On-Line Supplement for

"Mean-Variance Portfolio Selection with 'at-Risk' Constraints and Discrete Distributions"

by

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Sections 1 and 2 of this Appendix contain, respectively, the proofs of Theorems 1–4 and a description of the numerical procedure used in the paper "Mean-Variance Portfolio Selection with 'at-Risk' Constraints and Discrete Distributions" published in the *Journal of Banking and Finance* 31, 3761—3781, December 2007.

1. Proofs

Proof of Theorem 1. Suppose that $\boldsymbol{w}_{E,\alpha,V}$ exists. First, assume $V_{\alpha,\boldsymbol{w}_E} \leq V$. The desired result follows from Eq. (1). This completes the first part of our proof.

Second, assume $V_{\alpha,\boldsymbol{w}_E} > V$. We begin by showing that

$$V_{\alpha,\boldsymbol{w}_{E,\alpha,V}} = V. \tag{14}$$

Suppose by way of a contradiction that $V_{\alpha,\boldsymbol{w}_{E,\alpha,V}} < V$. Let $\boldsymbol{w}_* \equiv \varepsilon \boldsymbol{w}_E + (1-\varepsilon)\boldsymbol{w}_{E,\alpha,V}$, where $\varepsilon > 0$ is arbitrarily small. Since $\overline{R}_{\boldsymbol{w}_E} = \overline{R}_{\boldsymbol{w}_{E,\alpha,V}} = E$,

$$\overline{R}_{\boldsymbol{w}_*} = E. \tag{15}$$

Using the fact that the portfolio on the unconstrained boundary with expected return E is unique, we have $\sigma^2_{\boldsymbol{w}_E} < \sigma^2_{\boldsymbol{w}}$ for any portfolio $\boldsymbol{w}$ with an expected return of E and $\boldsymbol{w} \neq \boldsymbol{w}_E$. Thus, since $\overline{R}_{\boldsymbol{w}_{E,\alpha,V}} = E$ and $\boldsymbol{w}_{E,\alpha,V} \neq \boldsymbol{w}_E$, we have $\sigma^2_{\boldsymbol{w}_E} < \sigma^2_{\boldsymbol{w}_{E,\alpha,V}}$. Let $g : \mathbb{R}^J \to \mathbb{R}$ be defined by $g(\boldsymbol{w}) = \sigma^2_{\boldsymbol{w}}$. Since $\boldsymbol{V}$ is positive definite, $g(\cdot)$ is strictly convex. Thus,

$$\begin{aligned} \sigma^2_{\boldsymbol{w}_*} &< \varepsilon\sigma^2_{\boldsymbol{w}_E} + (1-\varepsilon)\sigma^2_{\boldsymbol{w}_{E,\alpha,V}} \\ &< \sigma^2_{\boldsymbol{w}_{E,\alpha,V}}, \end{aligned} \tag{16}$$

where the last inequality follows from the fact that $\sigma^2_{\boldsymbol{w}_E} < \sigma^2_{\boldsymbol{w}_{E,\alpha,V}}$.

Next, we show that

$$V_{\alpha,\boldsymbol{w}_*} < V. \tag{17}$$

For any portfolio $\boldsymbol{w}$, let $\underline{\Omega}(\boldsymbol{w}) \equiv \{s \in \Omega : \boldsymbol{w}^\top \boldsymbol{R}_s < -V_{\alpha,\boldsymbol{w}}\}$, $\underline{\overline{\Omega}}(\boldsymbol{w}) \equiv \{s \in \Omega : \boldsymbol{w}^\top \boldsymbol{R}_s = -V_{\alpha,\boldsymbol{w}}\}$, and $\overline{\Omega}(\boldsymbol{w}) \equiv \{s \in \Omega : \boldsymbol{w}^\top \boldsymbol{R}_s > -V_{\alpha,\boldsymbol{w}}\}$. Since ε is arbitrarily small, $\boldsymbol{w}_*^\top \boldsymbol{R}_s$ is arbitrarily close to $\boldsymbol{w}_{E,\alpha,V}^\top \boldsymbol{R}_s$ for any $s \in \Omega$. Hence,

$$\boldsymbol{w}_*^\top \boldsymbol{R}_{\underline{s}} < \boldsymbol{w}_*^\top \boldsymbol{R}_{\underline{\overline{s}}} < \boldsymbol{w}_*^\top \boldsymbol{R}_{\overline{s}} \ \forall (\underline{s}, \underline{\overline{s}}, \overline{s}) \in \underline{\Omega}(\boldsymbol{w}_{E,\alpha,V}) \times \underline{\overline{\Omega}}(\boldsymbol{w}_{E,\alpha,V}) \times \overline{\Omega}(\boldsymbol{w}_{E,\alpha,V}). \tag{18}$$

Note that

$$P[\widetilde{R}_{\boldsymbol{w}_*} > \boldsymbol{w}_*^\top \boldsymbol{R}_{\underline{s}}] \geq P[\widetilde{R}_{\boldsymbol{w}_{E,\alpha,V}} \geq -V_{\alpha,\boldsymbol{w}_{E,\alpha,V}}] \geq \alpha \ \forall \underline{s} \in \underline{\Omega}(\boldsymbol{w}_{E,\alpha,V}), \tag{19}$$

where the first and second inequalities follow from, respectively, Eqs. (18) and (4). Also,

$$P[\widetilde{R}_{\boldsymbol{w}_*} \geq \boldsymbol{w}_*^\top \boldsymbol{R}_{\overline{s}}] \leq P[\widetilde{R}_{\boldsymbol{w}_{E,\alpha,V}} > -V_{\alpha,\boldsymbol{w}_{E,\alpha,V}}] < \alpha \ \forall \overline{s} \in \overline{\Omega}(\boldsymbol{w}_{E,\alpha,V}), \tag{20}$$

