An On-Line Supplement for

"A Comparison of VaR and CVaR Constraints on Portfolio Selection with the Mean-Variance Model"

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This Supplement is organized as follows: Section 1 presents an examination of the non-normality case. Section 2 provides an example that illustrates the theoretical results of our paper. Section 3 contains summary tables on the portfolio-choice implications of VaR and CVaR constraints.

1. Non-Normality

Suppose that security rates of return do not have a multivariate normal or t-distribution. We now present an example of a family of probability distribution functions under which the implications are even more perverse than those described in Section 3.[1] Consider a sequence of portfolios with rates of return $\{r_n\}_{n=1}^{\infty}$. For every $n \geq 1$, suppose that rate of return r_n has: (i) a uniform distribution on $[-6\% - n/100, -1\%]$ conditional on the rate of return being lower than or equal to -1%, (ii) a uniform distribution on $(-1\%, 11\%)$ conditional on the rate of return being greater than -1% but lower than 11%, and (iii) a uniform distribution on $[11\%, 16\% + n/100]$ conditional on the rate of return being greater than or equal to 11%. Furthermore, assume that $\text{Prob}[r_n \leq -1\%] = \text{Prob}[r_n \geq 11\%] = 0.5\%$.[2]

It is easy to show that for every $n \geq 1$, $E[r_n] = 5\%$ and, by examining the probability distribution function of each portfolio that $V[0.99, r_n] = 0.94\%$. However, $\sigma^2[r_n] \approx 10^{-7}\left(3n^2 + 93n + 12{,}623\right)$,

[1] Alexander and Baptista (2003) similarly examine arbitrary distributions in evaluating portfolio performance using VaR.

[2] While these distributions are symmetric, examples with similar results can be constructed in which the distributions are skewed.

which converges to infinity as $n \to \infty$.[3] Furthermore, $L[0.99, r_n] \approx 10^{-4}(25n + 223)$, which also converges to infinity as $n \to \infty$.

This example illustrates that the implications of a VaR-based risk management system can be even more perverse under non-normality. Specifically, a portfolio manager facing a VaR constraint may select a portfolio with an arbitrarily large standard deviation and CVaR. In contrast, a portfolio manager facing a CVaR constraint would not be able to select a portfolio with an arbitrarily large standard deviation. Under these circumstances, a CVaR constraint is therefore a more effective risk management tool than a VaR constraint.

2. Example

The following example illustrates that our theoretical results are plausible in practical applications. The example consists of solving an agent's problem of finding how to allocate his or her wealth among ten asset classes: five involve U.S. securities (large stocks, small stocks, corporate bonds, Treasury bonds, and real estate investment trusts (REITs)), and five involve foreign securities (stocks in Canada, U.K., Germany, Switzerland, and emerging markets).[4] The following indices are used to measure the rates of return on these asset classes: the S&P 500 index (large stocks), the Russell 2000 index (small stocks), the Merrill Lynch U.S. corporate and Treasury bond indices, the index for all publicly traded REITs provided by the National Association of Real Estate Investment Trusts, and the Morgan Stanley Capital International (MSCI) indices for Canada, U.K., Germany, Switzerland, and emerging markets.[5] Sample means, variances, and covariances

[3] Note that the portfolio's VaR cannot be computed using a linear transformation of the two first moments of its rate of return distribution as shown in equation (3). Consequently, a VaR constraint in this example does not have the simple representation provided in equation (10). All portfolios in the example lie on a single point in mean-VaR space and on a horizontal line in mean-standard deviation space.

[4] This is a common use of the mean-variance model as noted by, e.g., Sharpe (1987), Solnik (1991), Black and Litterman (1992), and Michaud (1998). For a practical example, see the website of Russell/Mellon Analytical Services (http://www.russellmellon.com/products/desktoptools/atm/ATM.pdf).

