Online appendix A: proofs

The following three lemmas are useful in the proofs of our theoretical results.

Lemma 1. *If $\alpha < \alpha^\varepsilon$, then the portfolio with minimum estimated VaR at confidence level $1-\alpha$, denoted by $\boldsymbol{w}_\alpha$, has an estimated VaR at this confidence level of $V^\varepsilon_{1-\alpha} \equiv -H^\varepsilon_\alpha$.*

Proof. Suppose that $\alpha < \alpha^\varepsilon$. Using Eq. (4), portfolio $\boldsymbol{w}_\alpha$ is on the estimated MV frontier. It follows from Eqs. (4) and (8) that $E^\varepsilon[r_{\boldsymbol{w}_\alpha}]$ solves:

$$\min_{E\in\mathbb{R}} \quad z_\alpha\sqrt{1/C^\varepsilon + \frac{(E - A^\varepsilon/C^\varepsilon)^2}{D^\varepsilon/C^\varepsilon}} - E. \tag{33}$$

A first-order condition for $E^\varepsilon[r_{\boldsymbol{w}_\alpha}]$ to solve problem (33) is:

$$\frac{z_\alpha\left(E^\varepsilon[r_{\boldsymbol{w}_\alpha}] - A^\varepsilon/C^\varepsilon\right)/\left(D^\varepsilon/C^\varepsilon\right)}{\sqrt{1/C^\varepsilon + \left(E^\varepsilon[r_{\boldsymbol{w}_\alpha}] - A^\varepsilon/C^\varepsilon\right)^2/\left(D^\varepsilon/C^\varepsilon\right)}} - 1 = 0. \tag{34}$$

It follows from Eq. (34) that:

$$E^\varepsilon[r_{\boldsymbol{w}_\alpha}] = \sqrt{\frac{(D^\varepsilon)^2/\left(C^\varepsilon\right)^3}{z_\alpha^2 - D^\varepsilon/C^\varepsilon}} + A^\varepsilon/C^\varepsilon. \tag{35}$$

Using Eqs. (8) and (35), we have:

$$\sigma^\varepsilon[r_{\boldsymbol{w}_\alpha}] = \sqrt{\frac{z_\alpha^2/C^\varepsilon}{z_\alpha^2 - D^\varepsilon/C^\varepsilon}}. \tag{36}$$

Eqs. (4), (10), (35), and (36) imply that $V^\varepsilon[1-\alpha, r_{\boldsymbol{w}_\alpha}] = \sqrt{\frac{z_\alpha^2 - D^\varepsilon/C^\varepsilon}{C^\varepsilon}} - A^\varepsilon/C = -H^\varepsilon_\alpha$. ■

Lemma 2. *Fix any account $m \in \{1, ..., M\}$. If $\alpha_m < \alpha^\varepsilon$ and $H_m \le H^\varepsilon_{\alpha_m}$, then the optimal portfolio within account m, $\boldsymbol{w}^\varepsilon_m$, is on the estimated MV frontier. Furthermore, we have $E^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}] > A^\varepsilon/C^\varepsilon$ and $V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}^\varepsilon_m}] = -H_m$.*

Proof. Fix any account $m \in \{1, ..., M\}$. Suppose that $\alpha_m < \alpha^\varepsilon$ and $H_m \le H^\varepsilon_{\alpha_m}$. First, we show that portfolio $\boldsymbol{w}^\varepsilon_m$ is on the estimated MV frontier. Assume by way of a contradiction that it is not. Then, there exists a portfolio $\boldsymbol{w}$ with $E^\varepsilon[r_{\boldsymbol{w}}] = E^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}]$ and $\sigma^\varepsilon[r_{\boldsymbol{w}}] < \sigma^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}]$. Let $\boldsymbol{w}^* \equiv \zeta\boldsymbol{w}^\varepsilon_{E_1} + (1-\zeta)\boldsymbol{w}$ where $\zeta > 0$ is arbitrarily small and $E_1 > E^\varepsilon[r_{\boldsymbol{w}}]$. Note that $E^\varepsilon[r_{\boldsymbol{w}^*}] > E^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}]$ and $\sigma^\varepsilon[r_{\boldsymbol{w}^*}] < \sigma^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}]$. Hence, it follows from Eq. (4), that $V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}^*}] < V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}^\varepsilon_m}]$. Inequalities $E^\varepsilon[r_{\boldsymbol{w}^*}] > E^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}]$ and $V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}^*}] < V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}^\varepsilon_m}]$ contradict the fact that $\boldsymbol{w}^\varepsilon_m$ is the optimal portfolio within account m. This completes the first part of our proof.

Second, we show that $E^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}] > A^\varepsilon/C^\varepsilon$. Letting $E \equiv E^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}]$, Eqs. (4) and (8) imply that:

$$V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}^\varepsilon_E}] = z_{\alpha_m}\sqrt{1/C^\varepsilon + \left(E^\varepsilon[r_{\boldsymbol{w}^\varepsilon_E}] - A^\varepsilon/C^\varepsilon\right)^2/\left(D^\varepsilon/C^\varepsilon\right)} - E^\varepsilon[r_{\boldsymbol{w}^\varepsilon_E}]. \tag{37}$$

It follows from Eq. (37) that:

$$\frac{\partial V^{\varepsilon}[1-\alpha_m, r_{\boldsymbol{w}^{\varepsilon}_{E}}]}{\partial E^{\varepsilon}[r_{\boldsymbol{w}^{\varepsilon}_{E}}]} = \frac{z_{\alpha_m}\left(E^{\varepsilon}[r_{\boldsymbol{w}^{\varepsilon}_{E}}] - A^{\varepsilon}/C^{\varepsilon}\right)/\left(D^{\varepsilon}/C^{\varepsilon}\right)}{\sqrt{1/C^{\varepsilon} + \left(E^{\varepsilon}[r_{\boldsymbol{w}^{\varepsilon}_{E}}] - A^{\varepsilon}/C^{\varepsilon}\right)^2/\left(D^{\varepsilon}/C^{\varepsilon}\right)}} - 1. \tag{38}$$

Since $z_{\alpha_m} > 0$, Eq. (38) implies that if $E^{\varepsilon}[r_{\boldsymbol{w}^{\varepsilon}_{m}}] \leq A^{\varepsilon}/C^{\varepsilon}$, then $\partial V^{\varepsilon}[1-\alpha_m, r_{\boldsymbol{w}^{\varepsilon}_{E}}]/\partial E^{\varepsilon}[r_{\boldsymbol{w}^{\varepsilon}_{E}}] < 0$. Hence, we have $E^{\varepsilon}[r_{\boldsymbol{w}^{\varepsilon}_{m}}] > A^{\varepsilon}/C^{\varepsilon}$. This completes the second part of our proof.