where the first and second inequalities follow from, respectively, Eqs. (18) and (5). Eqs. (4), (5), (19), and (20) imply that $V_{\alpha,\boldsymbol{w}_*} = -\boldsymbol{w}_*^\top \boldsymbol{R}_{\underline{\overline{s}}}$ for some $\underline{\overline{s}} \in \underline{\overline{\Omega}}(\boldsymbol{w}_{E,\alpha,V})$. Since $-\boldsymbol{w}_*^\top \boldsymbol{R}_{\underline{\overline{s}}}$ is arbitrarily close to $-\boldsymbol{w}_{E,\alpha,V}^\top \boldsymbol{R}_{\underline{\overline{s}}} = V_{\alpha,\boldsymbol{w}_{E,\alpha,V}}$ for any $\underline{\overline{s}} \in \underline{\overline{\Omega}}(\boldsymbol{w}_{E,\alpha,V})$, $V_{\alpha,\boldsymbol{w}_*}$ is arbitrarily close to $V_{\alpha,\boldsymbol{w}_{E,\alpha,V}}$. Thus, Eq. (17) holds since $V_{\alpha,\boldsymbol{w}_{E,\alpha,V}} < V$. Eqs. (15)–(17) contradict the fact that $\boldsymbol{w}_{E,\alpha,V}$ is on the VaR-constrained boundary.

It follows from Eq. (14) that $\boldsymbol{w}_{E,\alpha,V}$ solves

$$\min_{\boldsymbol{w}\in\mathbb{R}^J} \frac{1}{2}\boldsymbol{w}^\top \boldsymbol{V}\boldsymbol{w} \tag{21}$$

$$s.t. \quad \boldsymbol{w}^\top \mathbf{1} = 1 \tag{22}$$

$$\boldsymbol{w}^\top \overline{\boldsymbol{R}} = E \tag{23}$$

$$\boldsymbol{w}^\top \boldsymbol{R}_s = -V \quad \forall s \in \underline{\overline{\Omega}}(\boldsymbol{w}_{E,\alpha,V}) \tag{24}$$

$$\boldsymbol{w}^\top \boldsymbol{R}_s \leq -V \quad \forall s \in \underline{\Omega}(\boldsymbol{w}_{E,\alpha,V}) \tag{25}$$

$$\boldsymbol{w}^\top \boldsymbol{R}_s \geq -V \quad \forall s \in \overline{\Omega}(\boldsymbol{w}_{E,\alpha,V}). \tag{26}$$

Using the definition of $\overline{\Omega}(\boldsymbol{w}_{E,\alpha,V})$ and $\underline{\Omega}(\boldsymbol{w}_{E,\alpha,V})$, constraints (25) and (26) do not bind. First-order conditions for $\boldsymbol{w}_{E,\alpha,V}$ to solve problem (21) subject to constraints (22)–(24) are

$$\boldsymbol{V}\boldsymbol{w}_{E,\alpha,V} - \lambda_1 \mathbf{1} - \lambda_2 \overline{\boldsymbol{R}} - \sum_{k=3}^{K+2} \lambda_k \boldsymbol{R}_{s_k} = 0 \tag{27}$$

$$\boldsymbol{w}_{E,\alpha,V}^\top \mathbf{1} = 1 \tag{28}$$

$$\boldsymbol{w}_{E,\alpha,V}^\top \overline{\boldsymbol{R}} = E \tag{29}$$

$$\boldsymbol{w}_{E,\alpha,V}^\top \boldsymbol{R}_{s_k} = -V, \; k = 3, ..., K+2, \tag{30}$$

where $\lambda_1, ..., \lambda_{K+2}$ are Lagrange multipliers associated with these constraints and $\{s_3, ..., s_{K+2}\} = \underline{\overline{\Omega}}(\boldsymbol{w}_{E,\alpha,V})$. Since $rank(\boldsymbol{V}) = J$, Eq. (27) implies that

$$\boldsymbol{w}_{E,\alpha,V} = \lambda_1(\boldsymbol{V}^{-1}\mathbf{1}) + \lambda_2(\boldsymbol{V}^{-1}\overline{\boldsymbol{R}}) + \sum_{k=3}^{K+2} \lambda_k(\boldsymbol{V}^{-1}\boldsymbol{R}_{s_k}). \tag{31}$$

Eq. (31) implies that Eq. (6) holds with $(\beta_1, ..., \beta_{K+2}) = (\lambda_1 c, \lambda_2 a, \lambda_3 c_{s_3}, ..., \lambda_{K+2} c_{s_{K+2}})$. We now find $\boldsymbol{L}_{E,V} \equiv [\lambda_1 \; \cdots \; \lambda_{K+2}]^\top$. Premultiplying Eq. (31) by $\mathbf{1}^\top$ and using Eq. (28), we have

$$\lambda_1(\mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1}) + \lambda_2(\mathbf{1}^\top \boldsymbol{V}^{-1}\overline{\boldsymbol{R}}) + \sum_{k=3}^{K+2} \lambda_k(\mathbf{1}^\top \boldsymbol{V}^{-1}\boldsymbol{R}_{s_k}) = 1. \tag{32}$$

Premultiplying Eq. (31) by $\overline{\boldsymbol{R}}^\top$ and using Eq. (29), we obtain

$$\lambda_1(\overline{\boldsymbol{R}}^\top \boldsymbol{V}^{-1}\mathbf{1}) + \lambda_2(\overline{\boldsymbol{R}}^\top \boldsymbol{V}^{-1}\overline{\boldsymbol{R}}) + \sum_{k=3}^{K+2} \lambda_k(\overline{\boldsymbol{R}}^\top \boldsymbol{V}^{-1}\boldsymbol{R}_{s_k}) = E. \tag{33}$$

Premultiplying Eq. (31) by $\boldsymbol{R}_s^\top$ and using Eq. (30), we have

$$\lambda_1(\boldsymbol{R}_s^\top \boldsymbol{V}^{-1}\mathbf{1}) + \lambda_2(\boldsymbol{R}_s^\top \boldsymbol{V}^{-1}\overline{\boldsymbol{R}}) + \sum_{k=3}^{K+2} \lambda_k(\boldsymbol{R}_s^\top \boldsymbol{V}^{-1}\boldsymbol{R}_{s_k}) = -V,\ s = s_3, ..., s_{K+2}. \tag{34}$$