[5] We include dividends in the computation of the rates of return for the stock and REITs indices. The Merrill

that are associated with the indices were computed using annual data for the period 1987-2001 and then used as optimization inputs.[6]

2.1. Efficient Frontiers

With the aforementioned inputs, $\sqrt{D/C} = 1.55$. Since $z_{0.9393} = k_{0.8483} = 1.55$, Corollary 4 implies the following. First, if $t \leq 0.8483$, then the mean-VaR and mean-CVaR efficient frontiers are empty. Second, if $0.8483 < t \leq 0.9393$, then the mean-VaR efficient frontier is empty, but the mean-CVaR efficient frontier is a non-empty proper subset of the mean-variance efficient frontier. Finally, if $t > 0.9393$, then the mean-VaR efficient frontier is a non-empty proper subset of the mean-CVaR efficient frontier, and in turn the mean-CVaR efficient frontier is a non-empty proper subset of the mean-variance efficient frontier.

For brevity, suppose that $t = 0.99$. The minimum variance, minimum CVaR, and minimum VaR portfolios are characterized by:

$$E[r_{m_\sigma}] = 2.97\%, \sigma[r_{m_\sigma}] = 3.78\%, \tag{A1}$$

$$E[r_{m_{L(0.99)}}] = 7.16\%, \sigma[r_{m_{L(0.99)}}] = 4.65\%, \tag{A2}$$

$$E[r_{m_{V(0.99)}}] = 8.20\%, \sigma[r_{m_{V(0.99)}}] = 5.07\%. \tag{A3}$$

Proposition 2, Corollary 4, and equations (A1), (A2), and (A3) imply the following. First, any mean-variance efficient portfolio w with $2.97\% \leq E[r_w] < 7.16\%$ is neither mean-VaR nor mean-CVaR efficient. Second, any mean-variance efficient portfolio w with $7.16\% \leq E[r_w] < 8.20\%$ is mean-VaR inefficient but mean-CVaR efficient. Finally, any mean-variance efficient portfolio with $E[r_w] \geq 8.20\%$ is both mean-VaR and mean-CVaR efficient.

Lynch U.S. corporate bond and Treasury bond indices include bonds with maturity equal to or greater than one year. The MSCI emerging markets index includes stocks that are traded in 26 countries.

[6] Michaud (1998, p. 12) notes that this is sometimes done in practical applications. For simplicity, we do not make an adjustment for estimation risk (see, e.g., Michaud (1998)).

2.2. Portfolio-Choice Implications of VaR and CVaR Constraints

We now illustrate the portfolio-choice implications of the constraints when $L = V$. Using equations (A1), (A2), and (A3), the bound L is: (i) large if $L \geq L[0.99, r_{m_\sigma}] = 7.11\%$, (ii) moderate if $V[0.99, r_{m_\sigma}] = 5.82\% \leq L < 7.11\% = L[0.99, r_{m_\sigma}]$, and (iii) small if $L[0.99, r_{m_{L(0.99)}}] = 5.23\% \leq L < 5.82\% = V[0.99, r_{m_\sigma}]$. Hence, we use the values 8%, 6%, and 5.25% as examples of, respectively, large, moderate, and small bounds.

2.2.1. Highly Risk-Averse Agent

Consider a highly risk-averse agent whose unconstrained optimal portfolio w^h has an expected rate of return relatively close to $E[r_{m_\sigma}] = 2.97\%$, say $E[r_{w^h}] = 3.25\%$. Since $\sigma[r_{w^h}] = 3.79\%$, equations (3) and (4) imply that $V[0.99, r_{w^h}] = 5.56\%$ and $L[0.99, r_{w^h}] = 6.84\%$. First, assume the bound is large, i.e., $L = 8\%$. Since $L > L[0.99, r_{w^h}] > V[0.99, r_{w^h}]$, imposing either the VaR or CVaR constraint does not change the optimal portfolio.

Second, assume the bound is moderate, i.e., $L = 6\%$. Since $L > V[0.99, r_{w^h}]$, imposing the VaR constraint does not change the optimal portfolio. However, since $L < L[0.99, r_{w^h}]$, equation (16) implies that the standard deviation of the CVaR-constrained optimal portfolio is 3.89%, so its relative increase is 2.63% [= (3.89 − 3.79)/3.79].

