Online Appendix for

"Portfolio Selection with Mental Accounts and Background Risk"

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This Appendix contains proofs of the theoretical results in the paper "Portfolio Selection with Mental Accounts and Background Risk" published in the *Journal of Banking and Finance* **36**, 968–980, April 2012.

The following three results are useful in our proof of Theorem 1.

Lemma 1. *Fix an account* $m \in \{1, ..., M\}$ *and a level of expected return* $E \in \mathbb{R}$ *for it. The portfolio that minimizes account* m*'s variance subject to the restriction that the account has an expected return of* E *is given by:*

$$\boldsymbol{w}_E \equiv \underline{\boldsymbol{w}}_m + \phi_E \left(\boldsymbol{w}_1 - \boldsymbol{w}_0 \right) \tag{16}$$

where $\phi_E = \frac{E - \underline{E}_m}{B/A - A/C}$. *Furthermore, we have:*

$$\sigma[r_{\boldsymbol{w}_E,m}] = \sqrt{\underline{\sigma}_m^2 + \frac{(E[r_{\boldsymbol{w}_E,m}] - \underline{E}_m)^2}{D/C}}. \tag{17}$$

Proof. Fix an account $m \in \{1, ..., M\}$ and a level of expected return $E \in \mathbb{R}$ for it. The portfolio that minimizes account m's variance subject to the restriction that the account has an expected return of E solves:

$$\min_{\boldsymbol{w} \in \mathbb{R}^N} \ \frac{1}{2} \left(\boldsymbol{w}' \boldsymbol{\Sigma} \boldsymbol{w} + \Omega_{mm} + 2 \boldsymbol{w}' \boldsymbol{\Psi}_m \right) \tag{18}$$

$$s.t. \quad \boldsymbol{w}' \mathbf{1} = 1 \tag{19}$$

$$\boldsymbol{w}' \boldsymbol{\mu} = E - \nu_m. \tag{20}$$

A first-order condition for $\boldsymbol{w}_E$ to solve problem (18) subject to constraints (19) and (20) is:

$$\boldsymbol{\Sigma} \boldsymbol{w}_E + \boldsymbol{\Psi}_m - \varphi_1 \mathbf{1} - \varphi_2 \boldsymbol{\mu} = \mathbf{0}, \tag{21}$$

where $\mathbf{0}$ is the $N \times 1$ vector $[0 \quad \cdots \quad 0]'$, and φ_1 and φ_2 are multipliers associated to these constraints. Using Eq. (21), we have:

$$\boldsymbol{w}_E = \varphi_1 \boldsymbol{\Sigma}^{-1}\mathbf{1} + \varphi_2 \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\Psi}_m. \tag{22}$$

Premultiplying Eq. (21) by $\mathbf{1}'$ and using Eq. (19), we obtain:

$$1 = \varphi_1 C + \varphi_2 A - A_m. \tag{23}$$

Premultiplying Eq. (21) by $\boldsymbol{\mu}'$ and using Eq. (20), we obtain:

$$E - \nu_m = \varphi_1 A + \varphi_2 B - B_m, \tag{24}$$

where $B_m \equiv \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Psi}_m$. Eqs. (23) and (24) imply that:

$$\varphi_1 = \frac{1 + A_m - \varphi_2 A}{C} \tag{25}$$

and

$$\varphi_2 = \frac{E - (1 + A_m)\, A/C - \nu_m + B_m}{B - A^2/C}. \tag{26}$$

Noting that $\underline{E}_m = (1 + A_m)\, \frac{A}{C} + \nu_m - B_m$, Eq. (16) follows from Eqs. (22), (25), and (26), and the definitions of $\underline{\boldsymbol{w}}_m$, $\boldsymbol{w}_0$, and $\boldsymbol{w}_1$. Using Eq. (22), we have:

$$\sigma[r_{\boldsymbol{w}_E,m}] = \sqrt{\varphi_1^2 C + 2\varphi_1\varphi_2 A + \varphi_2^2 B + \Omega_{mm} - C_m}, \tag{27}$$

where $C_m \equiv \boldsymbol{\Psi}_m'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Psi}_m$. Eqs. (25) and (27) imply that:

$$\sigma[r_{\boldsymbol{w}_E,m}] = \sqrt{\left(\frac{1 + A_m - \varphi_2 A}{C}\right)^2 C + 2\left(\frac{1 + A_m - \varphi_2 A}{C}\right)\varphi_2 A + \varphi_2^2 B + \Omega_{mm} - C_m}. \tag{28}$$