Let $\boldsymbol{K}_{E,V} \equiv [1 \ \ E \ \ -V \ \ \cdots \ -V]^\top$, $\boldsymbol{M}_{E,V} \equiv [\mathbf{1} \ \ \overline{\boldsymbol{R}} \ \ \boldsymbol{R}_{s_3} \ \ \cdots \ \ \boldsymbol{R}_{s_{K+2}}]$, and $\boldsymbol{N}_{E,V} \equiv \boldsymbol{M}_{E,V}^\top \boldsymbol{V}^{-1}\boldsymbol{M}_{E,V}$. Using Eqs. (32)–(34), $\boldsymbol{N}_{E,V}\boldsymbol{L}_{E,V} = \boldsymbol{K}_{E,V}$. Since $rank(\boldsymbol{V}) = J$ and $rank(\boldsymbol{M}_{E,V}) = K + 2$, $rank(\boldsymbol{N}_{E,V}) = K + 2$. Hence, $\boldsymbol{L}_{E,V} = \boldsymbol{N}_{E,V}^{-1}\boldsymbol{K}_{E,V}$. This completes the second part of our proof. ■

Proof of Theorem 2. Suppose that $\boldsymbol{w}_{E,\alpha,C}$ exists. First, assume $C_{\alpha,\boldsymbol{w}_E} \leq C$. The desired result follows from Eq. (1). This completes the first part of our proof.

Second, assume $C_{\alpha,\boldsymbol{w}_E} > C$. We begin by showing that

$$C_{\alpha,\boldsymbol{w}_{E,\alpha,C}} = C. \tag{35}$$

Suppose by way of a contradiction that $C_{\alpha,\boldsymbol{w}_{E,\alpha,C}} < C$. Let $\boldsymbol{w}_* \equiv \varepsilon \boldsymbol{w}_E + (1-\varepsilon)\boldsymbol{w}_{E,\alpha,C}$, where $\varepsilon > 0$ is arbitrarily small. Using arguments similar to those in the proof of Theorem 1,

$$\overline{R}_{\boldsymbol{w}_*} = E, \tag{36}$$

$$\sigma^2_{\boldsymbol{w}_*} < \sigma^2_{\boldsymbol{w}_{E,\alpha,C}}. \tag{37}$$

Next, we show that

$$C_{\alpha,\boldsymbol{w}_*} < C. \tag{38}$$

Using Eq. (7), we have

$$C_{\alpha,\boldsymbol{w}_{E,\alpha,C}} = \frac{1}{1-\alpha}\left\{\left[1 - \frac{\underline{S}(\boldsymbol{w}_{E,\alpha,C})}{S} - \alpha\right] V_{\alpha,\boldsymbol{w}_{E,\alpha,C}} - \sum_{s\in\underline{\Omega}(\boldsymbol{w}_{E,\alpha,C})} \frac{\boldsymbol{w}_{E,\alpha,C}^\top \boldsymbol{R}_s}{S}\right\}. \tag{39}$$

Using the same arguments as in the proof of Theorem 1, $\boldsymbol{w}_*^\top \boldsymbol{R}_s$ is arbitrarily close to $\boldsymbol{w}_{E,\alpha,C}^\top \boldsymbol{R}_s$ for any $s \in \Omega$, $V_{\alpha,\boldsymbol{w}_*}$ is arbitrarily close to $V_{\alpha,\boldsymbol{w}_{E,\alpha,C}}$, and

$$\boldsymbol{w}_*^\top \boldsymbol{R}_{\underline{s}} < \boldsymbol{w}_*^\top \boldsymbol{R}_{\underline{\overline{s}}} < \boldsymbol{w}_*^\top \boldsymbol{R}_{\overline{s}} \ \forall(\underline{s}, \underline{\overline{s}}, \overline{s}) \in \underline{\Omega}(\boldsymbol{w}_{E,\alpha,C}) \times \underline{\overline{\Omega}}(\boldsymbol{w}_{E,\alpha,C}) \times \overline{\Omega}(\boldsymbol{w}_{E,\alpha,C}). \tag{40}$$

Since $V_{\alpha,\boldsymbol{w}_*}$ is arbitrarily close to $V_{\alpha,\boldsymbol{w}_{E,\alpha,C}}$, it follows from Eq. (40) that

$$\underline{\Omega}(\boldsymbol{w}_*) = [\underline{\Omega}(\boldsymbol{w}_*) \cap \overline{\underline{\Omega}}(\boldsymbol{w}_{E,\alpha,C})] \cup \underline{\Omega}(\boldsymbol{w}_{E,\alpha,C}). \tag{41}$$

Using Eqs. (7) and (41), we have

$$C_{\alpha,\boldsymbol{w}_*} = \frac{1}{1-\alpha}\left\{\left[1 - \frac{\underline{S}(\boldsymbol{w}_*)}{S} - \alpha\right] V_{\alpha,\boldsymbol{w}_*} - \sum_{s\in\underline{\Omega}(\boldsymbol{w}_*)\cap\overline{\underline{\Omega}}(\boldsymbol{w}_{E,\alpha,C})} \frac{\boldsymbol{w}_*^\top \boldsymbol{R}_s}{S} - \sum_{s\in\underline{\Omega}(\boldsymbol{w}_{E,\alpha,C})} \frac{\boldsymbol{w}_*^\top \boldsymbol{R}_s}{S}\right\}. \tag{42}$$

