

## ON THE EXISTENCE OF MARKOV PARTITIONS FOR $\mathbb{Z}^d$ ACTIONS

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### 1. Introduction

The theory of higher-dimensional shifts of finite type is still largely an open area of investigation. Recent years have seen much activity, but fundamental questions remain unanswered. In this paper we consider the following basic question. Given a shift of finite type (SFT), under what topological mixing conditions are we guaranteed the existence of Bernoulli (or even  $K$ , mixing, or weakly mixing) invariant measures?

This question has been answered for shifts satisfying strong mixing conditions such as the uniform filling property (UFP) or strong irreducibility. In particular, the authors have shown [4] that a shift of finite type satisfying the uniform filling property has mixing and  $K$  invariant measures. Moreover these shifts of finite type enjoy a strong variational principle where the supremum can be taken over only mixing measures, or even only  $K$  measures. If in addition to having the uniform filling property a shift of finite type has dense periodic points, then the supremum can be taken over only Bernoulli measures [5]. The same result holds under the stronger mixing assumption of strong irreducibility without any assumptions on the periodic points [1]. Such examples, and the question we pose in this paper, are of interest because, in contrast to the one-dimensional case, it is not necessarily true that the measure of maximal entropy for a higher-dimensional topologically mixing shift of finite type will even be weak mixing [2].

In this paper we study shifts of finite type that have what we call a uniform filling set  $Z$ . Informally,  $Z$  is a subset of the shift of finite type that does not consist of a single fixed point, with the property that for any given pair of points  $z, z' \in Z$  we can paste an arbitrary block of  $z$  into a hole in  $z'$ , provided that we leave a filling collar of width  $\ell$ , where  $\ell$  is constant for all points in  $Z$ . A shift finite type with the uniform filling property is precisely the case where the entire shift of finite type is a uniform filling set. It is worth noting that the property of having a uniform filling set is not only weaker than the uniform filling property, but it is also weaker than the shift of finite type having a subshift with the uniform filling property. In particular, once we interpolate between two points  $z, z' \in Z$ , we do not expect the new point  $y$  to belong to  $Z$ , but just to the original shift of finite type.

As we will show, many interesting and well studied shifts that do not have the uniform filling property do have a uniform filling set. These include planar domino tilings, square ice, the lozenges, the  $n$ -color chessboards, and Rudolph tilings. We discuss these examples in detail below.

The main result of the paper is the following.

**THEOREM 1.1.** *Let  $(X, \mu, T)$  be a free ergodic measure-preserving  $\mathbb{Z}^d$  dynamical system, and let  $(Y, S)$  be a  $\mathbb{Z}^d$  shift of finite type with a uniform filling set. Then there is an  $S$ -invariant Borel probability measure  $\nu$  on  $Y$  such that  $(Y, \nu, S)$  is a non-atomic factor of  $(X, \mu, T)$ .*

An ergodic property is a property of a free ergodic measure-preserving  $\mathbb{Z}^d$  dynamical system  $(X, \mu, T)$  that is invariant under metric isomorphism. An ergodic property is *factorial* if it is inherited by factors. Some examples of factorial ergodic properties are the Bernoulli property, the  $K$ -property, the weak mixing property, the pure discrete spectrum property, and the property of having entropy zero. We have the following immediate corollary to Theorem 1.1.

**COROLLARY 1.2.** *If a  $\mathbb{Z}^d$  shift of finite type  $(Y, S)$  has a uniform filling set, then, given any factorial ergodic property, there is an invariant measure  $\nu$  on  $Y$  so that  $(Y, \nu, S)$  has this property. In particular,  $(Y, S)$  has a Bernoulli invariant measure.*

Theorem 1.1 is obtained by proving a tiling result. Let  $A$  be the alphabet of the shift of finite type  $Y$ . We show that, given any ergodic measure-preserving  $\mathbb{Z}^d$  action  $(X, \mu, T)$ , one can find a measurable partition  $\phi : X \rightarrow A$  so that for  $\mu$ -a.e.  $x \in X$ , there exists a point  $y_x \in Y$  such that  $y_x[\vec{v}] = \phi(T^{\vec{v}}x)$ . In particular, we tile almost every orbit of  $(X, \mu, T)$  with configurations from  $(Y, S)$ . The map  $\phi$  is the desired factor map. We call the corresponding partition of  $X$  a *Markov partition of type  $(Y, S)$*  for  $(X, \mu, T)$  (see [5]).