Third, we show that $V^{\varepsilon}[1-\alpha_m, r_{\boldsymbol{w}^{\varepsilon}_{m}}] = -H_m$. Eq. (5) implies that $V^{\varepsilon}[1-\alpha_m, r_{\boldsymbol{w}^{\varepsilon}_{m}}] \leq -H_m$. Assume by way of a contradiction that $V^{\varepsilon}[1-\alpha_m, r_{\boldsymbol{w}^{\varepsilon}_{m}}] < -H_m$. Let $\boldsymbol{w}^{**} \equiv \delta \boldsymbol{w}^{\varepsilon}_{E_2} + (1-\delta)\boldsymbol{w}^{\varepsilon}_{m}$ where $\delta > 0$ is arbitrarily small and $E_2 > E^{\varepsilon}[r_{\boldsymbol{w}^{\varepsilon}_{m}}]$. Note that $E^{\varepsilon}[r_{\boldsymbol{w}^{**}}] > E^{\varepsilon}[r_{\boldsymbol{w}^{\varepsilon}_{m}}]$ and $V^{\varepsilon}[1-\alpha_m, r_{\boldsymbol{w}^{**}}] < -H_m$, which contradict the fact that $\boldsymbol{w}^{\varepsilon}_{m}$ is the optimal portfolio within account m. This completes the third part of our proof. ■

Lemma 3. *Fix any $\gamma > 0$ and an objective function $f : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by:*

$$f(E^{\varepsilon}[r_{\boldsymbol{w}}], \sigma^{\varepsilon}[r_{\boldsymbol{w}}]) = E^{\varepsilon}[r_{\boldsymbol{w}}] - \frac{\gamma}{2}\left(\sigma^{\varepsilon}[r_{\boldsymbol{w}}]\right)^2. \tag{39}$$

Letting $E_{\gamma,f}$ denote the estimated expected return of the optimal portfolio associated with γ and f, we have $\frac{D^{\varepsilon}/C^{\varepsilon}}{E_{\gamma,f} - A^{\varepsilon}/C^{\varepsilon}} = \gamma$.

Proof of Lemma 3. Fix any $\gamma > 0$ and an objective function $f : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by Eq. (39). Note that the corresponding optimal portfolio is on the estimated MV frontier. Using Eqs. (8) and (39), $E_{\gamma,f}$ solves:

$$\max_{E \in \mathbb{R}} \quad E - \frac{\gamma}{2}\left[1/C^{\varepsilon} + \frac{(E - A^{\varepsilon}/C^{\varepsilon})^2}{D^{\varepsilon}/C^{\varepsilon}}\right]. \tag{40}$$

A first-order condition for $E_{\gamma,f}$ to solve (40) is $1 - \gamma\frac{E_{\gamma,f} - A^{\varepsilon}/C^{\varepsilon}}{D^{\varepsilon}/C^{\varepsilon}} = 0$. Hence, $\frac{D^{\varepsilon}/C^{\varepsilon}}{E_{\gamma,f} - A^{\varepsilon}/C^{\varepsilon}} = \gamma$. ■

Proof of Theorem 1. Fix any account $m \in \{1, ..., M\}$. First, we show (i). Suppose that $\alpha_m \geq \alpha^{\varepsilon}$. Using the definition of z_{α_m} and (9), we have:

$$0 < z_{\alpha_m} \leq \sqrt{D^{\varepsilon}/C^{\varepsilon}}. \tag{41}$$

Fix any level of estimated expected return $E \in \mathbb{R}$. Note that:

$$\frac{\left(E^{\varepsilon}[r_{\boldsymbol{w}^{\varepsilon}_{E}}] - A^{\varepsilon}/C^{\varepsilon}\right)/\left(D^{\varepsilon}/C^{\varepsilon}\right)}{\sqrt{1/C^{\varepsilon} + \left(E^{\varepsilon}[r_{\boldsymbol{w}^{\varepsilon}_{E}}] - A^{\varepsilon}/C^{\varepsilon}\right)^2/\left(D^{\varepsilon}/C^{\varepsilon}\right)}} < \frac{1}{\sqrt{D^{\varepsilon}/C^{\varepsilon}}}. \tag{42}$$

Using Eqs. (38), (41), and (42), we have $\frac{\partial V^{\varepsilon}[1-\alpha_m, r_{\boldsymbol{w}^{\varepsilon}_{E}}]}{\partial E^{\varepsilon}[r_{\boldsymbol{w}^{\varepsilon}_{E}}]} < 0$. It follows that the optimal portfolio within account m does not exist.

Suppose now that $\alpha_m < \alpha^{\varepsilon}$ and $H_m > H^{\varepsilon}_{\alpha_m}$. Note that $-H_m < -H^{\varepsilon}_{\alpha_m} = V^{\varepsilon}_{1-\alpha_m}$. Hence, there exists no portfolio $\boldsymbol{w}$ that meets constraint (5). Therefore, the optimal portfolio within account m does not exist. This completes our proof of part (i).

Second, we show part (ii). Suppose that $\alpha_m < \alpha^\varepsilon$ and $H_m \leq H^\varepsilon_{\alpha_m}$. Lemma 2 and Eq. (8) imply that:

$$E^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}] = A^\varepsilon/C^\varepsilon + \sqrt{(D^\varepsilon/C^\varepsilon)\left[\left(\sigma^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}]\right)^2 - 1/C^\varepsilon\right]}. \tag{43}$$

Using Eqs. (4) and (43) along with Lemma 2, we have:

$$z_{\alpha_m}\sigma^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}] - A^\varepsilon/C^\varepsilon - \sqrt{(D^\varepsilon/C^\varepsilon)\left[\left(\sigma^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}]\right)^2 - 1/C^\varepsilon\right]} = -H_m. \tag{44}$$

It follows from Eq. (44) that:

$$K_1\left(\sigma^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}]\right)^2 + K_2\sigma^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}] + K_3 = 0, \tag{45}$$

where $K_1 \equiv z^2_{\alpha_m} - D^\varepsilon/C^\varepsilon$, $K_2 \equiv -2z_{\alpha_m}\left(A^\varepsilon/C^\varepsilon - H_m\right)$, and $K_3 \equiv \left(A^\varepsilon/C^\varepsilon - H_m\right)^2 + D^\varepsilon/\left(C^\varepsilon\right)^2$. Using Eq. (45), we have:

$$\sigma^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}] = \frac{z_{\alpha_m}\left(A^\varepsilon/C^\varepsilon - H_m\right) \pm \sqrt{(D^\varepsilon/C^\varepsilon)\left[\left(A^\varepsilon/C^\varepsilon - H_m\right)^2 - \left(z^2_{\alpha_m} - D^\varepsilon/C^\varepsilon\right)/C^\varepsilon\right]}}{z^2_{\alpha_m} - D^\varepsilon/C^\varepsilon}. \tag{46}$$

It follows from Eq. (10) that $H^\varepsilon_{\alpha_m} < A^\varepsilon/C^\varepsilon$. Noting that $H_m \leq H^\varepsilon_{\alpha_m} < A^\varepsilon/C^\varepsilon$, we have $A^\varepsilon/C^\varepsilon - H_m > 0$. Using the fact that $\alpha_m < \alpha^\varepsilon$ and Eq. (9), we obtain $z^2_{\alpha_m} - D^\varepsilon/C^\varepsilon > 0$. Since $A^\varepsilon/C^\varepsilon - H_m > 0$, $z^2_{\alpha_m} - D^\varepsilon/C^\varepsilon > 0$, and $\boldsymbol{w}^\varepsilon_m$ solves maximization problem (1) subject to constraints (2) and (5), Eqs. (43) and (46) imply that:

$$\sigma^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}] = \frac{z_{\alpha_m}\left(A^\varepsilon/C^\varepsilon - H_m\right) + \sqrt{(D^\varepsilon/C^\varepsilon)\left[\left(A^\varepsilon/C^\varepsilon - H_m\right)^2 - \left(z^2_{\alpha_m} - D^\varepsilon/C^\varepsilon\right)/C^\varepsilon\right]}}{z^2_{\alpha_m} - D^\varepsilon/C^\varepsilon}. \tag{47}$$