Using Eq. (28) and elementary algebra, we have:

$$\sigma[r_{\boldsymbol{w}_E,m}] = \sqrt{\frac{(1 + A_m)^2}{C} + \Omega_{mm} - C_m + \varphi_2^2\left(B - \frac{A^2}{C}\right)}. \tag{29}$$

Noting that $\underline{\sigma}_m^2 = \frac{(1+A_m)^2}{C} + \Omega_{mm} - C_m$, Eq. (17) follows from Eqs. (26) and (29).□

Lemma 2. *If* $\alpha_m < \Phi(-\sqrt{D/C})$, *then* $V[1 - \alpha_m, r_{\underline{\boldsymbol{w}}_m,m}] = -H_{\alpha_m}$.

Proof. Suppose that $\alpha_m < \Phi(-\sqrt{D/C})$. Using Eq. (4), $\underline{\underline{\boldsymbol{w}}}_m$ minimizes account m's variance subject to the restriction that the account has an expected return of $E[r_{\underline{\underline{\boldsymbol{w}}}_m,m}]$. Lemma 1 implies that $E[r_{\underline{\underline{\boldsymbol{w}}}_m,m}]$ solves:

$$\min_{E \in \mathbb{R}} \quad z_{\alpha_m}\sqrt{\underline{\sigma}_m^2 + \frac{(E - \underline{E}_m)^2}{D/C}} - E. \tag{30}$$

A first-order condition for $E[r_{\underline{\underline{\boldsymbol{w}}}_m,m}]$ to solve problem (30) is:

$$z_{\alpha_m}\frac{(E[r_{\underline{\underline{\boldsymbol{w}}}_m,m}] - \underline{E}_m)/(D/C)}{\sqrt{\underline{\sigma}_m^2 + (E[r_{\underline{\underline{\boldsymbol{w}}}_m,m}] - \underline{E}_m)^2/(D/C)}} - 1 = 0. \tag{31}$$

It follows from Eq. (31) that:

$$E[r_{\underline{\underline{\boldsymbol{w}}}_m,m}] = \sqrt{\frac{(D/C)^2\,\underline{\sigma}_m^2}{z_{\alpha_m}^2 - D/C}} + \underline{E}_m. \tag{32}$$

Using Eqs. (17) and (32), we have:

$$\sigma[r_{\underline{\underline{\boldsymbol{w}}}_m,m}] = \sqrt{\frac{z_{\alpha_m}^2\underline{\sigma}_m^2}{z_{\alpha_m}^2 - D/C}}. \tag{33}$$

Eqs. (4), (32), and (33) imply the desired result.□

Lemma 3. *Fix any account $m \in \{1, ..., M\}$ with $\alpha_m < \Phi(-\sqrt{D/C})$ and $H_m \leq H_{\alpha_m}$. The optimal portfolio within account m is given by $\boldsymbol{w}_m = \boldsymbol{w}_E$ for same $E \in \mathbb{R}$ with $E > \underline{E}_m$. Furthermore, we have $V[1-\alpha_m, r_{\boldsymbol{w}_m,m}] = -H_m$.*