**COROLLARY 1.3.** *Let  $(Y, S)$  be a  $\mathbb{Z}^d$  shift of finite type which has a uniform filling set. Then any free ergodic measure-preserving  $\mathbb{Z}^d$  dynamical system  $(X, \mu, T)$  has a type  $(Y, S)$  Markov partition  $\phi$ .*

A natural question to ask is whether the factor map  $\phi$  can be made a metric isomorphism. The authors previously showed [5, Theorem 1.1] that it can, provided that  $(Y, S)$  has the uniform filling property and dense periodic points and if  $h_\mu(T) < h(Y)$ . It should be noted that Theorem 1.1 of this paper is used as a key lemma in the proof of [5, Theorem 1.1] (see [5, immediately following Lemma 4.8]). However, the general question remains open as other parts of the proof in [5] depend heavily on the uniform filling property in a way that does not seem to extend to the case of uniform filling sets.

The organization of the paper is as follows. In Section 2 we introduce the basic notation used in the paper. In Subsection 2.1 we recall the basic definitions relating to shifts of finite type. Section 3 contains the definition of a uniform filling set and a discussion of some of the properties of a shift of finite type with a uniform filling set. In Section 4 we discuss in detail the specific examples of shifts of finite type mentioned above. Finally, in Section 5 we prove Theorem 1.1.

## 2. Basic notation

For a vector  $\vec{v} \in \mathbb{R}^d$  let  $\|\vec{v}\| = \max |v_i|$ . We say that  $\vec{v} < \vec{w}$  if  $v_i < w_i$  for  $i = 1, \dots, d$ . If  $v_i > w_i$  for all  $i$ , then we write  $\vec{v} > \vec{w}$ . For  $\vec{v} < \vec{w}$ , we define the box

$$[\vec{v}, \vec{w}] = \prod_{j=1}^d [v_j, w_j],$$

with similar notation for (half) open boxes. We write  $B_n = [\vec{0}, \vec{n}]$  for  $\vec{n} = (n, n, \dots, n)$ ,  $n \geq 0$ . For a real number  $r$ , let  $[\vec{v} - r, \vec{w} + r] = \prod_{j=1}^d [v_j - r, w_j + r]$ . When  $r > 0$ , this is the box  $[\vec{v}, \vec{w}]$  together with a *collar* of thickness  $r$ . When  $r < 0$ , note that  $[\vec{v} - r, \vec{w} + r] \subseteq [\vec{v}, \vec{w}]$ .

For  $\ell \in \mathbb{N}$  and a box  $[\vec{v}, \vec{w}]$ , we set

$$\partial^\ell[\vec{v}, \vec{w}] = [\vec{v} - \ell, \vec{w} + \ell] \setminus [\vec{v}, \vec{w}].$$

This is an *outer* collar of width  $\ell$  around  $[\vec{v}, \vec{w}]$ . By  $\partial^{-\ell}[\vec{v}, \vec{w}]$ , we denote a collar of width  $\ell$  *inside*  $[\vec{v}, \vec{w}]$ .

Let  $(X, \mu, T)$  be a measure-preserving  $\mathbb{Z}^d$  action. A *Rohlin tower* with *shape*  $R \subseteq \mathbb{Z}^d$  and *base*  $F \subseteq X$  is a disjoint union  $T^R F = \bigcup_{\vec{v} \in R} T^{\vec{v}} F$ . If  $\mu(T^R F) > 1 - \delta$ , then the tower has *error*  $< \delta$ . For  $E \subseteq F$ , we call  $T^R E$  a *slice* of the tower  $T^R F$ . We call  $T^R \{x\}$ ,  $x \in X$  the *slice based at*  $x$ . The *Rohlin lemma* says that if  $(X, \mu, T)$  is aperiodic and ergodic, then, for any  $\delta > 0$  and for any  $n \in \mathbb{N}$ , there is a Rohlin tower with shape  $B_n$  and error less than  $\delta$  [3].