Eqs. (11)–(13) follow from Lemma 2 along with Eqs. (7), (43), and (47). This completes our proof of part (ii).■

Proof of Corollary 1. Fix any account $m \in \{1, ..., M\}$ with $\alpha_m < \alpha^\varepsilon$ and $H_m \leq H^\varepsilon_{\alpha_m}$. Eq. (16) follows from Theorem 1 and Lemma 3. ■

Proof of Theorem 2. Suppose that $\alpha_m < \alpha^\varepsilon$ and $H_m \leq H^\varepsilon_{\alpha_m}$ for any account $m \in \{1, ..., M\}$. Eqs. (18) and (19) follow from Theorem 1. Using Eqs. (7) and (18), the aggregate portfolio is on the estimated MV frontier. Hence, Eq. (20) follows from Eqs. (8) and (19). ■

Proof of Corollary 2. Suppose that $\alpha_m < \alpha^\varepsilon$ and $H_m \leq H^\varepsilon_{\alpha_m}$ for any $m \in \{1, ..., M\}$. Eq. (23) follows from Theorem 2 and Lemma 3. ■

Proof of Theorem 3. Fix any account $m \in \{1, ..., M\}$ and any constant $\gamma^i_m > 0$. Suppose that $\widetilde{\alpha}_m$ and $\widetilde{H}_m$ satisfy, respectively, Eqs. (25) and (26). Noting that $\gamma^i_m > 0$, Eqs. (9) and (24) imply that $\alpha^{\varepsilon,\gamma^i_m} < \alpha^\varepsilon$. Since $\alpha^{\varepsilon,\gamma^i_m} < \alpha^\varepsilon$ and $\widetilde{\alpha}_m \leq \alpha^{\varepsilon,\gamma^i_m}$, we have $\widetilde{\alpha}_m < \alpha^\varepsilon$.

We claim that $\widetilde{H}_m \leq H^{\varepsilon}_{\widetilde{\alpha}_m}$. In order to prove this claim, it suffices to show that:

$$\widetilde{H}_m - H^{\varepsilon}_{\widetilde{\alpha}_m} = 0 \text{ if } z_{\widetilde{\alpha}_m} = \sqrt{[D^{\varepsilon} + (\gamma^i_m)^2]/C^{\varepsilon}} \tag{48}$$

and:

$$\left.\frac{\partial(\widetilde{H}_m - H^{\varepsilon}_{\widetilde{\alpha}_m})}{\partial z_{\widetilde{\alpha}_m}}\right|_{z_{\widetilde{\alpha}_m}=z} \leq 0 \text{ for any } z \geq \sqrt{[D^{\varepsilon} + (\gamma^i_m)^2]/C^{\varepsilon}}. \tag{49}$$

Assume that $z_{\widetilde{\alpha}_m} = \sqrt{[D^{\varepsilon} + (\gamma^i_m)^2]/C^{\varepsilon}}$. It follows from Eq. (26) that $\widetilde{H}_m = \frac{A^{\varepsilon}}{C^{\varepsilon}} - \frac{\gamma^i_m}{C^{\varepsilon}}$. Using Eq. (10) with $\alpha = \widetilde{\alpha}_m$, we have $H^{\varepsilon}_{\widetilde{\alpha}_m} = \frac{A^{\varepsilon}}{C^{\varepsilon}} - \frac{\gamma^i_m}{C^{\varepsilon}}$. Hence, Eq. (48) holds. Eqs. (10) and (26) imply that:

$$\left.\frac{\partial(\widetilde{H}_m - H^{\varepsilon}_{\widetilde{\alpha}_m})}{\partial z_{\widetilde{\alpha}_m}}\right|_{z_{\widetilde{\alpha}_m}=z} = -\sqrt{\frac{1}{C^{\varepsilon}}\left[1 + \frac{D^{\varepsilon}}{(\gamma^i_m)^2}\right]} + \sqrt{\frac{1}{C^{\varepsilon}}\left(\frac{z^2}{z^2 - D^{\varepsilon}/C^{\varepsilon}}\right)}. \tag{50}$$

Using Eq. (50), we have:

$$\left.\frac{\partial(\widetilde{H}_m - H^{\varepsilon}_{\widetilde{\alpha}_m})}{\partial z_{\widetilde{\alpha}_m}}\right|_{z_{\widetilde{\alpha}_m}=\sqrt{[D^{\varepsilon}+(\gamma^i_m)^2]/C^{\varepsilon}}} = 0. \tag{51}$$

Note that:

$$\frac{\partial\sqrt{\frac{1}{C^{\varepsilon}}\left(\frac{z^2}{z^2-D^{\varepsilon}/C^{\varepsilon}}\right)}}{\partial z} \leq 0. \tag{52}$$

Eqs. (50)-(52) imply that Eq. (49) holds.

Since $\widetilde{\alpha}_m < \alpha^{\varepsilon}$ and $\widetilde{H}_m \leq H^{\varepsilon}_{\widetilde{\alpha}_m}$, part (ii) of Theorem 1 is applicable. Using $\widetilde{\alpha}_m$ and $\widetilde{H}_m$ instead of, respectively, α_m and H_m in Eq. (13), and Eq. (26), the standard deviation of portfolio $\widetilde{\boldsymbol{w}}^{\varepsilon}_m$ is:

$$\widetilde{\sigma}^{\varepsilon}_m = \frac{-\frac{z_{\widetilde{\alpha}_m} D^{\varepsilon}}{\gamma^i_m C^{\varepsilon}} + z^2_{\widetilde{\alpha}_m}\sqrt{\frac{1}{C^{\varepsilon}} + \frac{D^{\varepsilon}}{(\gamma^i_m)^2 C^{\varepsilon}}} + \sqrt{\frac{D^{\varepsilon}}{C^{\varepsilon}}\left[\left(\frac{D^{\varepsilon}}{\gamma^i_m C^{\varepsilon}} - z_{\widetilde{\alpha}_m}\sqrt{\frac{1}{C^{\varepsilon}} + \frac{D^{\varepsilon}}{(\gamma^i_m)^2 C^{\varepsilon}}}\right)^2 - \frac{z^2_{\widetilde{\alpha}_m} - D^{\varepsilon}/C^{\varepsilon}}{C^{\varepsilon}}\right]}}{z^2_{\widetilde{\alpha}_m} - \frac{D^{\varepsilon}}{C^{\varepsilon}}}. \tag{53}$$

It follows from Eq. (53) and elementary algebra that:

$$\widetilde{\sigma}^{\varepsilon}_m = \frac{-\frac{z_{\widetilde{\alpha}_m} D^{\varepsilon}}{\gamma^i_m C^{\varepsilon}} + z^2_{\widetilde{\alpha}_m}\sqrt{\frac{1}{C^{\varepsilon}} + \frac{D^{\varepsilon}}{(\gamma^i_m)^2 C^{\varepsilon}}} + \frac{D^{\varepsilon}}{C^{\varepsilon}}\sqrt{\left[\frac{z_{\widetilde{\alpha}_m}}{\gamma^i_m} - \sqrt{\frac{1}{C^{\varepsilon}} + \frac{D^{\varepsilon}}{(\gamma^i_m)^2 C^{\varepsilon}}}\right]^2}}{z^2_{\widetilde{\alpha}_m} - \frac{D^{\varepsilon}}{C^{\varepsilon}}}. \tag{54}$$