Finally, assume the bound is small, i.e., $L = 5.25\%$. Note that $L < V[0.99, r_{w^h}] < L[0.99, r_{w^h}]$. Since equation (18) implies that the standard deviation of the VaR-constrained optimal portfolio is 3.80%, its relative increase is only 0.45% [= (3.80 − 3.79)/3.79]. Similarly, since equation (16) implies that the standard deviation of the CVaR-constrained optimal portfolio is 4.47%, its relative increase is 17.96% [= (4.47 − 3.79)/3.79].

2.2.2. Slightly Risk-Averse Agent

Consider a slightly risk-averse agent whose unconstrained optimal portfolio w^s has a relatively large expected rate of return, say $E[r_{w^s}] = 16\%$. Since $\sigma[r_{w^s}] = 9.22\%$, equations (3) and (4) imply that $V[0.99, r_{w^s}] = 5.45\%$ and $L[0.99, r_{w^s}] = 8.58\%$. First, assume the bound is large, i.e., $L = 8\%$. Note that $L > V[0.99, r_{w^s}]$, so imposing the VaR constraint does not change the optimal portfolio. However, $L < L[0.99, r_{w^s}]$. Hence, equation (15) implies that the standard deviation of the CVaR-constrained optimal portfolio is 8.62%, for a relative decrease of 6.57% [= (9.22 – 8.62)/9.22].

Second, assume the bound is moderate, i.e., $L = 6\%$. The analysis is similar to that when $L = 8\%$. Imposing the VaR constraint does not change the optimal portfolio. However, since equation (15) implies that the standard deviation of the CVaR-constrained optimal portfolio is 6.28%, its relative decrease is 31.88% [= (9.22 – 6.28)/9.22].

Finally, assume the bound is small, i.e., $L = 5.25\%$. Note that $L < V[0.99, r_{w^s}] < L[0.99, r_{w^s}]$. Since equation (17) implies that the standard deviation of the VaR-constrained optimal portfolio is 8.90%, its relative decrease is only 3.54% [= (9.22 – 8.90)/9.22]. Similarly, equation (15) implies that the standard deviation of the CVaR-constrained optimal portfolio is 4.85%, for a relative large decrease of 47.39% [= (9.22 – 4.85)/9.22].

2.3. Implementing the CVaR Constraint

Suppose now that equations (19) and (20) are used to find the CVaR bound when $V = 5.25\%$. We consider (i) the highly risk-averse agent who, as mentioned earlier, had an unconstrained optimal portfolio w^h with $E[r_{w^h}] = 3.25\%$, $\sigma[r_{w^h}] = 3.79\%$, $V[0.99, r_{w^h}] = 5.56\%$, and $L[0.99, r_{w^h}] = 6.84\%$, and (ii) the slightly risk-averse agent who, again as mentioned earlier, had an unconstrained optimal portfolio w^s with $E[r_{w^s}] = 16\%$, $\sigma[r_{w^s}] = 9.22\%$, $V[0.99, r_{w^s}] = 5.45\%$, and $L[0.99, r_{w^s}] = 8.58\%$.

First, using equation (19), we have $\underline{L} = 6.54\%$ so that the constrained optimal portfolios coincide for the highly risk-averse agent, but not for the slightly risk-averse agent. Since equation (15) implies that the standard deviation of the CVaR-constrained optimal portfolio of the slightly risk-averse agent is 6.97%, its relative decrease is 24.36% [= (9.22 – 6.97)/9.22]. However, the standard deviation of the agent's VaR-constrained optimal portfolio is again 8.90%, for a relative decrease of only 3.54% [= (9.22 – 8.90)/9.22].

Second, using equation (20), we have $\overline{L} = 8.26\%$ so that the constrained optimal portfolios coincide for the slightly risk-averse agent, but not for the highly risk-averse agent. Since $\overline{L} > L[0.99, r_{w^h}] = 6.84\%$, imposing the CVaR constraint does not change the optimal portfolio of the highly risk-averse agent. However, the standard deviation of the agent's VaR-constrained optimal portfolio is again 3.80%, for a relative increase of 0.45% [= (3.80 − 3.79)/3.79].