### 2.1. Shifts of finite type

Let  $A$  be a finite set (the *alphabet*) and consider  $A^{\mathbb{Z}^d}$ , which is a compact metric space in the product topology. For  $\vec{w} \in \mathbb{Z}^d$ , we denote the  $\vec{w}$ th entry of  $y \in A^{\mathbb{Z}^d}$  by  $y[\vec{w}]$ . Let  $S$  be the  $\mathbb{Z}^d$  *shift* on  $A^{\mathbb{Z}^d}$ , defined by  $(S^{\vec{v}} y)[\vec{w}] = y[\vec{w} + \vec{v}]$  for  $\vec{v}, \vec{w} \in \mathbb{Z}^d$ . A  $\mathbb{Z}^d$  *subshift*  $(Y, S)$  is the restriction of  $S$  to a closed  $S$ -invariant subspace  $Y \subseteq A^{\mathbb{Z}^d}$ .

A *shift of finite type* is a subshift  $(Y, S)$  consisting of those elements of  $(A^{\mathbb{Z}^d}, S)$  which omit a given finite collection of finite blocks. To make this precise we need a little terminology. If  $R \subseteq \mathbb{Z}^d$ , then we call  $\mathbf{b} \in A^R$  a *block* with *shape*  $R$ . A block is *finite* if  $R$  is finite. The block obtained by restricting  $y \in A^{\mathbb{Z}^d}$  to  $R$  is denoted  $y[R]$ . We denote the complement of a shape  $R$  by  $R^c$ .

If  $\mathcal{F} = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$  is a finite collection of finite blocks with shapes  $R_1, \dots, R_n$ , then we define the shift of finite type  $Y_{\mathcal{F}} = \{y \in A^{\mathbb{Z}^d} : (S^{\vec{v}} y)[R_j] \neq \mathbf{f}_j \text{ for any } \mathbf{f}_j \in \mathcal{F} \text{ and } \vec{v} \in \mathbb{Z}^d\}$ . We refer to  $\mathcal{F}$  as the set of *forbidden blocks*. Given  $\mathcal{F}$ , let  $m = \max_j \{\text{diam}(R_j)\}$  with respect to  $\|\cdot\|$ . Let  $s = (m + 1)/2$  if  $m$  is odd and  $s = m/2$  if  $m$  is even. We call  $s$  the *step size* of  $Y = Y_{\mathcal{F}}$ . Without loss of generality, we can assume that every  $\mathbf{f} \in \mathcal{F}$  satisfies  $\mathbf{f} \in A^{B_s}$ . We say a block  $\mathbf{b}$  of shape  $R$  is *extendable* if  $\mathbf{b} = y[R]$  for some  $y \in Y$ .

### 3. The uniform filling set

The main property that we shall consider in this paper is the following.

**DEFINITION 3.1.** We say that a shift of finite type  $(Y, S)$  has a *uniform filling set* if there exists a translation invariant subset  $Z$  of  $Y$  with  $|Z| > 1$  and an  $\ell \in \mathbb{N}$  so that, given any  $z, z' \in Z$  and any box  $[\vec{v}, \vec{w}]$ , there exists  $y \in Y$  such that

$$y[[\vec{v}, \vec{w}]] = z[[\vec{v}, \vec{w}]] \quad \text{and} \quad y[[\vec{v} - \ell, \vec{w} + \ell]^c] = z'[[\vec{v} - \ell, \vec{w} + \ell]^c].$$

We call  $\ell$  the *filling length* of the set  $Z$ .

The set  $\partial^\ell[\vec{v}, \vec{w}]$  is called the *filling collar*.

**DEFINITION 3.2.** A shift of finite type  $(Y, S)$  has the *uniform filling property* if  $(Y, S)$  itself is a uniform filling set.

The following results on shifts of finite type with uniform filling sets are immediate.