Since (a) $V_{\alpha,\boldsymbol{w}_*}$ is arbitrarily close to $V_{\alpha,\boldsymbol{w}_{E,\alpha,C}}$, (b) $-\boldsymbol{w}_*^\top \boldsymbol{R}_s$ is arbitrarily close to $V_{\alpha,\boldsymbol{w}_{E,\alpha,C}}$ for any $s \in\underline{\Omega}(\boldsymbol{w}_*)\cap\overline{\underline{\Omega}}(\boldsymbol{w}_{E,\alpha,C})$, (c) $\underline{S}(\boldsymbol{w}_*)$ is equal to the number of states in $\underline{\Omega}(\boldsymbol{w}_*)\cap\overline{\underline{\Omega}}(\boldsymbol{w}_{E,\alpha,C})$ plus $\underline{S}(\boldsymbol{w}_{E,\alpha,C})$, and (d) $\boldsymbol{w}_*^\top \boldsymbol{R}_s$ is arbitrarily close to $\boldsymbol{w}_{E,\alpha,C}^\top \boldsymbol{R}_s$ for any $s \in\underline{\Omega}(\boldsymbol{w}_{E,\alpha,C})$, Eqs. (39) and (42) imply that $C_{\alpha,\boldsymbol{w}_*}$ is arbitrarily close to $C_{\alpha,\boldsymbol{w}_{E,\alpha,C}}$. Thus, Eq. (38) holds since $C_{\alpha,\boldsymbol{w}_{E,\alpha,C}} < C$. Eqs. (36)–(38) contradict the fact that $\boldsymbol{w}_{E,\alpha,C}$ is on the CVaR-constrained boundary.

For brevity, let $V = V_{\alpha,\boldsymbol{w}_{E,\alpha,C}}$ and $\Phi = \underline{\Omega}(\boldsymbol{w}_{E,\alpha,C})$. Suppose that $P[\Phi] = 1-\alpha$. It follows from Eq. (35) that $\boldsymbol{w}_{E,\alpha,C}$ solves

$$\min_{\boldsymbol{w}\in\mathbb{R}^J} \frac{1}{2}\boldsymbol{w}^\top \boldsymbol{V}\boldsymbol{w} \tag{43}$$

$$s.t.\quad \boldsymbol{w}^\top \mathbf{1} = 1 \tag{44}$$

$$\boldsymbol{w}^\top \overline{\boldsymbol{R}} = E \tag{45}$$

$$\boldsymbol{w}^\top \boldsymbol{R}_\Phi = -C \tag{46}$$

$$\boldsymbol{w}^\top \boldsymbol{R}_s \le -V \quad \forall s \in \Phi \tag{47}$$

$$\boldsymbol{w}^\top \boldsymbol{R}_s \ge -V - \delta \quad \forall s \in \Phi^c, \tag{48}$$

where Φ^c is the complement of Φ and $\delta > 0$ is arbitrarily small. Using the definition of V, δ, and Φ, constraints (47) and (48) do not bind. First-order conditions for $\boldsymbol{w}_{E,\alpha,C}$ to solve problem (43) subject to constraints (44)–(46) are

$$\boldsymbol{V}\boldsymbol{w}_{E,\alpha,C} - \lambda_1 \mathbf{1} - \lambda_2 \overline{\boldsymbol{R}} - \lambda_3 \boldsymbol{R}_\Phi = 0 \tag{49}$$

$$\boldsymbol{w}_{E,\alpha,C}^\top \mathbf{1} = 1 \tag{50}$$

$$\boldsymbol{w}_{E,\alpha,C}^\top \overline{\boldsymbol{R}} = E \tag{51}$$

$$\boldsymbol{w}_{E,\alpha,C}^\top \boldsymbol{R}_\Phi = -C, \tag{52}$$

where λ_1, λ_2, and λ_3 are Lagrange multipliers associated with these constraints. Since $rank(\boldsymbol{V}) = J$, Eq. (49) implies that

$$\boldsymbol{w}_{E,\alpha,C} = \lambda_1(\boldsymbol{V}^{-1}\mathbf{1}) + \lambda_2(\boldsymbol{V}^{-1}\overline{\boldsymbol{R}}) + \lambda_3(\boldsymbol{V}^{-1}\boldsymbol{R}_\Phi). \tag{53}$$

Eq. (53) implies that Eq. (8) holds with $(\beta_1, \beta_2, \beta_3) = (\lambda_1 c, \lambda_2 a, \lambda_3 c_\Phi)$. We now find $\boldsymbol{L}_{E,C} \equiv [\lambda_1 \quad \lambda_2 \quad \lambda_3]^\top$. Using arguments similar to those in the proof of Theorem 1, we have $\boldsymbol{L}_{E,C} = \boldsymbol{N}_{E,C}^{-1}\boldsymbol{K}_{E,C}$, where $\boldsymbol{K}_{E,C} \equiv [1 \quad E \quad -C]^\top$, $\boldsymbol{M}_{E,C} \equiv \boldsymbol{N}_{E,C}^\top \boldsymbol{V}^{-1}\boldsymbol{N}_{E,C}$, and $\boldsymbol{N}_{E,C} \equiv [\mathbf{1} \quad \overline{\boldsymbol{R}} \quad \boldsymbol{R}_\Phi]$.

Suppose now that $P[\Phi] < 1-\alpha$. Let $F = [(1-\alpha)S/\underline{S}(\boldsymbol{w}_{E,\alpha,C})-1]V - C(1-\alpha)S/\underline{S}(\boldsymbol{w}_{E,\alpha,C})$. Note that $\boldsymbol{w}_{E,\alpha,C}$ solves

$$\min_{w\in\mathbb{R}^J} \frac{1}{2}\boldsymbol{w}^\top \boldsymbol{V}\boldsymbol{w} \tag{54}$$

$$s.t. \quad \boldsymbol{w}^\top \mathbf{1} = 1 \tag{55}$$

$$\boldsymbol{w}^\top \overline{\boldsymbol{R}} = E \tag{56}$$

$$\boldsymbol{w}^\top \boldsymbol{R}_\Phi = F \tag{57}$$

$$\boldsymbol{w}^\top \boldsymbol{R}_s = -V \quad \forall s \in \underline{\overline{\Omega}}(\boldsymbol{w}_{E,\alpha,C}) \tag{58}$$

$$\boldsymbol{w}^\top \boldsymbol{R}_s \leq -V \quad \forall s \in \Phi \tag{59}$$

$$\boldsymbol{w}^\top \boldsymbol{R}_s \geq -V \quad \forall s \in \Phi^c \backslash \underline{\overline{\Omega}}(\boldsymbol{w}_{E,\alpha,C}). \tag{60}$$

