On-Line Appendix for
"Portfolio Selection with a Drawdown Constraint"
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This Appendix provides proofs of Theorems 1–3 in the paper "Portfolio Selection with a Drawdown Constraint" published in the *Journal of Banking and Finance* 30, 3171–3189, November 2006.

Proof of Theorem 1. First, assume $D[r_{w_E}] \leq D$. The desired result follows from Eq. (1). This completes the first part of our proof.

Second, assume $D[r_{w_E}] > D$. By definition, $D[r_{w_{E,D}}] \leq D$. We claim that $D[r_{w_{E,D}}] = D$. Suppose by way of a contradiction that $D[r_{w_{E,D}}] < D$. Since w_E is on the mean-variance boundary, $\sigma[r_{w_E}] < \sigma[r_{w_{E,D}}]$. Let $w^* \equiv \varepsilon w_E + (1-\varepsilon) w_{E,D}$, where $\varepsilon > 0$ is arbitrarily small. Note that $E[r_{w^*}] = E$, $\sigma[r_{w^*}] < \sigma[r_{w_{E,D}}]$, and $D[r_{w^*}] < D$, a contradiction of the fact that $w_{E,D}$ is on the constrained mean-variance boundary. Hence, $D[r_{w_{E,D}}] = D$.

Let $s_3, ..., s_{K+2}$ be the states at which the constraint binds. Then, $w_{E,D}$ solves

$$\min_{w \in \mathbb{R}^J} \frac{1}{2} w^\top \Sigma w \tag{12}$$

$$s.t. \quad w^\top \iota = 1 \tag{13}$$

$$w^\top \mu = E \tag{14}$$

$$w^\top r_s = -D \quad \forall s \in \{s_3, ..., s_{K+2}\} \tag{15}$$

$$w^\top r_s \geq -D \quad \forall s \notin \{s_3, ..., s_{K+2}\}. \tag{16}$$

Note that constraint (16) does not bind. First-order necessary and sufficient conditions for $w_{E,D}$ to solve problem (12) subject to constraints (13)–(15) are

$$\begin{aligned} \Sigma w_{E,D} - \lambda_1 \iota - \lambda_2 \mu - \sum_{k=3}^{K+2} \lambda_k r_{s_k} &= 0 && (17) \\ w_{E,D}^{\top} \iota &= 1 && (18) \\ w_{E,D}^{\top} \mu &= E && (19) \\ w_{E,D}^{\top} r_s &= -D,\ s = s_3, ..., s_{K+2}, && (20) \end{aligned}$$

where $\lambda_1, ..., \lambda_{K+2}$ are Lagrange multipliers associated with constraints (13)–(15). Since $rank(\Sigma) = J$, Eq. (17) implies that

$$w_{E,D} = \lambda_1 \left(\Sigma^{-1} \iota\right) + \lambda_2 \left(\Sigma^{-1} \mu\right) + \sum_{k=3}^{K+2} \lambda_k \left(\Sigma^{-1} r_{s_k}\right). \tag{21}$$

Eq. (21) implies that Eq. (3) holds with

$$[\varphi_1 \quad \varphi_2 \quad \varphi_3 \quad \cdots \quad \varphi_{K+2}] \equiv [\lambda_1 c \quad \lambda_2 a \quad \lambda_3 e_{s_3} \quad \cdots \quad \lambda_{K+2} e_{s_{K+2}}].$$

We now find the $(K+2) \times 1$ vector $L \equiv [\lambda_1 \quad \cdots \quad \lambda_{K+2}]^{\top}$. Premultiplying Eq. (21) by $\iota^{\top}$ and using Eq. (18), we have

$$\lambda_1 \left(\iota^{\top} \Sigma^{-1} \iota\right) + \lambda_2 \left(\iota^{\top} \Sigma^{-1} \mu\right) + \sum_{k=3}^{K+2} \lambda_k \left(\iota^{\top} \Sigma^{-1} r_{s_k}\right) = 1. \tag{22}$$

Premultiplying Eq. (21) by $\mu^{\top}$ and using Eq. (19), we obtain

$$\lambda_1 \left(\mu^{\top} \Sigma^{-1} \iota\right) + \lambda_2 \left(\mu^{\top} \Sigma^{-1} \mu\right) + \sum_{k=3}^{K+2} \lambda_k \left(\mu^{\top} \Sigma^{-1} r_{s_k}\right) = E. \tag{23}$$

Premultiplying Eq. (21) by $r_s^{\top}$ and using Eq. (20), we have

$$\lambda_1 \left(r_s^{\top} \Sigma^{-1} \iota\right) + \lambda_2 \left(r_s^{\top} \Sigma^{-1} \mu\right) + \sum_{k=3}^{K+2} \lambda_k \left(r_s^{\top} \Sigma^{-1} r_{s_k}\right) = -D,\ \ s = s_3, ..., s_{K+2}. \tag{24}$$