Finally, assume $6.54\% = \underline{L} < L < \overline{L} = 8.26\%$, say $L = 7.40\%$. Since $L > L[0.99, r_{w^h}] = 6.84\%$, imposing the CVaR constraint does not change the optimal portfolio of the highly risk-averse agent, but the VaR-constrained optimal portfolio has again a standard deviation of 3.80%, for a relative increase of 0.45% [= (3.80 − 3.79)/3.79]. Given that equation (15) implies that the standard deviation of the CVaR-constrained optimal portfolio of the slightly risk-averse agent is 7.97%, its relative decrease is 13.58% [= (9.22 − 7.97)/9.22]. However, the standard deviation of the agent's VaR-constrained optimal portfolio is again 8.90%, for a relative decrease of only 3.54% [= (9.22 − 8.90)/9.22]. Hence, the CVaR constraint results in portfolios with smaller standard deviations for both types of agents.

2.4. Adding a Riskfree Security

In December 2001, the yield on a one-year Treasury security was approximately 2%.[7] Hence, we assume that a riskfree security with rate of return $r_f = 2\%$ is available. With this input, $\sqrt{H} = 1.57$. Since $z_{0.9418} = k_{0.8545} = 1.57$, Corollary 5 implies the following. First, if $t \leq 0.8545$, then the mean-VaR and mean-CVaR efficient frontiers are empty. Second, if $0.8545 < t \leq 0.9418$, then the mean-VaR efficient frontier is empty, but the mean-CVaR efficient frontier coincides with the mean-variance efficient frontier. Finally, if $t > 0.9418$, then the mean-VaR, mean-CVaR, and mean-variance efficient frontiers coincide.

For brevity, assume that $t = 0.99$ and $L = V = 4\%$.[8] First, consider a highly risk-averse

[7]See http://www.federalreserve.gov/releases/h15/data/m/tcm1y.txt.

[8]If, for example, $L = V = -3\% < -r_f = -2\%$ and $t = 0.85$, then the perverse results described in footnote 18

agent whose unconstrained optimal portfolio w^h has a relatively small expected rate of return, say $E[r_{w^h}] = 8\%$. Since $\sigma[r_{w^h}] = 3.82\%$, equations (3) and (4) imply that $V[0.99, r_{w^h}] = 0.89\%$ and $L[0.99, r_{w^h}] = 2.19\%$. Note that $L > L[0.99, r_{w^h}] > V[0.99, r_{w^h}]$. Hence, imposing either the VaR or CVaR constraint does not change the optimal portfolio.

Second, consider a slightly risk-averse agent whose unconstrained optimal portfolio w^s has a relatively large expected rate of return, say $E[r_{w^s}] = 16\%$. Since $\sigma[r_{w^s}] = 8.92\%$, equations (3) and (4) imply that $V[0.99, r_{w^s}] = 4.74\%$ and $L[0.99, r_{w^s}] = 7.77\%$. Note that both constraints bind, as $L < V[0.99, r_{w^s}] < L[0.99, r_{w^s}]$. The standard deviation of the VaR-constrained optimal portfolio is 7.93%, for a relative decrease of 11.04% [= (8.92 − 7.93)/8.92]. Similarly, the standard deviation of the CVaR-constrained optimal portfolio is 5.48%, for a relative decrease of 38.57% [= (8.92 − 5.48)/8.92]. Hence, the standard deviation of the CVaR-constrained optimal portfolio is notably smaller than that of the VaR-constrained optimal portfolio.[9]

hold. However, it is debatable whether such situation would arise in practice as it is akin to requiring an agent to select a portfolio that with 85% confidence will outperform the riskfree security by 1%.

[9]If the unconstrained optimal portfolio has an expected rate of return of 12%, then it satisfies the VaR constraint but does not satisfy the CVaR constraint. The relative decrease of the optimal portfolio's standard deviation arising from the imposing the CVaR constraint is 14% [= (6.37 − 5.48)/6.37].