Proof. Fix any account $m \in \{1, ..., M\}$ with $\alpha_m < \Phi(-\sqrt{D/C})$ and $H_m \leq H_{\alpha_m}$. First, we show that $\boldsymbol{w}_m = \boldsymbol{w}_E$ for some $E \in \mathbb{R}$. Suppose by way of a contradiction that $\boldsymbol{w}_m \neq \boldsymbol{w}_E$, where $E = E[r_{\boldsymbol{w}_m,m}]$. It follows from Lemma 1 that $\sigma[r_{\boldsymbol{w}_E,m}] < \sigma[r_{\boldsymbol{w}_m,m}]$. Since $E[r_{\boldsymbol{w}_E,m}] = E[r_{\boldsymbol{w}_m,m}]$ and $\sigma[r_{\boldsymbol{w}_E,m}] < \sigma[r_{\boldsymbol{w}_m,m}]$, Eq. (4) implies that:

$$V[1-\alpha_m, r_{\boldsymbol{w}_E,m}] < V[1-\alpha_m, r_{\boldsymbol{w}_m,m}]. \tag{34}$$

Fix any $E_1 \in \mathbb{R}$ with $E_1 > E[r_{\boldsymbol{w}_m,m}]$. Let $\varepsilon > 0$ be arbitrarily small. Consider portfolio $\boldsymbol{w}_\varepsilon \equiv \varepsilon\boldsymbol{w}_{E_1} + (1-\varepsilon)\boldsymbol{w}_E$. Note that:

$$E[r_{\boldsymbol{w}_\varepsilon,m}] > E[r_{\boldsymbol{w}_E,m}]. \tag{35}$$

Since ε is arbitrarily small, Eq. (34) implies that:

$$\begin{aligned} V[1-\alpha_m, r_{\boldsymbol{w}_\varepsilon,m}] &< V[1-\alpha_m, r_{\boldsymbol{w}_m,m}] \\ &\leq -H_m, \end{aligned} \tag{36}$$

where the second inequality follows from the definition of $\boldsymbol{w}_m$. Eqs. (35) and (36) contradict the fact that $\boldsymbol{w}_m$ is the optimal portfolio within account m. This completes the first part of our proof.

Second, we show that $E > \underline{E}_m$. Using Eqs. (4) and (17), we have:

$$V[1-\alpha_m, r_{\boldsymbol{w}_E,m}] = z_{\alpha_m}\sqrt{\underline{\sigma}_m^2 + \left(E[r_{\boldsymbol{w}_E,m}] - \underline{E}_m\right)^2 / (D/C)} - E[r_{\boldsymbol{w}_E,m}]. \tag{37}$$

It follows from Eq. (37) that:

$$\frac{\partial V[1-\alpha_m, r_{\boldsymbol{w}_E,m}]}{\partial E[r_{\boldsymbol{w}_E,m}]} = z_{\alpha_m}\frac{\left(E[r_{\boldsymbol{w}_E,m}] - \underline{E}_m\right) / (D/C)}{\sqrt{\underline{\sigma}_m^2 + \left(E[r_{\boldsymbol{w}_E,m}] - \underline{E}_m\right)^2 / (D/C)}} - 1. \tag{38}$$

Since $z_{\alpha_m} > 0$, Eq. (38) implies that if $E[r_{\boldsymbol{w}_E,m}] \leq \underline{E}_m$, then $\partial V[1-\alpha_m, r_{\boldsymbol{w}_E,m}]/\partial E[r_{\boldsymbol{w}_E,m}] < 0$. Hence, we have $E > \underline{E}_m$. This completes the second part of our proof.

Third, we show that $V[1-\alpha_m, r_{\boldsymbol{w}_m,m}] = -H_m$. Suppose by way of a contradiction that $V[1-\alpha_m, r_{\boldsymbol{w}_m,m}] < -H_m$. Fix any $E_2 \in \mathbb{R}$ with $E_2 > E[r_{\boldsymbol{w}_m,m}]$. Let $\xi > 0$ be arbitrarily small. Consider portfolio $\boldsymbol{w}_\xi \equiv \xi \boldsymbol{w}_{E_2} + (1-\xi)\boldsymbol{w}_m$. Note that:

$$E[r_{\boldsymbol{w}_\xi,m}] > E[r_{\boldsymbol{w}_m,m}] \tag{39}$$

and

$$V[1-\alpha_m, r_{\boldsymbol{w}_\xi,m}] < -H_m. \tag{40}$$

Eqs. (39) and (40) contradict the fact that $\boldsymbol{w}_m$ is the optimal portfolio within account m. This completes the third part of our proof.□