Noting that $\widetilde{\alpha}_m \leq \alpha^{\varepsilon,\gamma^i_m}$, we have $z_{\widetilde{\alpha}_m} \geq \sqrt{[D^{\varepsilon} + (\gamma^i_m)^2]/C^{\varepsilon}}$. Since $z_{\widetilde{\alpha}_m} \geq \sqrt{[D^{\varepsilon} + (\gamma^i_m)^2]/C^{\varepsilon}}$ and $\gamma^i_m > 0$, we obtain $\frac{z_{\widetilde{\alpha}_m}}{\gamma^i_m} \geq \sqrt{\frac{1}{C^{\varepsilon}} + \frac{D^{\varepsilon}}{(\gamma^i_m)^2 C^{\varepsilon}}}$. Hence, it follows from Eq. (54) that Eq. (29) holds. ■

Proof of Theorem 4. For any account $m \in \{1, ..., M\}$, suppose that $\widetilde{\alpha}_m$ and $\widetilde{H}_m$ satisfy, respectively, Eqs. (25) and (26) for some constant $\gamma^i_m > 0$. Eq. (30) follows from Eq. (27). Eqs. (7) and (30) imply that the aggregate portfolio is on the estimated MV frontier. Using Eq. (28) and the fact that $\sum_{m=1}^{M} y_m = 1$, the estimated expected return of the aggregate portfolio is:

$$\widetilde{E}^{\varepsilon}_a = \frac{A^{\varepsilon}}{C^{\varepsilon}} + \left(\sum_{m=1}^{M} y_m/\gamma^i_m\right)\frac{D^{\varepsilon}}{C^{\varepsilon}}. \tag{55}$$

Applying Lemma 3 with $E_{\gamma,f} = \frac{A^\varepsilon}{C^\varepsilon} + \left(\sum_{m=1}^{M} y_m/\gamma_m^i\right) \frac{D^\varepsilon}{C^\varepsilon}$, the implied risk aversion coefficient of the aggregate portfolio is $\gamma_a^i = \left(\sum_{m=1}^{M} y_m/\gamma_m^i\right)^{-1}$. Hence, Eq. (55) implies that Eq. (31) holds. Since the aggregate portfolio is on the estimated MV frontier, Eq. (32) follows from Eqs. (8) and (31). ■

Online appendix B: out-of-sample performance of optimal portfolios within accounts and aggregate portfolios when using larger risk aversion coefficients

In this section, we assess the robustness of our main results when using larger risk aversion coefficients than those used in DMSS. Specifically, we consider risk aversion coefficients of 10, 5, and 2 (instead of 4, 3, and 1) in determining the average CERs of optimal portfolios within accounts 1, 2, and 3, respectively. Moreover, we consider a risk aversion coefficient of 5 (instead of 2.4) in determining the average CERs of aggregate portfolios.[54]

B1. Short selling allowed

Suppose that short selling is allowed. First, consider the use of simulated data. Panel A of Table B1 shows the average CERs of optimal portfolios within accounts with fixed thresholds. Compared to panel A of Table 2, average CERs are smaller, endogenous threshold returns are larger (see the middle three rows), and endogenous threshold probabilities are smaller (see the last three rows). Panel C of Table B1 shows the average CERs of optimal portfolios within accounts with variable thresholds. Compared to panel C of Table 2, average CERs are smaller except with 60 draws and exogenous implied risk aversion coefficients (see the first three rows), and endogenous implied risk aversion coefficients are larger (see the last three rows). Importantly, average CERs with either fixed thresholds (panel A of Table B1) or endogenous implied risk aversion coefficients (last three rows of panel C) still exceed the average CERs of optimal portfolios in the MV model (first three rows of panel C).[55]

Panels A and C of Table B2 report the average CERs of aggregate portfolios with, respectively, fixed and variable thresholds. Compared to panels A and C of Table 3, average CERs are smaller except for the case with 60 draws, exogenous implied risk aversion coefficients, and exogenous fractions of wealth in the accounts (see the first row of panel C), whereas for any given account the corresponding endogenous fraction of wealth might decrease, remain unchanged, or increase.

Second, consider the use of empirical data. Panels A and C of Table B3 present the average CERs of optimal portfolios within accounts with, respectively, fixed and variable thresholds. Compared to panels A and C of Table 4, average CERs are smaller except with exogenous implied risk aversion coefficients (see the first three rows of panel C). However, as before, the extent to which average CERs associated with the use of the DMSS model (in panel A of Table B3) exceed those associated with the use of the MV model (in the first three rows of panel C of Table B3) is larger than such

[54] As noted earlier, in the absence of estimation risk, the risk aversion coefficient implied by the aggregate portfolio is $\gamma_a = 1/(\sum_{m=1}^{M} y_m/\gamma_m)$. Recalling that (1) there are $M = 3$ accounts, (2) the exogenous fractions of wealth in the accounts are given by $(y_1, y_2, y_3) = (0.6, 0.2, 0.2)$, and (3) average CERs of optimal portfolios within accounts are determined by using risk aversion coefficients of $(\gamma_1, \gamma_2, \gamma_3) = (10, 5, 2)$, we have $1/(\sum_{m=1}^{M} y_m/\gamma_m) = 1/(0.6/10 + 0.2/5 + 0.2/2) = 5$.

[55] In assessing the statistical significance of the difference between the distributions of CERs for optimal portfolios within accounts and those for optimal portfolios in the MV model, we utilize: (i) the two-sample Kolmogorov-Smirnov test and (ii) the Wilcoxon rank sum test. We find that the difference is statistically significant (at the 1% level) in all cases.

an extent with simulated data (compare panel A of Table B1 to the first three rows of panel C of the same table).[56]

Panels A and C of Table B4 report the average CERs of aggregate portfolios with, respectively, fixed and variable thresholds. Compared to panels A and C of Table 5, average CERs are smaller with fixed thresholds and the majority of the cases with variable thresholds, whereas for any given account the corresponding endogenous fraction of wealth might decrease, remain unchanged, or increase.

B2. Short selling disallowed

Suppose that short selling is disallowed. First, consider the use of simulated data. Panels B and D of Table B1 show the average CERs of optimal portfolios within accounts with, respectively, fixed and variable thresholds. Compared to panels B and D of Table 2, average CERs are smaller. Also, increases in average CERs arising from using the DMSS model with fixed thresholds instead of the MV model (compare panel B and the first three rows of panel D of Table B1) are smaller than those in the case where short selling is allowed (compare panel A and the first three rows of panel C of Table B1). Panels B and D of Table B2 show the average CERs of aggregate portfolios with, respectively, fixed and variable thresholds. Compared to panels B and D of Table 3, average CERs are smaller.

Second, consider the use of empirical data. Panels B and D of Table B3 provide the average CERs of optimal portfolios within accounts with, respectively, fixed and variable thresholds. Panels B and D of Table B4 provide the average CERs of aggregate portfolios with, respectively, fixed and variable thresholds. The results mainly differ from those reported for the simulated data in panels B and D of Tables B1 and B2 in that average CERs are smaller.