LEMMA 3.3. *If  $(Y, S)$  has a uniform filling set, then  $(Y, S)$  has positive topological entropy.*

LEMMA 3.4. *Let  $X$  be a shift of finite type with a uniform filling set  $Z$  and filling length  $\ell$ . Suppose that  $Y$  is a shift of finite type and  $\phi : X \rightarrow Y$  is a factor map with  $|\phi(Z)| > 1$ . Then  $\phi(Z)$  is a uniform filling set for  $Y$ .*

A key property of a shift of finite type  $(Y, S)$  is that it is possible to paste together parts of different points of  $(Y, S)$  if they have sufficient overlap. For shifts of finite type with a uniform filling set, this property allows us to glue blocks from any point from the shift into a point from the uniform filling set  $Z$  as long as the block we have chosen has sufficient overlap with a point from  $Z$ . The next result is a formal statement of this fact.

LEMMA 3.5. *Suppose that  $(Y, S)$  is a shift of finite type with step size  $s$  and filling set  $Z$  with filling length  $\ell$ . Let  $\mathcal{F}$  denote the forbidden blocks defining  $Y$ . Fix a box  $[\vec{v}, \vec{w}]$ . Suppose that  $y \in Y$  is such that there exists  $z \in Z$  with*

$$y[\partial^{-s} [[\vec{v}, \vec{w}]]] = z[\partial^{-s} [\vec{v}, \vec{w}]].$$

Then, given  $z' \in Z$ , we can find a point  $y^* \in Y$  such that

$$y^* [[\vec{v}, \vec{w}]] = y [[\vec{v}, \vec{w}]] \quad \text{and} \quad y^* [[\vec{v} - \ell, \vec{w} + \ell]^c] = z' [[\vec{v} - \ell, \vec{w} + \ell]^c]. \tag{3.1}$$

*Proof.* Since  $z, z' \in Z$ , we know there exists a point  $y'' \in Y$  with the property

$$y'' [[\vec{v}, \vec{w}]] = z [[\vec{v}, \vec{w}]] \quad \text{and} \quad y'' [[\vec{v} - \ell, \vec{w} + \ell]^c] = z' [[\vec{v} - \ell, \vec{w} + \ell]^c].$$

We define  $y^*$  as follows.

$$y^* [\vec{v}] = \begin{cases} y[\vec{v}] & \vec{v} \in [\vec{v}, \vec{w}] \\ y''[\vec{v}] & \vec{v} \in \partial^\ell [\vec{v}, \vec{w}] \\ z'[\vec{v}] & \text{otherwise.} \end{cases}$$

To see that  $y^* \in Y$ , we note that for all  $\vec{v} \in \mathbb{Z}^d$  the block  $y^*[B_s + \vec{v}]$  is entirely contained in  $y, z'$  or  $y''$ , and thus cannot be an element of  $\mathcal{F}$ . □

#### 4. Examples

In this section we consider specific examples of shifts of finite type that have a uniform filling set but do not have the uniform filling property. We begin with the following.

DEFINITION 4.1. Let  $(Y, S)$  be a  $\mathbb{Z}^d$  shift of finite type. A point  $z^f \in Y$  is *frozen* if, for any  $z \in Y$  that differs from  $z^f$  on at most finitely many coordinates,  $z = z^f$ .

Informally, the only way way to fill in a finite hole in  $z^f$  is to fill it back to  $z^f$  itself. Clearly, if a shift of finite type  $(Y, S)$  has a frozen point  $z^f$ , then it does not have the uniform filling property.

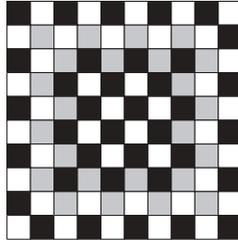


FIGURE 1. The point constructed in the proof of Proposition 4.2 where a block from  $z$  is interpolated into the ambient configuration from  $z'$  with a 3-color filling collar of width  $\ell = 2$ .

#### 4.1. The chessboard systems

Let  $A = \{0, 1, 2\}$ . Define  $X^3 \subseteq A^{\mathbb{Z}^2}$  by forbidding any symbol to occur horizontally or vertically adjacent to itself.  $X^3$  is called the *infinite three color chessboard shift*.