Using the definition of Φ and $\underline{\overline{\Omega}}(\boldsymbol{w}_{E,\alpha,C})$, constraints (59) and (60) do not bind. First-order conditions for $\boldsymbol{w}_{E,\alpha,C}$ to solve problem (54) subject to constraints (55)–(58) are

$$\boldsymbol{V}\boldsymbol{w}_{E,\alpha,C} - \lambda_1 \mathbf{1} - \lambda_2 \overline{\boldsymbol{R}} - \lambda_3 \boldsymbol{R}_\Phi - \sum_{k=4}^{K+3} \lambda_k \boldsymbol{R}_{s_k} = 0 \quad (61)$$

$$\boldsymbol{w}_{E,\alpha,C}^\top \mathbf{1} = 1 \quad (62)$$

$$\boldsymbol{w}_{E,\alpha,C}^\top \overline{\boldsymbol{R}} = E \quad (63)$$

$$\boldsymbol{w}_{E,\alpha,C}^\top \boldsymbol{R}_\Phi = F \quad (64)$$

$$\boldsymbol{w}_{E,\alpha,C}^\top \boldsymbol{R}_{s_k} = -V,\ k = 4, ..., K+3, \quad (65)$$

where $\lambda_1, ..., \lambda_{K+3}$ are Lagrange multipliers associated with these constraints, and $\{s_4, ..., s_{K+3}\} = \overline{\underline{\Omega}}(\boldsymbol{w}_{E,\alpha,C})$. Since $rank(\boldsymbol{V}) = J$, Eq. (61) implies that

$$\boldsymbol{w}_{E,\alpha,C} = \lambda_1(\boldsymbol{V}^{-1}\mathbf{1}) + \lambda_2(\boldsymbol{V}^{-1}\overline{\boldsymbol{R}}) + \lambda_3(\boldsymbol{V}^{-1}\boldsymbol{R}_\Phi) + \sum_{k=4}^{K+3} \lambda_k(\boldsymbol{V}^{-1}\boldsymbol{R}_{s_k}). \quad (66)$$

Eq. (66) implies that Eq. (9) holds with $(\beta_1, ..., \beta_{K+3}) = (\lambda_1 c, \lambda_2 a, \lambda_3 c_\Phi, \lambda_4 c_{s_4}, ..., \lambda_{K+3} c_{s_{K+3}})$. We now find $\boldsymbol{L}_{E,C} \equiv [\lambda_1 \quad \cdots \quad \lambda_{K+3}]^\top$. Using arguments similar to those in the proof of Theorem 1, we have $\boldsymbol{L}_{E,C} = \boldsymbol{N}_{E,C}^{-1}\boldsymbol{K}_{E,C}$, where $\boldsymbol{K}_{E,C} \equiv [1 \quad E \quad F \quad -V \quad \cdots \quad -V]^\top$, $\boldsymbol{M}_{E,C} \equiv \boldsymbol{N}_{E,C}^\top \boldsymbol{V}^{-1} \boldsymbol{N}_{E,C}$, and $\boldsymbol{N}_{E,C} \equiv [\mathbf{1} \quad \overline{\boldsymbol{R}} \quad \boldsymbol{R}_\Phi \quad \boldsymbol{R}_{s_4} \quad \cdots \quad \boldsymbol{R}_{s_{K+3}}]$. This completes the second part of our proof. ■

Proof of Theorem 3. Suppose that $\boldsymbol{w}_{E,\alpha,V}$ exists. First, assume $V_{\alpha,\boldsymbol{w}_E} \leq V$. The desired result follows from Eq. (10). This completes the first part of our proof.

Second, assume $V_{\alpha,\boldsymbol{w}_E} > V$. Using arguments similar to those in the proof of Theorem 1, we have Eq. (14). For any portfolio $\boldsymbol{w} \in \mathbb{R}^{J+1}$, let $\underline{\Omega}_f(\boldsymbol{w}) \equiv \{s \in \Omega : \overline{\boldsymbol{w}}^\top \boldsymbol{R}_s + (1 - \overline{\boldsymbol{w}}^\top \mathbf{1})R_f < -V_{\alpha,\boldsymbol{w}}\}$, $\overline{\underline{\Omega}}_f(\boldsymbol{w}) \equiv \{s \in \Omega : \overline{\boldsymbol{w}}^\top \boldsymbol{R}_s + (1 - \overline{\boldsymbol{w}}^\top \mathbf{1})R_f = -V_{\alpha,\boldsymbol{w}}\}$, and $\overline{\Omega}_f(\boldsymbol{w}) \equiv \{s \in \Omega : \overline{\boldsymbol{w}}^\top \boldsymbol{R}_s + (1 - \overline{\boldsymbol{w}}^\top \mathbf{1})R_f > -V_{\alpha,\boldsymbol{w}}\}$, where $\overline{\boldsymbol{w}}$ denotes the $J \times 1$ vector given by the first J entries of $\boldsymbol{w}$. It follows from Eq. (14) that $\overline{\boldsymbol{w}}_{E,\alpha,V}$ solves

$$\min_{w\in\mathbb{R}^J} \frac{1}{2}\boldsymbol{w}^\top \boldsymbol{V}\boldsymbol{w} \tag{67}$$

$$s.t. \quad \boldsymbol{w}^\top\overline{\boldsymbol{R}} + (1-\boldsymbol{w}^\top\mathbf{1})R_f = E \tag{68}$$

$$\boldsymbol{w}^\top\boldsymbol{R}_s + (1-\boldsymbol{w}^\top\mathbf{1})R_f = -V \quad \forall s \in \overline{\underline{\Omega}}_f(\boldsymbol{w}_{E,\alpha,V}) \tag{69}$$

$$\boldsymbol{w}^\top\boldsymbol{R}_s + (1-\boldsymbol{w}^\top\mathbf{1})R_f \le -V \quad \forall s \in \underline{\Omega}_f(\boldsymbol{w}_{E,\alpha,V}) \tag{70}$$

$$\boldsymbol{w}^\top\boldsymbol{R}_s + (1-\boldsymbol{w}^\top\mathbf{1})R_f \ge -V \quad \forall s \in \overline{\Omega}_f(\boldsymbol{w}_{E,\alpha,V}). \tag{71}$$