B3. Summary

Our main results are robust to using larger risk aversion coefficients than those used in DMSS. The use of the DMSS model reduces estimation risk relative to the use of the MV model with such coefficients, particularly when short selling is allowed.

[56] When using the exogenous thresholds in the second and third rows of panel A of Table B3 and 60 months to find the estimated optimization inputs, the average CERs of the optimal portfolios within accounts 2 and 3 are negative (-0.99% and -1.78%). However, they still notably exceed the corresponding average CERs of the optimal portfolios in the MV model in the second and third rows of panel C (-4.52% and -12.54%). Moreover, the average CERs of the optimal portfolios within accounts 2 and 3 become positive when using either: (a) the exogenous threshold probabilities and endogenous threshold returns in the fifth and sixth rows of panel A (0.92% and 1.35%); or (b) the endogenous threshold probabilities and exogenous threshold returns in the eighth and ninth rows of such a panel (0.97% and 1.46%). Hence, there is still a wide range of thresholds for which optimal portfolios in the DMSS model notably outperform optimal portfolios in the MV model with the risk aversion coefficients used in this appendix.

Table B1: Average CERs of optimal portfolios within accounts using simulated data

This table shows average CERs of optimal portfolios within accounts using simulated data. The number of draws used to find the estimated optimization inputs is either 60 or 120. While panels A and B use fixed threshold probabilities and returns, panels C and D use variable thresholds. Short selling is allowed (disallowed) in panels A and C (B and D). In the first three rows of panels A and B, threshold probabilities and returns are exogenous. In the next three rows, threshold probabilities are exogenous, whereas threshold returns are endogenously set by maximizing average CERs. Similarly, in the last three rows, threshold returns are exogenous, whereas threshold probabilities are endogenously set by maximizing average CERs. In the first three rows of panels C and D, threshold probabilities and returns are set so that the risk aversion coefficients implied by the optimal portfolios within accounts 1, 2, and 3 are exogenously given by, respectively, 10, 5, and 2. In the last three rows, they are endogenously set by maximizing average CERs. In determining the CERs for accounts 1, 2, and 3, all panels use risk aversion coefficients of, respectively, 10, 5, and 2 (except for the first three rows of panels C and D, these coefficients generally differ from the implied risk aversion coefficients).

	Threshold		Avg.	Threshold		Avg.
	probability (%)	return (%)	CER (%)	probability (%)	return (%)	CER (%)
Account	Number of draws = 60			Number of draws = 120		
Panel A: Fixed thresholds, short selling allowed						
1	1.00	−5.00	0.78	1.00	−5.00	0.94
2	5.00	−8.00	0.67	5.00	−8.00	1.25
3	10.00	−10.00	1.56	10.00	−10.00	2.46
1	1.00	−3.94	0.81	1.00	−5.00	0.94
2	5.00	−4.06	1.12	5.00	−5.76	1.35
3	10.00	−6.45	1.87	10.00	−10.02	2.46
1	0.27	−5.00	0.81	0.99	−5.00	0.94
2	0.49	−8.00	1.13	1.91	−8.00	1.36
3	5.14	−10.00	1.91	9.84	−10.00	2.46
Panel B: Fixed thresholds, short selling disallowed						
1	1.00	−5.00	0.58	1.00	−5.00	0.62
2	5.00	−8.00	0.64	5.00	−8.00	0.70
3	10.00	−10.00	1.00	10.00	−10.00	1.10
1	1.00	−4.05	0.62	1.00	−3.82	0.65
2	5.00	−3.61	0.76	5.00	−3.93	0.80
3	10.00	−5.49	1.01	10.00	−6.05	1.10
1	0.25	−5.00	0.62	0.16	−5.00	0.65
2	0.09	−8.00	0.76	0.16	−8.00	0.80
3	1.67	−10.00	1.02	2.35	−10.00	1.10

	Implied risk aversion coefficient	Avg. CER (%)	Implied risk aversion coefficient	Avg. CER (%)
Account	Number of draws = 60		Number of draws = 120	
Panel C: Variable thresholds, short selling allowed				
1	10.00	0.09	10.00	0.73
2	5.00	−0.33	5.00	0.94
3	2.00	−1.74	2.00	1.44
1	27.41	0.81	17.47	0.92
2	13.69	1.11	8.73	1.33
3	5.47	1.86	3.49	2.41
Panel D: Variable thresholds, short selling disallowed				
1	10.00	0.54	10.00	0.61
2	5.00	0.72	5.00	0.76
3	2.00	0.99	2.00	1.06
1	23.82	0.62	17.09	0.64
2	9.77	0.73	7.56	0.77
3	0.53	1.00	0.21	1.10

Table B2: Average CERs of aggregate portfolios using simulated data

This table reports average CERs of aggregate portfolios using simulated data. The number of draws used to find the estimated optimization inputs is either 60 or 120. In the top half of each panel, the fractions of wealth in the accounts are exogeneously given. In the bottom half of each panel, such fractions are endogeneously set by maximizing the average CERs of aggregate portfolios. While short selling is allowed in panels A and C, it is disallowed in panels B and D. Panels A, B, C, and D use the same threshold probabilities and returns as, respectively, panels A, B, C, and D of Table B1.

Threshold						Fraction			Avg.	Threshold						Fraction			Avg.
probability (%)			return (%)			of wealth (%)			CER	probability (%)			return (%)			of wealth (%)			CER
Account									(%)	Account									(%)
1	2	3	1	2	3	1	2	3		1	2	3	1	2	3	1	2	3	
Number of draws = 60										Number of draws = 120									
Panel A: Fixed thresholds, short selling allowed																			
1.00	5.00	10.00	−5.00	−8.00	−10.00	60	20	20	0.97	1.00	5.00	10.00	−5.00	−8.00	−10.00	60	20	20	1.35
1.00	5.00	10.00	−3.94	−4.06	−6.45	60	20	20	1.12	1.00	5.00	10.00	−5.00	−5.76	−10.02	60	20	20	1.35
0.27	0.49	5.14	−5.00	−8.00	−10.00	60	20	20	1.13	0.99	1.91	9.84	−5.00	−8.00	−10.00	60	20	20	1.35
1.00	5.00	10.00	−5.00	−8.00	−10.00	76	24	0	1.13	1.00	5.00	10.00	−5.00	−8.00	−10.00	43	57	0	1.36
1.00	5.00	10.00	−3.94	−4.06	−6.45	2	98	0	1.12	1.00	5.00	10.00	−5.00	−5.76	−10.02	1	99	0	1.35
0.27	0.49	5.14	−5.00	−8.00	−10.00	0	100	0	1.13	0.99	1.91	9.84	−5.00	−8.00	−10.00	0	100	0	1.36
Panel B: Fixed thresholds, short selling disallowed																			
1.00	5.00	10.00	−5.00	−8.00	−10.00	60	20	20	0.76	1.00	5.00	10.00	−5.00	−8.00	−10.00	60	20	20	0.81
1.00	5.00	10.00	−4.05	−3.61	−5.49	60	20	20	0.77	1.00	5.00	10.00	−3.82	−3.93	−6.05	60	20	20	0.80
0.25	0.09	1.67	−5.00	−8.00	−10.00	60	20	20	0.77	0.16	0.16	2.35	−5.00	−8.00	−10.00	60	20	20	0.80
1.00	5.00	10.00	−5.00	−8.00	−10.00	81	19	0	0.77	1.00	5.00	10.00	−5.00	−8.00	−10.00	71	29	0	0.81
1.00	5.00	10.00	−4.05	−3.61	−5.49	67	0	33	0.77	1.00	5.00	10.00	−3.82	−3.93	−6.05	58	0	42	0.81
0.25	0.09	1.67	−5.00	−8.00	−10.00	65	0	35	0.77	0.16	0.16	2.35	−5.00	−8.00	−10.00	57	0	43	0.81