PROPOSITION 4.2. *The three color chessboard shift  $(X^3, S)$  does not have the uniform filling property but it does have a uniform filling set.*

*Proof.* First we observe that the periodic point  $z^f \in X^3$ , defined by  $z^f[(i, j)] = |i| + |j| \pmod{3}$ , is frozen. Thus  $(X^3, S)$  does not have the uniform filling property.

To see that  $(X^3, S)$  has a uniform filling set, we show that the set  $Z$  consisting of the two element orbit of a periodic point with any two colors, say 0 and 1, satisfies Definition 3.1 with  $\ell = 2$ .

Given  $z$  and  $z'$  in  $Z$ , and a box  $R = [\vec{v}, \vec{w}]$ , we note that there are two possibilities: either  $z[R] = z'[R]$ , or  $z[R] = S^{\vec{e}_1}(z')[R]$ . In the former case, no interpolation is necessary. In the latter case, assume that  $z$  satisfies  $z[(i, j)] = |i| + |j| \pmod{2}$ , so that  $z'$  then satisfies  $z'[(i, j)] = |i| + |j| + 1 \pmod{2}$ .

We use two layers around  $R$  and the remaining color to switch the parity of the colors from  $z$  to  $z'$ . In particular, we define  $y \in X^3$  by first setting  $y[R] = z[R]$  and  $y[[\vec{v} - 2, \vec{w} + 2]^c] = z'[[\vec{v} - 2, \vec{w} + 2]^c]$ . Then for  $(i, j) \in \partial^1(R)$  we set  $y[(i, j)] = 2(|i| + |j| \pmod{2})$ , which guarantees that adjacent locations in  $\partial^1(R)$  and  $R$  have been assigned different colors. Now for  $(i, j) \in \partial^2(R) \setminus \partial^1(R)$  we set  $y[(i, j)] = 1 + |i| + |j| \pmod{2}$ . The assignments in  $\partial^2(R)$  clearly satisfy the chessboard rule, and moreover the locations of color 1 are compatible with its occurrence in  $z'$  (see Figure 1).  $\square$

By starting with the alphabet  $\{0, 1, \dots, n - 1\}$ , and using the same adjacency rules, one can also define the *infinite  $n$ -color chessboard shift*  $(X^n, S)$ .

COROLLARY 4.3. *The  $n$ -color chessboard shift of finite type  $(X^n, S)$  has a uniform filling set for all  $n \geq 3$ .*

*Proof.* Since  $X^n \subseteq X^m$  for all  $m > n$ , the result then follows immediately from Proposition 4.2.  $\square$

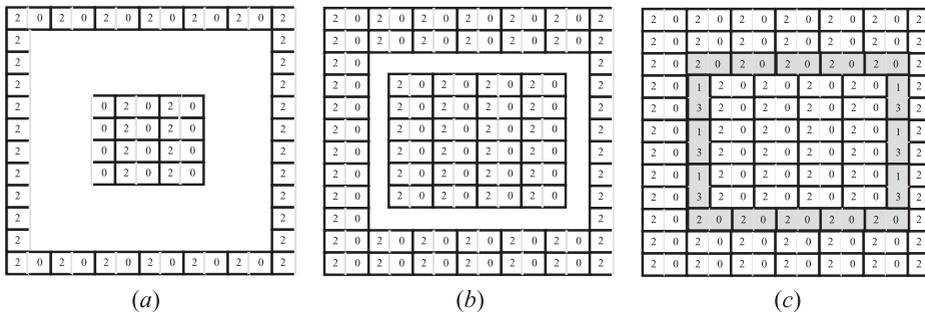


FIGURE 2. The filling operation for dominos. (a) The filling collar of width  $\ell = 3$ . (b) Filled to make whole dominos, to reduce the width to  $\ell = 1$ , and to extend the inner configuration to even height. (c) The final tiling corresponding to  $x \in X^D$ .