Using the definition of $\underline{\Omega}_f(\boldsymbol{w}_{E,\alpha,V})$ and $\overline{\Omega}_f(\boldsymbol{w}_{E,\alpha,V})$, constraints (70) and (71) do not bind. First-order conditions for $\overline{\boldsymbol{w}}_{E,\alpha,V}$ to solve problem (67) subject to constraints (68)–(69) are

$$\boldsymbol{V}\overline{\boldsymbol{w}}_{E,\alpha,V} - \lambda_2(\overline{\boldsymbol{R}} - \mathbf{1}R_f) - \sum_{k=3}^{K+2}\lambda_k(\boldsymbol{R}_{s_k} - \mathbf{1}R_f) = 0 \tag{72}$$

$$\overline{\boldsymbol{w}}_{E,\alpha,V}^\top(\overline{\boldsymbol{R}} - \mathbf{1}R_f) = E - R_f \tag{73}$$

$$\overline{\boldsymbol{w}}_{E,\alpha,V}^\top(\boldsymbol{R}_{s_k} - \mathbf{1}R_f) = -V - R_f, \; k = 3, ..., K+2, \tag{74}$$

where $\lambda_2, ..., \lambda_{K+2}$ are Lagrange multipliers associated with these constraints and $\{s_3, ..., s_{K+2}\} = \overline{\underline{\Omega}}_f(\boldsymbol{w}_{E,\alpha,V})$. Since $rank(\boldsymbol{V}) = J$, Eq. (72) implies that

$$\overline{\boldsymbol{w}}_{E,\alpha,V} = \lambda_2[\boldsymbol{V}^{-1}(\overline{\boldsymbol{R}} - \mathbf{1}R_f)] + \sum_{k=3}^{K+2}\lambda_k[\boldsymbol{V}^{-1}(\boldsymbol{R}_{s_k} - \mathbf{1}R_f)]. \tag{75}$$

Eq. (75) implies that Eq. (11) holds with $(\theta_1, ..., \theta_{K+2}) = (1-\theta_2- \cdots - \theta_{K+2}, \lambda_2(a - cR_f), \lambda_3(c_{s_3} - cR_f), ..., \lambda_{K+2}(c_{s_{K+2}} - cR_f))$. We now find $\boldsymbol{L}_{E,V} \equiv [\lambda_2 \quad \cdots \quad \lambda_{K+2}]^\top$. Using arguments similar to those in the proof of Theorem 1, we have $\boldsymbol{L}_{E,V} = \boldsymbol{N}_{E,V}^{-1}\boldsymbol{K}_{E,V}$, where $\boldsymbol{K}_{E,V} \equiv [E-R_f \quad -V-R_f \quad \cdots \quad -V-R_f]^\top$, $\boldsymbol{M}_{E,V} \equiv \boldsymbol{N}_{E,V}^\top \boldsymbol{V}^{-1}\boldsymbol{N}_{E,V}$, and $\boldsymbol{N}_{E,V} \equiv [\overline{\boldsymbol{R}}-\mathbf{1}R_f \quad \boldsymbol{R}_{s_3}-\mathbf{1}R_f \quad \cdots \quad \boldsymbol{R}_{s_{K+2}}-\mathbf{1}R_f]$. This completes the second part of our proof. ■

Proof of Theorem 4. Suppose that $\boldsymbol{w}_{E,\alpha,C}$ exists. First, assume $C_{\alpha,\boldsymbol{w}_E} \le C$. The desired result follows from Eq. (10). This completes the first part of our proof.

Second, assume $C_{\alpha,\boldsymbol{w}_E} > C$. Using arguments similar to those in the proof of Theorem 2, we have Eq. (35). For brevity, let $V = V_{\alpha,\boldsymbol{w}_{E,\alpha,C}}$ and $\Phi = \underline{\Omega}_f(\boldsymbol{w}_{E,\alpha,C})$. Suppose that $P[\Phi] = 1 - \alpha$. It follows from Eq. (35) that $\overline{\boldsymbol{w}}_{E,\alpha,C}$ solves

$$\min_{\boldsymbol{w}\in\mathbb{R}^J} \frac{1}{2}\boldsymbol{w}^\top \boldsymbol{V}\boldsymbol{w} \tag{76}$$

$$s.t. \quad \boldsymbol{w}^\top\overline{\boldsymbol{R}} + (1 - \boldsymbol{w}^\top\mathbf{1})R_f = E \tag{77}$$

$$\boldsymbol{w}^\top\boldsymbol{R}_\Phi + (1 - \boldsymbol{w}^\top\mathbf{1})R_f = -C \tag{78}$$

$$\boldsymbol{w}^\top\boldsymbol{R}_s + (1 - \boldsymbol{w}^\top\mathbf{1})R_f \leq -V \quad \forall s \in \Phi \tag{79}$$

$$\boldsymbol{w}^\top\boldsymbol{R}_s + (1 - \boldsymbol{w}^\top\mathbf{1})R_f \geq -V - \delta \quad \forall s \in \Phi^c, \tag{80}$$

where $\delta > 0$ is arbitrarily small. Using the definition of V, δ, and Φ, constraints (79) and (80) do not bind. First-order conditions for $\overline{\boldsymbol{w}}_{E,\alpha,C}$ to solve problem (76) subject to constraints (77) and (78) are