Implied risk aversion coefficient			Fraction of wealth (%)			Avg.	Implied risk aversion coefficient			Fraction of wealth (%)			Avg.
Account						CER	Account						CER
1	2	3	1	2	3	(%)	1	2	3	1	2	3	(%)
Number of draws = 60							Number of draws = 120						
Panel C: Variable thresholds, short selling allowed													
10.00	5.00	2.00	60	20	20	−0.33	10.00	5.00	2.00	60	20	20	0.94
27.41	13.69	5.47	60	20	20	1.11	17.47	8.73	3.49	60	20	20	1.33
10.00	5.00	2.00	100	0	0	1.04	10.00	5.00	2.00	85	15	0	1.33
27.41	13.69	5.47	75	0	25	1.11	17.47	8.73	3.49	75	0	25	1.33
Panel D: Variable thresholds, short selling disallowed													
10.00	5.00	2.00	60	20	20	0.75	10.00	5.00	2.00	60	20	20	0.79
23.82	9.77	0.53	60	20	20	0.76	17.09	7.56	0.21	60	20	20	0.79
10.00	5.00	2.00	70	0	30	0.75	10.00	5.00	2.00	60	0	40	0.79
23.82	9.77	0.53	59	0	41	0.77	17.09	7.56	0.21	54	0	46	0.81

Table B3: Average CERs of optimal portfolios within accounts using empirical data

This table shows average CERs of optimal portfolios within accounts using empirical data. The number of months in the periods used to find the estimated optimization inputs is either 60 or 120. While panels A and B use fixed threshold probabilities and returns, panels C and D use variable thresholds. Short selling is allowed (disallowed) in panels A and C (B and D). In the first three rows of panels A and B, threshold probabilities and returns are exogenous. In the next three rows, threshold probabilities are exogenous, whereas threshold returns are endogenously set by maximizing average CERs. Similarly, in the last three rows, threshold returns are exogenous, whereas threshold probabilities are endogenously set by maximizing average CERs. In the first three rows of panels C and D, threshold probabilities and returns are set so that the risk aversion coefficients implied by the optimal portfolios within accounts 1, 2, and 3 are exogenously given by, respectively, 10, 5, and 2. In the last three rows, they are endogenously set by maximizing average CERs. In determining the CERs for accounts 1, 2, and 3, all panels use risk aversion coefficients of, respectively, 10, 5, and 2 (except for the first three rows of panels C and D, these coefficients generally differ from the implied risk aversion coefficients).

	Threshold		Avg.	Threshold		Avg.
	probability (%)	return (%)	CER (%)	probability (%)	return (%)	CER (%)
Account	Number of months = 60			Number of months = 120		
Panel A: Fixed thresholds, short selling allowed						
1	1.00	−5.00	0.42	1.00	−5.00	0.72
2	5.00	−8.00	−0.99	5.00	−8.00	0.67
3	10.00	−10.00	−1.78	10.00	−10.00	1.77
1	1.00	−2.76	0.70	1.00	−3.83	0.77
2	5.00	−2.19	0.92	5.00	−4.16	1.10
3	10.00	−2.84	1.35	10.00	−7.11	1.96
1	0.02	−5.00	0.74	0.26	−5.00	0.82
2	0.01	−8.00	0.97	0.53	−8.00	1.18
3	0.92	−10.00	1.46	5.97	−10.00	2.13
Panel B: Fixed thresholds, short selling disallowed						
1	1.00	−5.00	0.27	1.00	−5.00	0.45
2	5.00	−8.00	0.45	5.00	−8.00	0.53
3	10.00	−10.00	0.81	10.00	−10.00	0.92
1	1.00	−1.87	0.57	1.00	−3.28	0.55
2	5.00	−1.16	0.62	5.00	−2.42	0.65
3	10.00	−11.31	0.81	10.00	−6.12	0.92
1	0.96	−5.00	0.27	0.31	−5.00	0.51
2	0.76	−8.00	0.47	0.01	−8.00	0.64
3	9.85	−10.00	0.81	2.44	−10.00	0.92

	Implied risk aversion coefficient	Avg. CER (%)	Implied risk aversion coefficient	Avg. CER (%)
Account	Number of months = 60		Number of months = 120	
Panel C: Variable thresholds, short selling allowed				
1	10.00	−1.90	10.00	0.22
2	5.00	−4.52	5.00	−0.06
3	2.00	−12.54	2.00	−0.99
1	49.59	0.74	23.88	0.79
2	26.50	0.92	12.13	1.11
3	11.07	1.34	4.90	1.97
Panel D: Variable thresholds, short selling disallowed				
1	10.00	0.21	10.00	0.43
2	5.00	0.38	5.00	0.54
3	2.00	0.60	2.00	0.86
1	91.01	0.56	34.26	0.54
2	78.29	0.61	17.88	0.59
3	0.00	0.82	0.00	1.06

Table B4: Average CERs of aggregate portfolios using empirical data

This table reports average CERs of aggregate portfolios using empirical data. The number of months in the periods used to find the estimated optimization inputs is either 60 or 120. In the top half of each panel, the fractions of wealth in the accounts are exogeneously given. In the bottom half of each panel, such fractions are endogeneously set by maximizing the average CERs of aggregate portfolios. While short selling is allowed in panels A and C, it is disallowed in panels B and D. Panels A, B, C, and D use the same threshold probabilities and returns as, respectively, panels A, B, C, and D of Table B3.