Proof of Theorem 1. Fix any account $m \in \{1, ..., M\}$. First, we show part (i). Suppose that $\alpha_m \geq \Phi(-\sqrt{D/C})$. Then:

$$0 < z_{\alpha_m} \leq \sqrt{D/C}. \tag{41}$$

Fix any $E \in \mathbb{R}$. Note that:

$$\frac{(E[r_{\boldsymbol{w}_E,m}] - \underline{E}_m) \,/\, (D/C)}{\sqrt{\underline{\sigma}_m^2 + (E[r_{\boldsymbol{w}_E,m}] - \underline{E}_m)^2 \,/\, (D/C)}} < \frac{1}{\sqrt{D/C}}. \tag{42}$$

It follows from Eqs. (38), (41), and (42) that:

$$\frac{\partial V[1-\alpha_m, r_{\boldsymbol{w}_E,m}]}{\partial E[r_{\boldsymbol{w}_E,m}]} < 0. \tag{43}$$

Eq. (43) implies that the optimal portfolio within account m does not exist.

Suppose that $\alpha_m < \Phi(-\sqrt{D/C})$ and $H_m > H_{\alpha_m}$. Note that $-H_m < -H_{\alpha_m} = V[1-\alpha_m, r_{\underline{\underline{\boldsymbol{w}}}_m,m}]$. Hence, there exists no portfolio $\boldsymbol{w}$ that meets constraint (5). Therefore, the optimal portfolio within account m does not exist. This completes our proof of part (i).

Second, we show part (ii). Suppose that $\alpha_m < \Phi(-\sqrt{D/C})$ and $H_m \leq H_{\alpha_m}$. Using Lemma 3, we have $E[r_{\boldsymbol{w}_m,m}] > \underline{E}_m$. Hence, it follows from Lemma 3 and Eq. (17) that:

$$E[r_{\boldsymbol{w}_m,m}] = \underline{E}_m + \sqrt{D/C\,(\sigma^2[r_{\boldsymbol{w}_m,m}] - \underline{\sigma}_m^2)}. \tag{44}$$

Using Eqs. (4), (44) and Lemma 3, we have:

$$z_{\alpha_m}\sigma[r_{\boldsymbol{w}_m,m}] - \underline{E}_m - \sqrt{D/C\,(\sigma^2[r_{\boldsymbol{w}_m,m}] - \underline{\sigma}_m^2)} = -H_m. \tag{45}$$

It follows from Eq. (45) that:

$$\zeta_1\sigma^2[r_{\boldsymbol{w}_m,m}] + \zeta_2\sigma[r_{\boldsymbol{w}_m,m}] + \zeta_3 = 0, \tag{46}$$

where $\zeta_1 \equiv z_{\alpha_m}^2 - D/C$, $\zeta_2 \equiv -2z_{\alpha_m}(\underline{E}_m - H_m)$, and $\zeta_3 \equiv (\underline{E}_m - H_m)^2 + (D/C)\,\underline{\sigma}_m^2$. Using Eq. (46), we have:

$$\sigma[r_{\boldsymbol{w}_m,m}] = \frac{z_{\alpha_m}(\underline{E}_m - H_m) \pm \sqrt{(D/C)\left[(\underline{E}_m - H_m)^2 - (z_{\alpha_m}^2 - D/C)\underline{\sigma}_m^2\right]}}{z_{\alpha_m}^2 - D/C}. \tag{47}$$

Eqs. (8)–(11) follow from Lemmas 1 and 3, and Eqs. (44) and (47). This completes our proof of part (ii).□

The following result is used in our proof of Corollary 1.