Let the $(K+2) \times 1$ vector $A_{E,D}$ denote $[1 \quad E \quad -D \quad \cdots \quad -D]^{\top}$. Let $M \equiv [\iota \quad \mu \quad r_{s_3} \quad \cdots \quad r_{s_{K+2}}]$ and $N \equiv M^{\top} \Sigma^{-1} M$. Using Eqs. (22)–(24), $NL = A_{E,D}$. Since $rank(\Sigma) = J$ and

$rank(M) = K+2$, $rank(N) = K+2$. Hence, $L = N^{-1}A_{E,D}$. This completes the second part of our proof. ■

Proof of Theorem 2. First, assume $D[r_{w_E}] \leq D$. The desired result follows from Eq. (4). This completes the first part of our proof.

Second, assume $D[r_{w_E}] > D$. It follows from the argument used in the proof of Theorem 1 that $D[r_{w_{E,D}}] = D$. Let $s_3, ..., s_{K+2}$ be the states at which the constraint binds. Let $\overline{w}$ be the vector that consists of the first J elements of $w_{E,D}$. Then, $\overline{w}$ solves

$$\min_{w \in \mathbb{R}^J} \frac{1}{2} w^\top \Sigma w \tag{25}$$

$$s.t. \;\; w^\top \mu + (1 - w^\top \iota) r_f = E \tag{26}$$

$$w^\top r_s + (1 - w^\top \iota) r_f = -D \quad \forall s \in \{s_3, ..., s_{K+2}\} \tag{27}$$

$$w^\top r_s + (1 - w^\top \iota) r_f \geq -D \quad \forall s \notin \{s_3, ..., s_{K+2}\}. \tag{28}$$

Note that constraint (28) does not bind. First-order necessary and sufficient conditions for $\overline{w}$ to solve problem (25) subject to constraints (26) and (27) are

$$\Sigma \overline{w} - \gamma_1 (\mu - \iota r_f) - \sum_{k=3}^{K+2} \gamma_k (r_{s_k} - \iota r_f) = 0 \tag{29}$$

$$\overline{w}^\top (\mu - \iota r_f) = E - r_f \tag{30}$$

$$\overline{w}^\top (r_s - \iota r_f) = -D - r_f, \;\; s = s_3, ..., s_{K+2}, \tag{31}$$

where $\gamma_1, \gamma_3, ..., \gamma_{K+2}$ are Lagrange multipliers associated with constraints (26) and (27). Since $rank(\Sigma) = J$, Eq. (29) implies that

$$\overline{w} = \gamma_1 \left[\Sigma^{-1} (\mu - \iota r_f)\right] + \sum_{k=3}^{K+2} \gamma_k \left[\Sigma^{-1} (r_{s_k} - \iota r_f)\right]. \tag{32}$$

Eq. (32) implies that Eq. (6) holds with

$$[\theta_1 \quad \theta_2 \quad \theta_3 \quad \cdots \quad \theta_{K+2}] \equiv [1 - \sum_{k=2}^{K+2} \theta_k \quad \gamma_1(a - c r_f) \quad \gamma_3(e_{s_3} - c r_f) \quad \cdots \quad \gamma_{K+2}(e_{s_{K+2}} - c r_f)].$$

We now find the $(K+1)\times 1$ vector $Q \equiv [\gamma_1 \quad \gamma_3 \quad \cdots \quad \gamma_{K+2}]^\top$. Premultiplying Eq. (32) by $(\mu - \iota r_f)^\top$ and using Eq. (30), we have

$$\gamma_1 \left[(\mu - \iota r_f)^\top \Sigma^{-1} (\mu - \iota r_f) \right] + \sum_{k=3}^{K+2} \gamma_k \left[(\mu - \iota r_f)^\top \Sigma^{-1} (r_{s_k} - \iota r_f) \right] = E - r_f. \tag{33}$$

Premultiplying Eq. (32) by $(r_s - \iota r_f)^\top$ and using Eq. (31), we obtain

$$\gamma_1 \left[(r_s - \iota r_f)^\top \Sigma^{-1} (\mu - \iota r_f) \right] + \sum_{k=3}^{K+2} \gamma_k \left[(r_s - \iota r_f)^\top \Sigma^{-1} (r_{s_k} - \iota r_f) \right] = -D - r_f, \tag{34}$$

$s = s_3, ..., s_{K+2}$. Let the $(K+1)\times 1$ vector $B_{E,D}$ denote $[E - r_f \quad -D - r_f \quad \cdots \quad -D - r_f]^\top$. Let $R \equiv [\mu - \iota r_f \quad r_{s_3} - \iota r_f \quad \cdots \quad r_{s_{K+2}} - \iota r_f]$ and $T \equiv R^\top \Sigma^{-1} R$. Using Eqs. (33)–(34), $TQ = B_{E,D}$. The fact that $rank(M) = K+2$ implies that $rank(R) = K+1$. Since $rank(\Sigma) = J$ and $rank(R) = K+1$, $rank(T) = K+1$. Hence, $Q = T^{-1} B_{E,D}$. This completes the second part of our proof. ■

Proof of Theorem 3. First, assume $D^\varepsilon[r_{w^\varepsilon_E}] \leq D^\varepsilon$. The desired result follows from Eqs. (7) and (8). This completes the first part of our proof.