Threshold probability (%)			Threshold return (%)			Fraction of wealth (%)			Avg. CER (%)	Threshold probability (%)			Threshold return (%)			Fraction of wealth (%)			Avg. CER (%)
Account 1	2	3	1	2	3	1	2	3		Account 1	2	3	1	2	3	1	2	3	
Number of draws = 60										Number of draws = 120									
Panel A: Fixed thresholds, short selling allowed																			
1.00	5.00	10.00	−5.00	−8.00	−10.00	60	20	20	−0.21	1.00	5.00	10.00	−5.00	−8.00	−10.00	60	20	20	0.97
1.00	5.00	10.00	−2.76	−2.19	−2.84	60	20	20	0.91	1.00	5.00	10.00	−3.83	−4.16	−7.11	60	20	20	1.09
0.02	0.01	0.92	−5.00	−8.00	−10.00	60	20	20	0.96	0.26	0.53	5.97	−5.00	−8.00	−10.00	60	20	20	1.18
1.00	5.00	10.00	−5.00	−8.00	−10.00	100	0	0	0.90	1.00	5.00	10.00	−5.00	−8.00	−10.00	89	0	11	1.09
1.00	5.00	10.00	−2.76	−2.19	−2.84	0	100	0	0.92	1.00	5.00	10.00	−3.83	−4.16	−7.11	0	99	1	1.10
0.02	0.01	0.92	−5.00	−8.00	−10.00	0	99	1	0.97	0.26	0.53	5.97	−5.00	−8.00	−10.00	0	96	4	1.18
Panel B: Fixed thresholds, short selling disallowed																			
1.00	5.00	10.00	−5.00	−8.00	−10.00	60	20	20	0.51	1.00	5.00	10.00	−5.00	−8.00	−10.00	60	20	20	0.64
1.00	5.00	10.00	−1.87	−1.16	−11.31	60	20	20	0.67	1.00	5.00	10.00	−3.28	−2.42	−6.12	60	20	20	0.67
0.96	0.76	9.85	−5.00	−8.00	−10.00	60	20	20	0.51	0.31	0.01	2.44	−5.00	−8.00	−10.00	60	20	20	0.66
1.00	5.00	10.00	−5.00	−8.00	−10.00	58	0	42	0.52	1.00	5.00	10.00	−5.00	−8.00	−10.00	80	20	0	0.64
1.00	5.00	10.00	−1.87	−1.16	−11.31	66	0	34	0.68	1.00	5.00	10.00	−3.28	−2.42	−6.12	0	72	28	0.68
0.96	0.76	9.85	−5.00	−8.00	−10.00	58	0	42	0.52	0.31	0.01	2.44	−5.00	−8.00	−10.00	72	0	28	0.66

Implied risk aversion coefficient			Fraction of wealth (%)			Avg. CER (%)	Implied risk aversion coefficient			Fraction of wealth (%)			Avg. CER (%)
Account 1	2	3	1	2	3		Account 1	2	3	1	2	3	
Number of draws = 60							Number of draws = 120						
Panel C: Variable thresholds, short selling allowed													
10.00	5.00	2.00	60	20	20	−4.52	10.00	5.00	2.00	60	20	20	−0.06
49.59	26.50	11.07	60	20	20	0.92	23.88	12.13	4.90	60	20	20	1.11
10.00	5.00	2.00	100	0	0	0.12	10.00	5.00	2.00	100	0	0	1.08
49.59	26.50	11.07	75	0	25	0.92	23.88	12.13	4.90	75	0	25	1.11
Panel D: Variable thresholds, short selling disallowed													
10.00	5.00	2.00	60	20	20	0.42	10.00	5.00	2.00	60	20	20	0.59
91.01	78.29	0.00	60	20	20	0.66	34.26	17.88	0.00	60	20	20	0.65
10.00	5.00	2.00	100	0	0	0.44	10.00	5.00	2.00	59	0	41	0.60
91.01	78.29	0.00	64	0	36	0.67	34.26	17.88	0.00	55	0	45	0.68

Online appendix C: out-of-sample performance of optimal portfolios within accounts relative to that of estimated minimum-variance and equally-weighted portfolios

In order to reduce estimation risk within the MV model, some researchers suggest the use of either the estimated minimum-variance portfolio (see, e.g., Chan, Karceski, and Lakonishok (1999) and Jagannathan and Ma (2003)) or the equally-weighted portfolio (see, e.g., DeMiguel, Garlappi, and Uppal (2009)). While a full-scale examination of the out-of-sample performance of optimal portfolios within accounts relative to that of estimated minimum-variance and equally-weighted portfolios is beyond the scope of our paper, we next compare their out-of-sample performance in our setting.

C1. Short selling allowed

Suppose that short selling is allowed. First, consider the use of simulated data. Panel A of Table C1 reports average CERs of the estimated minimum-variance portfolio. Note that the average CERs of optimal portfolios within accounts with fixed thresholds displayed in panel A of Table 2 exceed those of the estimated minimum-variance portfolio.[57] Similarly, the average CERs of optimal portfolios within accounts with variable thresholds displayed in panel C of Table 2 also exceed those of the estimated minimum-variance portfolio, except with exogenous thresholds and 60 draws.

Panel C of Table C1 reports the average CERs of the equally-weighted portfolio. By design, this portfolio has the same CER in all simulations. Hence, its average CER does not depend on the number of draws.[58] It can be seen that the average CERs of optimal portfolios within accounts with fixed thresholds as shown in panel A of Table 2 exceed those of the equally-weighted portfolio.[59] Similarly, the average CERs of optimal portfolios within accounts with variable thresholds as shown in panel C of Table 2 also exceed those of the equally-weighted portfolio, except with exogenous thresholds and 60 draws.

Second, consider the use of empirical data. Panel A of Table C2 reports average CERs of the estimated minimum-variance portfolio. Note that the average CERs of optimal portfolios within accounts with fixed thresholds displayed in panel A of Table 4 exceed those of the estimated minimum-variance portfolio with a single exception. This exception involves account 2 and the use of 60 months to find the estimated optimization inputs (see the results below the column 'Number

[57] In assessing the statistical significance of the difference between the distributions of CERs for optimal portfolios within accounts and the estimated minimum-variance portfolio, we utilize: (i) the two-sample Kolmogorov-Smirnov test and (ii) the Wilcoxon rank sum test. We find that the difference is statistically significant (at the 1% level) in all cases.

[58] However, it depends on the account since the CERs of different accounts are determined by using different risk aversion coefficients as noted earlier.

[59] Note that the CERs of the optimal portfolio within a given account depend on the simulation, whereas the equally-weighted portfolio has the same CER in all simulations as noted earlier. Hence, the distribution of CERs for the former portfolio differs (by design) from the distribution of CERs for the latter. Therefore, we do not conduct tests to assess the statistical significance of the difference between such distributions.

of months = 60' in the second row of Table 4A and panel A of Table C2). Also, comparing the first three rows of Table 4C with panel A of Table C2, the average CERs of optimal portfolios within accounts with variable exogenous thresholds are smaller than those of the estimated minimum-variance portfolio. However, comparing the last three rows of Table 4C and panel A of Table C2, the average CERs of optimal portfolios within accounts with variable endogenous thresholds are larger than those of the estimated minimum-variance portfolio.

Panel C of Table C2 reports the average CERs of the equally-weighted portfolio. Using this panel and Table 4A, the average CERs of optimal portfolios within accounts with fixed thresholds exceed those the equally-weighted portfolio with a single exception. This exception involves account 2 and the use of 60 months to find the estimated optimization inputs; see the results below the column 'Number of months = 60' in the second row of Table 4A and panel C of Table C2. Also, using the first three rows of Table 4C and panel C of Table C2, the average CERs of optimal portfolios within accounts with variable exogenous thresholds are smaller than those of the equally-weighted portfolio. However, using the last three rows of Table 4C and panel C of Table C2, the average CERs of optimal portfolios within accounts with variable endogenous thresholds are larger than those of the equally-weighted portfolio.

C2. Short selling disallowed

Suppose that short selling is disallowed. First, consider the use of simulated data. Comparing the average CERs of optimal portfolios within accounts (in Tables 2B and 2D) and those of estimated minimum-variance and equally-weighted portfolios (in panels B and C of Table C1, respectively), the results differ from those presented when short selling is allowed in two respects. First, in the case of variable exogenous thresholds and 60 draws, the average CERs of optimal portfolios within accounts now exceed those of estimated minimum-variance and equally-weighted portfolios; focusing on the results under the column 'number of draws = 60,' compare the first three rows of Table 2D to, respectively, panels B and C of Table C1. Second, in other cases, the extent to which the average CERs of the former portfolios exceed those of the latter is smaller; for example, focusing on the case of fixed thresholds and the estimated minimum-variance portfolio, compare the differences between Table 2B and panel B of Table C1 to the differences between Table 2A and panel A of Table C1.