Lemma 4. *Consider an investor with a single account who faces account m's background risk and has an objective function given by Eq. (12). The investor's optimal portfolio is:*

$$\boldsymbol{w}_{\gamma_m} \equiv \underline{\boldsymbol{w}}_m + \frac{A}{\gamma_m}\left(\boldsymbol{w}_1 - \boldsymbol{w}_0\right). \tag{48}$$

Proof. Consider an investor with a single account who faces account m's background risk and has an objective function given by Eq. (12). The investor's optimal portfolio solves:

$$\max_{\boldsymbol{w}\in\mathbb{R}^N} \quad \boldsymbol{w}'\boldsymbol{\mu} + \nu_m - \frac{\gamma_m}{2}\left(\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} + \Omega_{mm} + 2\boldsymbol{w}'\boldsymbol{\Psi}_m\right) \tag{49}$$

$$s.t. \qquad \boldsymbol{w}'\mathbf{1} = 1. \tag{50}$$

A first-order condition for $\boldsymbol{w}_{\gamma_m}$ to solve problem (49) subject to constraint (50) is:

$$\boldsymbol{\mu} - \gamma_m\left(\boldsymbol{\Sigma}\boldsymbol{w}_{\gamma_m} + \boldsymbol{\Psi}_m\right) + \lambda_m\mathbf{1} = \mathbf{0}, \tag{51}$$

where λ_m is the multiplier associated with this constraint. Eq. (51) implies that:

$$\boldsymbol{w}_{\gamma_m} = \boldsymbol{\Sigma}^{-1}\left(\frac{\boldsymbol{\mu} + \lambda_m\mathbf{1}}{\gamma_m} - \boldsymbol{\Psi}_m\right). \tag{52}$$

Premultiplying Eq. (52) by $\mathbf{1}'$ and using Eq. (50), we have:

$$1 = \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \lambda_m\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\gamma_m} - \mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Psi}_m. \tag{53}$$

Eq. (53) implies that:

$$\lambda_m = \frac{\gamma_m\left(1 + A_m\right) - A}{C}. \tag{54}$$

It follows from Eqs. (53) and (54) that:

$$\boldsymbol{w}_{\gamma_m} = \left(1 + A_m\right)\frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{C} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\Psi}_m + \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{A}{C}\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\gamma_m} \tag{55}$$

The desired result follows from Eq. (55) and the definitions of $\underline{\boldsymbol{w}}_m$, $\boldsymbol{w}_0$, and $\boldsymbol{w}_1$.□

Proof of Corollary 1. Fix any account $m \in \{1, ..., M\}$ with $\alpha_m < \Phi(-\sqrt{D/C})$ and $H_m \leq H_{\alpha_m}$. The desired result follows from Eqs. (8) and (48).□

Proof of Corollary 2. Fix any account $m \in \{1, ..., M\}$ with $\alpha_m < \Phi(-\sqrt{D/C})$ and $H_m \leq H_{\alpha_m}$. First, we show the 'if' part. Suppose that $\boldsymbol{\Psi}_m = \delta_1 \mathbf{1} + \delta_2 \boldsymbol{\mu}$ for some constants δ_1 and δ_2. Using the definition of $\underline{\boldsymbol{w}}_m$ and the assumption that $\boldsymbol{\Psi}_m = \delta_1 \mathbf{1} + \delta_2 \boldsymbol{\mu}$, we have:

$$\underline{\boldsymbol{w}}_m = \left[1 + \mathbf{1}'\boldsymbol{\Sigma}^{-1}\left(\delta_1 \mathbf{1} + \delta_2 \boldsymbol{\mu}\right)\right] \boldsymbol{w}_0 - \boldsymbol{\Sigma}^{-1}\left(\delta_1 \mathbf{1} + \delta_2 \boldsymbol{\mu}\right). \tag{56}$$

It follows from Eq. (56) that:

$$\underline{\boldsymbol{w}}_m = \boldsymbol{w}_0 - A\delta_2 \left(\boldsymbol{w}_1 - \boldsymbol{w}_0\right). \tag{57}$$

Eqs. (8) and (57) imply that:

$$\boldsymbol{w}_m = \boldsymbol{w}_0 + \left(\eta_m - A\delta_2\right)\left(\boldsymbol{w}_1 - \boldsymbol{w}_0\right). \tag{58}$$

Merton (1972) shows that a portfolio $\boldsymbol{w}$ is on the mean-variance frontier if and only if:

$$\boldsymbol{w} = \theta \boldsymbol{w}_0 + (1-\theta)\boldsymbol{w}_1 \tag{59}$$

for some $\theta \in \mathbb{R}$. It follows from Eqs. (58) and (59) that portfolio $\boldsymbol{w}_m$ is on the mean-variance frontier. This completes the first part of our proof.