$$\boldsymbol{V}\overline{\boldsymbol{w}}_{E,\alpha,C} - \lambda_2(\overline{\boldsymbol{R}} - \mathbf{1}R_f) - \lambda_3(\boldsymbol{R}_\Phi - \mathbf{1}R_f) = 0 \tag{81}$$

$$\overline{\boldsymbol{w}}^\top_{E,\alpha,C}(\overline{\boldsymbol{R}} - \mathbf{1}R_f) = E - R_f \tag{82}$$

$$\overline{\boldsymbol{w}}^\top_{E,\alpha,C}(\boldsymbol{R}_\Phi - \mathbf{1}R_f) = -C - R_f, \tag{83}$$

where λ_2 and λ_3 are Lagrange multipliers associated with these constraints. Since $rank(\boldsymbol{V}) = J$, Eq. (81) implies that

$$\overline{\boldsymbol{w}}_{E,\alpha,C} = \lambda_2[\boldsymbol{V}^{-1}(\overline{\boldsymbol{R}} - \mathbf{1}R_f)] + \lambda_3[\boldsymbol{V}^{-1}(\boldsymbol{R}_\Phi - \mathbf{1}R_f)]. \tag{84}$$

Eq. (84) implies that Eq. (12) holds with $(\theta_1, \theta_2, \theta_3) = (1-\theta_2-\theta_3, \lambda_2(a-cR_f), \lambda_3(c_\Phi - cR_f))$. We now find $\boldsymbol{L}_{E,C} \equiv [\lambda_2 \quad \lambda_3]^\top$. Using arguments similar to those in the proof of Theorem 1, we have $\boldsymbol{L}_{E,C} = \boldsymbol{N}^{-1}_{E,C}\boldsymbol{K}_{E,C}$, where $\boldsymbol{K}_{E,C} \equiv [E - R_f \qquad -C - R_f]^\top$, $\boldsymbol{M}_{E,C} \equiv \boldsymbol{N}^\top_{E,C}\boldsymbol{V}^{-1}\boldsymbol{N}_{E,C}$, and $\boldsymbol{N}_{E,C} \equiv [\overline{\boldsymbol{R}} - \mathbf{1}R_f \quad \boldsymbol{R}_\Phi - \mathbf{1}R_f]$.

Suppose now that $P[\Phi] < 1-\alpha$. Let $F = [(1-\alpha)S/\underline{S}(\boldsymbol{w}_{E,\alpha,C})-1]V - C(1-\alpha)S/\underline{S}(\boldsymbol{w}_{E,\alpha,C})$. Note that $\overline{\boldsymbol{w}}_{E,\alpha,C}$ solves

$$\min_{w\in\mathbb{R}^J} \frac{1}{2}\boldsymbol{w}^\top \boldsymbol{V}\boldsymbol{w} \tag{85}$$

$$s.t. \quad \boldsymbol{w}^\top\overline{\boldsymbol{R}} + (1-\boldsymbol{w}^\top\mathbf{1})R_f = E \tag{86}$$

$$\boldsymbol{w}^\top\boldsymbol{R}_\Phi + (1-\boldsymbol{w}^\top\mathbf{1})R_f = F \tag{87}$$

$$\boldsymbol{w}^\top\boldsymbol{R}_s + (1-\boldsymbol{w}^\top\mathbf{1})R_f = -V \quad \forall s \in \underline{\Omega}_f(\boldsymbol{w}_{E,\alpha,C}) \tag{88}$$

$$\boldsymbol{w}^\top\boldsymbol{R}_s + (1-\boldsymbol{w}^\top\mathbf{1})R_f \leq -V \quad \forall s \in \Phi \tag{89}$$

$$\boldsymbol{w}^\top\boldsymbol{R}_s + (1-\boldsymbol{w}^\top\mathbf{1})R_f \geq -V \quad \forall s \in \Phi^c\backslash\underline{\Omega}_f(\boldsymbol{w}_{E,\alpha,C}). \tag{90}$$

Using the definition of Φ and $\underline{\Omega}_f(\boldsymbol{w}_{E,\alpha,C})$, constraints (89) and (90) do not bind. First-order conditions for $\overline{\boldsymbol{w}}_{E,\alpha,C}$ to solve problem (85) subject to constraints (86)–(88) are

$$\boldsymbol{V}\overline{\boldsymbol{w}}_{E,\alpha,C} - \lambda_2(\overline{\boldsymbol{R}} - \mathbf{1}R_f) - \lambda_3(\boldsymbol{R}_\Phi - \mathbf{1}R_f) - \sum_{k=4}^{K+3}\lambda_k(\boldsymbol{R}_{s_k} - \mathbf{1}R_f) = 0 \tag{91}$$

$$\overline{\boldsymbol{w}}_{E,\alpha,C}^\top(\overline{\boldsymbol{R}} - \mathbf{1}R_f) = E - R_f \tag{92}$$

$$\overline{\boldsymbol{w}}_{E,\alpha,C}^\top(\boldsymbol{R}_\Phi - \mathbf{1}R_f) = F - R_f \tag{93}$$

$$\overline{\boldsymbol{w}}_{E,\alpha,C}^\top(\boldsymbol{R}_{s_k} - \mathbf{1}R_f) = -V - R_f, \ k = 4, ..., K+3, \tag{94}$$

where $\lambda_1, ..., \lambda_{K+3}$ are Lagrange multipliers associated with these constraints, and $\{s_4, ..., s_{K+3}\} = \underline{\Omega}_f(\boldsymbol{w}_{E,\alpha,C})$. Since $rank(\boldsymbol{V}) = J$, Eq. (91) implies that

$$\overline{\boldsymbol{w}}_{E,\alpha,C} = \lambda_2[\boldsymbol{V}^{-1}(\overline{\boldsymbol{R}} - \mathbf{1}R_f)] + \lambda_3[\boldsymbol{V}^{-1}(\boldsymbol{R}_\Phi - \mathbf{1}R_f)] + \sum_{k=4}^{K+3}\lambda_k[\boldsymbol{V}^{-1}(\boldsymbol{R}_{s_k} - \mathbf{1}R_f)]. \tag{95}$$