Second, consider the use of empirical data. Comparing the average CERs of optimal portfolios within accounts (in Tables 4B and 4D) and those of the estimated minimum-variance portfolio (in panel B of Table C2), the results differ from those presented when short selling is allowed in that there are fewer cases where the average CERs of the former portfolios are smaller than those of the latter. In contrast, comparing the average CERs of optimal portfolios within accounts (again in Tables 4B and 4D) and those of the equally-weighted portfolio (in panel C of Table C2), the results differ from those presented when short selling is allowed in that there are more cases where the average CERs of the former portfolios are smaller than those of the latter.

C3. Summary

In our setting, we find that the out-of-sample performance of optimal portfolios within accounts typically exceeds those of estimated minimum-variance and equally-weighted portfolios (with certain exceptions discussed earlier). However, a detailed analysis of the relative out-of-sample performance of such portfolios in other settings (involving, e.g., different assets and/or sample periods) is left for future research.

Table C1: Average CERs of estimated minimum-variance and equally-weighted portfolios using simulated data

This table reports average CERs of estimated minimum-variance and equally-weighted portfolios using simulated data. The number of draws used to find the estimated optimization inputs is either 60 or 120. Panel A considers the estimated minimum-variance portfolio when short selling is allowed. Panel B considers the estimated minimum-variance portfolio when short selling is disallowed. Panel C considers the equally-weighted portfolio. By design, this portfolio has the same CER in all simulations. Hence, its average CER does not depend on the number of draws.

	Avg. CER (%)	
Account	Number of draws = 60	Number of draws = 120
Panel A: Estimated minimum-variance portfolio, short selling allowed		
1	0.64	0.65
2	0.66	0.66
3	0.68	0.68
Panel B: Estimated minimum-variance portfolio, short selling disallowed		
1	0.64	0.64
2	0.65	0.65
3	0.68	0.67
Panel C: Equally-weighted portfolio		
1	0.77	0.77
2	0.84	0.84
3	0.98	0.98

Table C2: Average CERs of estimated minimum-variance and equally-weighted portfolios using empirical data

This table reports average CERs of estimated minimum-variance and equally-weighted portfolios using empirical data. The number of months used to find the estimated optimization inputs is either 60 or 120. Panel A considers the estimated minimum-variance portfolio when short selling is allowed. Panel B considers the estimated minimum-variance portfolio when short selling is disallowed. Panel C considers the equally-weighted portfolio.

	Avg. CER (%)	
Account	Number of months = 60	Number of months = 120
Panel A: Estimated minimum-variance portfolio, short selling allowed		
1	0.64	0.55
2	0.65	0.55
3	0.68	0.57
Panel B: Estimated minimum-variance portfolio, short selling disallowed		
1	0.62	0.56
2	0.62	0.57
3	0.64	0.58
Panel C: Equally-weighted portfolio		
1	0.73	0.70
2	0.79	0.76
3	0.92	0.88

Online appendix D: extension of results to the case of non-normality

Our results assume that asset returns have a multivariate normal distribution. However, we next show that these results hold: (1) more generally when asset returns have a multivariate elliptical distribution with finite first and second moments; and (2) at least as an approximation when the multivariate distribution of asset returns is unknown, but has finite first and second moments.

D1. Elliptical distribution

It is well-known that the MV model is consistent with expected utility maximization when asset returns have a multivariate elliptical distribution with finite first and second moments (see, e.g., Ingersoll (1987, Ch. 4, Appendix B)). Hence, suppose that asset returns have such a distribution. For any portfolio $\boldsymbol{w}$, its estimated VaR at confidence level $1-\alpha$ is:

$$V^{\varepsilon,e}[1-\alpha, r_{\boldsymbol{w}}] = z_{\alpha}^{e}\sigma[r_w] - E[r_w], \tag{56}$$

where z_α^e denotes the quantile α of the corresponding univariate elliptical distribution standardized to have zero mean and unit variance. As an illustration of a multivariate elliptical distribution, consider a multivariate t-distribution with six degrees of freedom.[60] For example, if $\alpha = 1\%$, then $z_{0.01}^e = 2.57$ (in comparison, $z_{0.01} = 2.33$ under normality). Replacing $V^{\varepsilon}[1-\alpha, r_{\boldsymbol{w}}]$ with $V^{\varepsilon,e}[1-\alpha, r_{\boldsymbol{w}}]$ throughout our paper, it can be seen that our results hold when asset returns have a multivariate elliptical distribution with finite first and second moments.

D2. Unknown distribution

There is an extensive literature recognizing that the MV model is, at least as an approximation, consistent with expected utility maximization when no distributional assumption on asset returns is made (see, e.g., Markowitz (2000, pp. 52–70)). Hence, suppose that the multivariate distribution of asset returns is unknown, but has finite first and second moments. Let x_α denote the quantile α of the corresponding univariate distribution with mean μ_x and standard deviation σ_x. O'Cinneide (1990) notes that:

$$|x_\alpha - \mu_x| \leq \sigma_x \max\left\{\sqrt{\frac{1-\alpha}{\alpha}}, \sqrt{\frac{\alpha}{1-\alpha}}\right\}. \tag{57}$$

Since we assume that $\alpha \in (0, 1/2)$, we have $\sqrt{\frac{1-\alpha}{\alpha}} > \sqrt{\frac{\alpha}{1-\alpha}}$. It follows from Eq. (57) that for any portfolio $\boldsymbol{w}$, we have:

$$V^{\varepsilon}[1-\alpha, r_{\boldsymbol{w}}] \leq z_{\alpha}^{O}\sigma^{\varepsilon}[r_{\boldsymbol{w}}] - E^{\varepsilon}[r_{\boldsymbol{w}}], \tag{58}$$

[60] We also extend our results with simulated and empirical data to the case where asset returns have a multivariate t-distribution with six degrees of freedom. The results are similar to those presented earlier when asset returns have a multivariate normal distribution. Note that a t-distribution with six degrees of freedom has excess kurtosis of three, whereas the normal distribution has excess kurtosis of zero. Hence, our use of the former distribution allows for considerable fat tails in the distribution of asset returns.

where $z^O_\alpha \equiv \sqrt{\frac{1-\alpha}{\alpha}}$. For example, if $\alpha = 1\%$, then $z^O_{0.01} = \sqrt{\frac{1-0.01}{0.01}} = 9.95$ (in comparison, $z_{0.01} = 2.33$ under normality as noted earlier). Using Eq. (58), $z^O_\alpha \sigma^\varepsilon[r_{\boldsymbol{w}}] - E^\varepsilon[r_{\boldsymbol{w}}]$ is an upper bound to $V^\varepsilon[1-\alpha, r_{\boldsymbol{w}}]$. Replacing $V^\varepsilon[1-\alpha, r_{\boldsymbol{w}}]$ with this upper bound throughout our paper, it can be seen that our results also hold, at least as an approximation, when the multivariate distribution of asset returns is unknown, but has finite first and second moments.

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