/j_s}{(1+k_s m_s)(a_s - c_s r_f)/j_s - m_s(d_s - c_s r_{f,s})}$, and $m_s \equiv \sqrt{\frac{1/j_s}{y_\alpha^2 - \overline{g}_s}}$. Hence, portfolios on the M-SCVaR frontier at the $100\alpha\%$ confidence level with expected returns smaller than r_f exhibit two-fund separation with the two funds being $\overline{\boldsymbol{w}}_0$ and $\overline{\boldsymbol{w}}_{3,s}$. Similarly, suppose that $E \geq r_f$. Then, $\overline{\boldsymbol{w}}_{\alpha,E} = \overline{\theta}_{0,\alpha,E}\overline{\boldsymbol{w}}_0 + (1 - \overline{\theta}_{0,\alpha,E})\overline{\boldsymbol{w}}_{4,s}$ where $\overline{\boldsymbol{w}}_{4,s} \equiv \varphi \overline{\boldsymbol{w}}_{1,s} + (1 - \varphi)\overline{\boldsymbol{w}}_{2,s}$, and $\varphi \equiv \frac{(1-k_s m_s)(a_s - c_s r_f)/j_s}{(1-k_s m_s)(a_s - c_s r_f)/j_s + m_s(d_s - c_s r_{f,s})}$. Hence, portfolios on the M-SCVaR frontier at the $100\alpha\%$ confidence level with expected returns equal to or larger than r_f exhibit two-fund separation with the two funds being $\overline{\boldsymbol{w}}_0$ and $\overline{\boldsymbol{w}}_{4,s}$. Since $\overline{\boldsymbol{w}}_{3,s}$ generally differs from $\overline{\boldsymbol{w}}_{4,s}$, portfolios on the M-SCVaR frontier at the $100\alpha\%$ confidence level with all expected returns exhibit three-fund separation with the three funds being $\overline{\boldsymbol{w}}_0$, $\overline{\boldsymbol{w}}_{3,s}$, and $\overline{\boldsymbol{w}}_{4,s}$ (or, equivalently, $\overline{\boldsymbol{w}}_0$, $\overline{\boldsymbol{w}}_{1,s}$, and $\overline{\boldsymbol{w}}_{2,s}$).