Eq. (95) implies that Eq. (13) holds with $(\theta_1, ..., \theta_{K+3}) = (1 - \theta_2 - \cdots - \theta_{K+3}, \lambda_2(a - cR_f), \lambda_3(c_\Phi - cR_f), \lambda_4(c_{s_4} - cR_f), ..., \lambda_{K+3}(c_{s_{K+3}} - cR_f))$. We now find $\boldsymbol{L}_{E,C} \equiv [\lambda_2 \ \cdots \ \lambda_{K+3}]^\top$. Using arguments similar to those in the proof of Theorem 1, we have $\boldsymbol{L}_{E,C} = \boldsymbol{N}_{E,C}^{-1}\boldsymbol{K}_{E,C}$, where $\boldsymbol{K}_{E,C} \equiv [E - R_f \quad F - R_f \quad -V - R_f \quad \cdots \quad -V - R_f]^\top$, $\boldsymbol{M}_{E,C} \equiv \boldsymbol{N}_{E,C}^\top\boldsymbol{V}^{-1}\boldsymbol{N}_{E,C}$, and

$\boldsymbol{N}_{E,C} \equiv [\overline{\boldsymbol{R}}-\mathbf{1}R_f \quad \boldsymbol{R}_\Phi-\mathbf{1}R_f \quad \boldsymbol{R}_{s_4}-\mathbf{1}R_f \quad \cdots \quad \boldsymbol{R}_{s_{K+3}}-\mathbf{1}R_f]$. This completes the second part of our proof. ■

2. Numerical Procedure

Suppose that there is no riskfree security. In deriving the portfolios on the VaR-constrained boundary, we follow an optimization routine written in Matlab using the 'fmincon' function, which allows non-linear constraints. Given an initial guess for a solution to the variance minimization problem subject to expected return and VaR constraints, the use of this function may not lead to a global solution to the problem. This occurs when either (i) there exists no solution to the problem since the VaR constraint is too tight, or (ii) there exists a solution to the problem, but the use of the 'fmincon' function leads to a vector that does not satisfy all of the constraints in the problem, or (iii) its use leads to a local solution to the problem.[1] In order to arrive at a global solution to the problem, we use several starting values: (1) when available, a portfolio on the VaR-constrained boundary with an expected return smaller than but close to that of the portfolio that we are solving for, (2) the equally-weighted portfolio, (3) the portfolios with a weight of 100% in one of the securities, and (4) twenty randomly generated portfolios assuming either a normal or uniform distribution for security weights.[2] This part of the routine is fast and stable in the range of expected returns for which there are portfolios on the VaR-constrained boundary.

Our characterization of the VaR-constrained boundary is useful in the second part of our routine. Using Theorem 1, a portfolio on the VaR-constrained boundary requires $K+2$ funds,

[1] Krokhmal, Palmquist, and Uryasev (2002) recognize difficulties in optimization problems involving VaR when security returns have discrete distributions. In particular, they note the non-convexity of VaR and the possibility of existence of multiple local solutions to these problems. As Rockafellar and Uryasev (2000) note, one of the advantages of CVaR over VaR is that the former is convex.

[2] These portfolios are obtained as follows. A 10×1 vector is randomly generated by assuming that all of its entries have either (1) a standard normal distribution, or (2) a uniform distribution on the interval $[-0.5, 0.5]$. This vector is then normalized so that its entries sum to one (by dividing them by the sum of all entries). Each of these two distributional assumptions is used to generate ten portfolios. Our results remain unchanged when using 100 randomly generated portfolios (each of the two distributional assumptions is used to generate 50 portfolios).

where K is the number of states for which its return is $-V$. However, due to small numerical errors, the routine described above may lead to a portfolio for which the number of states where its return is $-V$ differs from K. In order to find the exact value of K, we focus on the set of states where this portfolio has 'a return close to $-V$,' denoted by $\Omega_{\approx V}$.[3] Suppose that $\Omega_{\approx V}$ contains $\overline{K}$ states. For each subset of states $\{s_3, ..., s_{\widehat{K}+2}\} \subset \Omega_{\approx V}$ where $1 \leq \widehat{K} \leq \overline{K}$, we use Eq. (6) to a obtain a portfolio. Next, we verify whether this portfolio satisfies the VaR constraint. Of the portfolios that satisfy the constraint, we choose the one with the smallest variance. Note that K is given by the number of states contained in the subset of states used to obtain this portfolio. This part of the routine is fast since $\widehat{K}$ is typically small. The procedure utilized to derive portfolios on the VaR-constrained boundary in the presence of a riskfree security is similar with the exception that Eq. (11) is used to find K.[4]

The procedure utilized to derive portfolios on the CVaR-constrained boundary is similar with two exceptions. First, the appropriate changes are made to set up the CVaR constraint using the 'fmincon' function. Second, Eqs. (9) and (13) are now used to find K in the cases of, respectively, absence and presence of a riskfree security.[5]

[3] The precise meaning of 'a return close to $-V$' is 'a return that belongs to the interval $(-V - 0.0005, -V + 0.0005)$.'
[4] We also utilized an alternative approach that can be used when K is relatively small. In this approach, we initially use Eq. (6) to find the portfolios associated with each subset of K states. Of these portfolios, we then find the set of those that satisfy the VaR constraint. Finally, we pick the portfolio on this set with the smallest variance, which is on the VaR-constrained boundary. Our results remain unchanged if this approach is utilized when either $K = 3$ or $K = 4$.
[5] We also utilized an alternative approach to find the portfolios on the CVaR-constrained boundary that follows a representation of the CVaR constraint provided by Krokhmal, Palmquist, and Uryasev (2002). Their representation uses dummy variables to capture the states for which portfolio losses are larger than the portfolio's VaR. Using this representation, optimization problems involving a CVaR constraint are convex and thus easier to solve. Our results remain unchanged when this approach is utilized.