Second, we show the 'only if' part. Suppose that $\boldsymbol{w}_m$ is on the mean-variance frontier. Using Eqs. (8) and (59), $\underline{\boldsymbol{w}}_m$ is also on this frontier. Hence, Eq. (59) implies that:

$$\underline{\boldsymbol{w}}_m = \underline{\theta}_m \boldsymbol{w}_0 + (1-\underline{\theta}_m)\boldsymbol{w}_1 \tag{60}$$

for some $\underline{\theta}_m \in \mathbb{R}$. Using the definition of $\underline{\boldsymbol{w}}_m$ in the left-hand side of Eq. (60), we obtain:

$$\left(1 + \mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Psi}_m\right)\boldsymbol{w}_0 - \boldsymbol{\Sigma}^{-1}\boldsymbol{\Psi}_m = \underline{\theta}_m \boldsymbol{w}_0 + (1-\underline{\theta}_m)\boldsymbol{w}_1, \tag{61}$$

or equivalently:

$$\boldsymbol{\Sigma}^{-1}\boldsymbol{\Psi}_m = \left(1 + \mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Psi}_m - \underline{\theta}_m\right)\boldsymbol{w}_0 - (1-\underline{\theta}_m)\boldsymbol{w}_1. \tag{62}$$

Premultiplying Eq. (62) by $\boldsymbol{\Sigma}$, we have:

$$\boldsymbol{\Psi}_m = \frac{1 + \mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Psi}_m - \underline{\theta}_m}{C}\mathbf{1} - \frac{1 - \underline{\theta}_m}{A}\boldsymbol{\mu}. \tag{63}$$

It follows from Eq. (63) that $\boldsymbol{\Psi}_m = \delta_1\mathbf{1} + \delta_2\boldsymbol{\mu}$ for some constants δ_1 and δ_2. This completes the second part of our proof.□

The following result is used in our proof of Corollary 3.

Lemma 5. *Consider an investor with a single account who does not face background risk and has an objective function given by Eq. (13). The investor's optimal portfolio is:*

$$\boldsymbol{w}_\gamma \equiv \boldsymbol{w}_0 + \frac{A}{\gamma}\left(\boldsymbol{w}_1 - \boldsymbol{w}_0\right). \tag{64}$$

Proof. Consider an investor with a single account who does not face background risk and has an objective function given by Eq. (13). The investor's optimal portfolio solves:

$$\max_{\boldsymbol{w}\in\mathbb{R}^N} \quad \boldsymbol{w}'\boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} \tag{65}$$

$$s.t. \qquad \boldsymbol{w}'\mathbf{1} = 1. \tag{66}$$

A first-order condition for $\boldsymbol{w}_\gamma$ to solve problem (65) subject to constraint (66) is:

$$\boldsymbol{\mu} - \gamma\boldsymbol{\Sigma}\boldsymbol{w}_\gamma - \lambda\mathbf{1} = \mathbf{0}, \tag{67}$$

where λ is the multiplier associated with this constraint. Eq. (67) implies that:

$$\boldsymbol{w}_\gamma = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \lambda\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\gamma}. \tag{68}$$

Premultiplying Eq. (68) by $\mathbf{1}'$ and using Eq. (66), we have:

$$1 = \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \lambda\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\gamma}. \tag{69}$$

Eq. (69) implies that:

$$\lambda = \frac{A-\gamma}{C}. \tag{70}$$

It follows from Eqs. (68) and (70) that:

$$\boldsymbol{w}_\gamma = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{C} + \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{A}{C}\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\gamma}. \tag{71}$$

The desired result follows from Eq. (71).□

Proof of Corollary 3. Fix any given account $m \in \{1, ..., M\}$ with $\alpha_m < \Phi(-\sqrt{D/C})$, $H_m \leq H_{\alpha_m}$, and $\boldsymbol{\Psi}_m = \delta_1\mathbf{1} + \delta_2\boldsymbol{\mu}$ for some constants δ_1 and δ_2. It follows that Eq. (58) holds. Eqs. (58) and (64) imply the desired result.□