3. Summary Tables

Table 1
Impact of a VaR or CVaR constraint within a mean-variance model on portfolio efficiency and selection[a]

(a) Mean-variance efficient portfolios that are precluded by the presence of a VaR or CVaR constraint[b]

		Bound L=V		
Confidence level t	*Constraint*	*Large*	*Moderate*	*Small*
Low	VaR	none	none	small σ
	CVaR	none	small σ	small σ
Moderate	VaR	none	none	small σ
	CVaR	large σ	small and large σ	small and large σ
High	VaR	large σ	large σ	small and large σ
	CVaR	large σ	small and large σ	small and large σ

(b) Impact of a VaR constraint on the standard deviation of an agent's optimal portfolio[c]

		Bound V		
Confidence level t	*Agent*	*Large*	*Moderate*	*Small*
Low or Moderate	Highly risk-averse	no effect	no effect	increase
	Slightly risk-averse	no effect	no effect	no effect
High	Highly risk-averse	no effect	no effect	increase
	Slightly risk-averse	decrease	decrease	decrease

(c) Impact of a CVaR constraint on the standard deviation of an agent's optimal portfolio[d]

		Bound L		
Confidence level t	*Agent*	*Large*	*Moderate*	*Small*
Low	Highly risk-averse	no effect	increase	increase
	Slightly risk-averse	no effect	no effect	no effect
Moderate or High	Highly risk-averse	no effect	increase	increase
	Slightly risk-averse	decrease	decrease	decrease

[a] We assume that there is no riskfree security. The confidence level is: (i) low if $k_t \leq \sqrt{D/C}$, (ii) moderate if $z_t \leq \sqrt{D/C} < k_t$, and (iii) high if $z_t > \sqrt{D/C}$. The bound is: (i) large if $L = V \geq L[t, r_{m_\sigma}]$, (ii) moderate if $V[t, r_{m_\sigma}] \leq L = V < L[t, r_{m_\sigma}]$, and (iii) small if $L = V < V[t, r_{m_\sigma}]$.

[b] The entries in the cells show the characteristics of the portfolios that are precluded.

[c] The impact is measured relative to the agent's optimal portfolio in the absence of the VaR constraint.

[d] The impact is measured relative to the agent's optimal portfolio in the absence of the CVaR constraint.

Table 2
Constraint that results in the smallest standard deviation for the optimal portfolio[a]

(a) CVaR and VaR bounds coincide

		Bound L		
Confidence level t	*Agent*	*Large*	*Moderate*	*Small*
Low	Highly risk-averse	both	VaR	VaR
	Slightly risk-averse	both	both	both
Moderate or High	Highly risk-averse	both	VaR	VaR
	Slightly risk-averse	CVaR	CVaR	CVaR

(b) CVaR bound appropriately larger than VaR bound[b]

		Bound L		
Confidence level t	*Agent*	*Maximum*	*Intermediate*	*Minimum*
Low	Highly risk-averse	NA	CVaR	both
	Slightly risk-averse	NA	both	both
Moderate	Highly risk-averse	NA	CVaR	both
	Slightly risk-averse	NA	CVaR	CVaR
High	Highly risk-averse	CVaR	CVaR	both
	Slightly risk-averse	both	CVaR	CVaR

[a] We assume that there is no riskfree security. The confidence level is: (i) low if $k_t \le \sqrt{D/C}$, (ii) moderate if $z_t \le \sqrt{D/C} < k_t$, and (iii) high if $z_t > \sqrt{D/C}$. The bound is: (i) large if $L = V \ge L[t, r_{m_\sigma}]$, (ii) moderate if $V[t, r_{m_\sigma}] \le L = V < L[t, r_{m_\sigma}]$, and (iii) small if $L = V < V[t, r_{m_\sigma}]$.

[b] The CVaR bound is determined by equations (19) and (20), i.e., $\underline{L} \le L \le \overline{L}$. 'NA' means 'does not apply.'

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