Second, assume $D^\varepsilon[r_{w^\varepsilon_E}] > D^\varepsilon$. It follows from arguments similar to those used in the proof of Theorem 1 that $D^\varepsilon[r_{w^\varepsilon_{E,D^\varepsilon}}] = D^\varepsilon$. Let $s_3, ..., s_{K+2}$ be the states at which the constraint binds. Then, $\overline{x} = w^\varepsilon_{E,D^\varepsilon} - w_b$ solves

$$\min_{x \in \mathbb{R}^J} \frac{1}{2} x^\top \Sigma x \tag{35}$$

$$s.t. \quad x^\top \iota = 0 \tag{36}$$

$$x^\top \mu = E - E[r_{w_b}] \tag{37}$$

$$x^\top r_s = -D^\varepsilon \quad \forall s \in \{s_3, ..., s_{K+2}\} \tag{38}$$

$$x^\top r_s \geq -D^\varepsilon \quad \forall s \notin \{s_3, ..., s_{K+2}\}. \tag{39}$$

Note that constraint (39) does not bind. First-order necessary and sufficient conditions for $\overline{x}$ to solve problem (35) subject to constraints (36)–(38) are

$$\begin{aligned}
\Sigma\overline{x} - \delta_1\iota - \delta_2\mu - \sum_{k=3}^{K+2}\delta_k r_{s_k} &= 0 && (40)\\
\overline{x}^\top\iota &= 0 && (41)\\
\overline{x}^\top\mu &= E - E[r_{w_b}] && (42)\\
\overline{x}^\top r_s &= -D^\varepsilon \ \forall s \in \{s_3, ..., s_{K+2}\}, && (43)
\end{aligned}$$

where $\delta_1, ..., \delta_{K+2}$ are Lagrange multipliers associated with constraints (36)–(38). Since $rank(\Sigma) = J$, Eq. (40) implies that

$$\overline{x} = \delta_1\left(\Sigma^{-1}\iota\right) + \delta_2\left(\Sigma^{-1}\mu\right) + \sum_{k=3}^{K+2}\delta_k\left(\Sigma^{-1}r_{s_k}\right). \tag{44}$$

Eq. (44) implies that Eqs. (10) and (11) hold with

$$[\pi_1 \quad \pi_2 \quad \pi_3 \quad \cdots \quad \pi_{K+2}] \equiv [\delta_1 c \quad \delta_2 a \quad \delta_3 e_{s_3} \quad \cdots \quad \delta_{K+2}e_{s_{K+2}}].$$

We now find the $(K+2)\times 1$ vector $U \equiv [\delta_1 \quad \cdots \quad \delta_{K+2}]^\top$. Premultiplying Eq. (44) by $\iota^\top$ and using Eq. (41), we have

$$\delta_1\left(\iota^\top\Sigma^{-1}\iota\right) + \delta_2\left(\iota^\top\Sigma^{-1}\mu\right) + \sum_{k=3}^{K+2}\delta_k\left(\iota^\top\Sigma^{-1}r_{s_k}\right) = 0. \tag{45}$$

Premultiplying Eq. (44) by $\mu^\top$ and using Eq. (42), we obtain

$$\delta_1\left(\mu^\top\Sigma^{-1}\iota\right) + \delta_2\left(\mu^\top\Sigma^{-1}\mu\right) + \sum_{k=3}^{K+2}\delta_k\left(\mu^\top\Sigma^{-1}r_{s_k}\right) = E - E[r_{w_b}]. \tag{46}$$

Premultiplying Eq. (44) by $r_s^\top$ and using Eq. (43), we have

$$\delta_1\left(r_s^\top\Sigma^{-1}\iota\right) + \delta_2\left(r_s^\top\Sigma^{-1}\mu\right) + \sum_{k=3}^{K+2}\delta_k\left(r_s^\top\Sigma^{-1}r_{s_k}\right) = -D^\varepsilon,\ s = s_3, ..., s_{K+2}. \tag{47}$$

Let the $(K+2)\times 1$ vector C_{E,D^ε} denote $[0 \quad E - E[r_{w_b}] \quad -D^\varepsilon \quad \cdots \ -D^\varepsilon]^\top$. Eqs. (45) and (46) imply that $NU = C_{E,D^\varepsilon}$. Since $rank(N) = K+2$, $U = N^{-1}C_{E,D^\varepsilon}$. This completes the second part of our proof. ■