Online Appendix for

"Optimal Delegated Portfolio Management with Background Risk"

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This Appendix contains proofs of Theorems 1–4 in the paper "Optimal Delegated Portfolio Management with Background Risk" published in the *Journal of Banking and Finance* **32**, 977–985, June 2008.

Proof of Theorem 1. Observe that $\boldsymbol{q}_I$ solves

$$\max_{\boldsymbol{q}\in\mathbb{R}^N} \quad \boldsymbol{q}'\boldsymbol{R} - \frac{\gamma_I}{2}\left(\boldsymbol{q}'\boldsymbol{V}\boldsymbol{q} + \sigma_Y^2 + 2\boldsymbol{q}'\boldsymbol{V}_Y\right) \tag{10}$$

$$s.t. \quad \boldsymbol{q}'\boldsymbol{1} = 1. \tag{11}$$

A first-order condition for portfolio $\boldsymbol{q}_I$ to solve problem (10) subject to constraint (11) is given by

$$\boldsymbol{R} - \gamma_I(\boldsymbol{V}\boldsymbol{q}_I + \boldsymbol{V}_Y) + \lambda_I\boldsymbol{1} = 0, \tag{12}$$

where λ_I is the Lagrange multiplier associated with the constraint. Eq. (12) implies that

$$\boldsymbol{q}_I = \frac{\boldsymbol{V}^{-1}(\boldsymbol{R} - \gamma_I\boldsymbol{V}_Y + \lambda_I\boldsymbol{1})}{\gamma_I}. \tag{13}$$

Premultiplying both sides of Eq. (13) by $\boldsymbol{1}'$ and using constraint (11), we have

$$1 = \frac{\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{R} + \lambda_I\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}}{\gamma_I} - \boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{V}_Y. \tag{14}$$

It follows from Eq. (14) and the definition of b and c that

$$\lambda_I = \frac{\gamma_I(1 + \boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{V}_Y) - b}{c}. \tag{15}$$

Hence, Eqs. (13) and (15) imply that Eq. (2) holds. ■

Proof of Theorem 2. Observe that $\boldsymbol{q}_P$ solves

$$\max_{\boldsymbol{q}\in\mathbb{R}^N} \ (\boldsymbol{q}-\boldsymbol{q}_B)'\boldsymbol{R} - \frac{\gamma_P}{2}(\boldsymbol{q}-\boldsymbol{q}_B)'\boldsymbol{V}(\boldsymbol{q}-\boldsymbol{q}_B) \tag{16}$$

$$s.t. \quad \boldsymbol{q}'\boldsymbol{1} = 1. \tag{17}$$

A first-order condition for portfolio $\boldsymbol{q}_P$ to solve problem (16) subject to constraint (17) is given by

$$\boldsymbol{R} - \gamma_P \boldsymbol{V}(\boldsymbol{q}_P - \boldsymbol{q}_B) + \lambda_P \boldsymbol{1} = 0, \tag{18}$$

where λ_P is the Lagrange multiplier associated with the constraint. Eq. (18) implies that

$$\boldsymbol{q}_P = \boldsymbol{q}_B + \frac{\boldsymbol{V}^{-1}(\boldsymbol{R}+\lambda_P\boldsymbol{1})}{\gamma_P}. \tag{19}$$

Premultiplying both sides of Eq. (19) by $\boldsymbol{1}'$ and using constraint (17), we have

$$1 = 1 + \frac{\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{R} + \lambda_P\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}}{\gamma_P}. \tag{20}$$

It follows from Eq. (20) and the definition of b and c that

$$\lambda_P = -\frac{b}{c}. \tag{21}$$

Hence, Eqs. (19) and (21) imply that Eq. (4) holds.[1] ■

Proof of Theorem 3. First, we show the 'if part.' Suppose that Eq. (6) holds. It follows from Eqs. (4) and (6) that

$$\begin{aligned} \boldsymbol{q}_P &= \boldsymbol{q}_Y + \frac{b}{\gamma_I}(\boldsymbol{q}_1 - \boldsymbol{q}_0) \\ &= \boldsymbol{q}_I, \end{aligned}$$

[1] Note that Eq. (4) is equivalent to Eq. (2) in Brennan (1993, p. 7).

where the last equality follows from Eq. (2). This completes the first part of our proof.

Second, we show the 'only if part.' Suppose that Eq. (5) holds. Eqs. (2), (4), and (5) imply that

$$\boldsymbol{q}_Y + \frac{b}{\gamma_I}(\boldsymbol{q}_1 - \boldsymbol{q}_0) = \boldsymbol{q}_B + \frac{b}{\gamma_P}(\boldsymbol{q}_1 - \boldsymbol{q}_0). \tag{22}$$

It follows from Eq. (22) that Eq. (6) holds. This completes the second part of our proof. ■

Proof of Theorem 4. First, we show the 'if part.' Suppose that Eq. (9) holds for some weights $w_{P_1}, ..., w_{P_M}$. Using Eq. (7), we have

$$\sum_{m=1}^{M} w_{P_m} \boldsymbol{q}_{P_m} = \sum_{m=1}^{M} w_{P_m} \boldsymbol{q}_{B_m} + b \sum_{m=1}^{M} \frac{w_{P_m}}{\gamma_{P_m}}(\boldsymbol{q}_1 - \boldsymbol{q}_0). \tag{23}$$

It follows from Eqs. (9) and (23) that

$$\begin{aligned} \sum_{m=1}^{M} w_{P_m} \boldsymbol{q}_{P_m} &= \boldsymbol{q}_Y + \frac{b}{\gamma_I}(\boldsymbol{q}_1 - \boldsymbol{q}_0) \\ &= \boldsymbol{q}_I, \end{aligned}$$

where the last equality follows from Eq. (2). This completes the first part of our proof.

Second, we show the 'only if part.' Suppose that Eq. (8) holds for some weights $w_{P_1}, ..., w_{P_M}$. Eqs. (2), (7), and (8) imply that

$$\boldsymbol{q}_Y + \frac{b}{\gamma_I}(\boldsymbol{q}_1 - \boldsymbol{q}_0) = \sum_{m=1}^{M} w_{P_m} \boldsymbol{q}_{B_m} + b \sum_{m=1}^{M} \frac{w_{P_m}}{\gamma_{P_m}}(\boldsymbol{q}_1 - \boldsymbol{q}_0). \tag{24}$$

It follows from Eq. (24) that Eq. (9) holds. This completes the second part of our proof. ■