Proof of Theorem 2. Suppose that $\alpha_m < \Phi(-\sqrt{D/C})$ and $H_m \leq H_{\alpha_m}$ for any account $m \in \{1, ..., M\}$. Using Eq. (8), we have:

$$\boldsymbol{w}_a = \sum_{m=1}^M y_m \underline{\boldsymbol{w}}_m + \sum_{m=1}^M y_m \eta_m \left(\boldsymbol{w}_1 - \boldsymbol{w}_0\right). \tag{72}$$

Noting that $\underline{\boldsymbol{w}}_a = \sum_{m=1}^M y_m \underline{\boldsymbol{w}}_m$, the desired result follows from Eq. (72).□

The following result is used in our proof of Corollary 4.

Lemma 6. *Consider an investor with a single account who faces the aggregate background risk and has an objective function given by Eq. (15). The investor's optimal portfolio is:*

$$\boldsymbol{w}_{\gamma_a} \equiv \underline{\boldsymbol{w}}_a + \frac{A}{\gamma_a}\left(\boldsymbol{w}_1 - \boldsymbol{w}_0\right). \tag{73}$$

Proof of Lemma 6. Similar to the proof of Lemma 4 and thus omitted.□

Proof of Corollary 4. Suppose that $\alpha_m < \Phi(-\sqrt{D/C})$ and $H_m \leq H_{\alpha_m}$ for any account $m \in \{1, ..., M\}$. The desired result follows from Eqs. (14) and (73).□

Proof of Corollary 5. Suppose that $\alpha_m < \Phi(-\sqrt{D/C})$ and $H_m \leq H_{\alpha_m}$ for any account $m \in \{1, ..., M\}$. First, we show the 'if' part. Suppose that $\boldsymbol{\Psi}_a = \delta_1 \mathbf{1} + \delta_2 \boldsymbol{\mu}$ for some constants δ_1 and δ_2. Using arguments similar to those in the proof of Corollary 2, we have:

$$\boldsymbol{w}_a = \boldsymbol{w}_0 + (\eta_a - A\delta_2)(\boldsymbol{w}_1 - \boldsymbol{w}_0). \tag{74}$$

It follows from Eqs. (59) and (74) that portfolio $\boldsymbol{w}_a$ is on the mean-variance frontier. This completes the first part of our proof.

Second, we show the 'only if' part. Suppose that $\boldsymbol{w}_a$ is on the mean-variance frontier. Using Eqs. (14) and (59), $\underline{\boldsymbol{w}}_a$ is also on this frontier. Hence, Eq. (59) implies that:

$$\underline{\boldsymbol{w}}_a = \underline{\theta}_a \boldsymbol{w}_0 + (1 - \underline{\theta}_a)\boldsymbol{w}_1 \tag{75}$$

for some $\underline{\theta}_a \in \mathbb{R}$. Using arguments similar to those used in the proof of Corollary 2, we have:

$$\boldsymbol{\Psi}_a = \frac{1 + \mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Psi}_a - \underline{\theta}_a}{C}\mathbf{1} - \frac{1 - \underline{\theta}_a}{A}\boldsymbol{\mu}. \tag{76}$$

It follows from Eq. (76) that $\boldsymbol{\Psi}_a = \delta_1 \mathbf{1} + \delta_2 \boldsymbol{\mu}$ for some constants δ_1 and δ_2. This completes the second part of our proof.$\square$

Proof of Corollary 6. Suppose that $\alpha_m < \Phi(-\sqrt{D/C})$ and $H_m \leq H_{\alpha_m}$ for any account $m \in \{1, ..., M\}$, and $\boldsymbol{\Psi}_a = \delta_1 \mathbf{1} + \delta_2 \boldsymbol{\mu}$ for some constants δ_1 and δ_2. It follows that Eq. (74) holds. Eqs. (64) and (74) imply the desired result.$\square$