4.2. Dominos

Consider all the tilings of the plane  $\mathbb{R}^2$  by  $1 \times 2$  and  $2 \times 1$  rectangles (called dominos), such that all the vertices lie on the integer lattice  $\mathbb{Z}^2$ . Such a domino tiling can be coded into a subshift  $X^D$  of  $\{0, 1, 2, 3\}^{\mathbb{Z}^2}$  as follows. Label the left and right halves of horizontal dominos 0 and 2, and label the top and bottom halves of the vertical dominos 1 and 3. Given a domino tiling, define  $z \in \{0, 1, 2, 3\}^{\mathbb{Z}^2}$  so that  $z[\vec{v}]$  is the symbol on the square with lower left vertex  $\vec{v}$ . The domino shift of finite type  $X^D \subseteq \{0, 1, 2, 3\}^{\mathbb{Z}^2}$  is defined by defining a set of forbidden blocks that insure left and right, and top and bottom, half dominos are always paired correctly. Clearly each  $z \in X^D$  corresponds to a unique domino tiling [9] and vice versa.

The shift of finite type  $(X^D, S)$  has been very well studied in a variety of contexts. In particular, Burton and Steif showed that the measure of maximal entropy is unique and Bernoulli [2]. We include a proof that it has a uniform filling set, as this it is an excellent warm-up for the remaining examples that we consider.

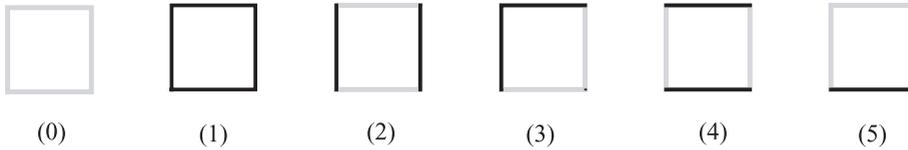
PROPOSITION 4.4. *The domino shift of finite type  $X^D$  does not have the uniform filling property, but it does have a uniform filling set.*

*Proof.* Since, as is easy to verify, the periodic point  $z^f[(i, j)] = 2(|i| + |j| \pmod 2)$  is frozen,  $X^D$  does not have the uniform filling property.

Now we show that the set  $Z$  consisting of the point  $z[(i, j)] = 2(|i| \pmod 2)$  and its shift  $z' = S_{\vec{e}_1} z$  is a filling set for  $X^D$  with filling length  $\ell = 3$ .

We need to prove the existence of a point  $x \in X^D$  satisfying  $x[\vec{v}, \vec{w}] = z'[\vec{v}, \vec{w}] = \mathbf{a}$  and  $x[\vec{v} - \ell, \vec{w} + \ell]^c = z[\vec{v} - \ell, \vec{w} + \ell]^c = \mathbf{b}$  for any box  $[\vec{v}, \vec{w}] \subset \mathbb{Z}^2$ .

Note that the filling collar consists of three columns and three rows as shown in Figure 2(a). We begin by reducing the number of untiled columns in the filling collar to one by completing any half dominos that are on the vertical boundaries of  $\mathbf{a}$  or  $\mathbf{b}$  or, if necessary, by using complete dominos to extend the inner or outer configurations. Next, we add a row of dominos across the top boundary of  $\mathbf{a}$  or the bottom boundary of  $\mathbf{b}$  as necessary to reduce the number of rows in the filling collar to one, and to make the height of the new  $\mathbf{a}$  even (see Figure 2(b)). We finally complete the tiling as shown in Figure 2(c) by filling the remaining two empty rows with horizontal dominos and the two empty columns with vertical dominos.  $\square$


 FIGURE 3. *The square ice Wang tiles.*

#### 4.3. Square ice

Consider the tiles shown in Figure 3, called the *square ice Wang tiles*. Also consider the set of all tilings of  $\mathbb{R}^2$  by copies of these tiles subject to the matching rule that the colors (black or gray) of adjacent edges match and such that the vertices lie in the integer lattice  $\mathbb{Z}^2$ . As in the case of the dominos, the square ice tilings of the plane can be modeled as a subshift of finite type of  $\{0, 1, 2, 3, 4, 5\}^{\mathbb{Z}^2}$ . This shift of finite type is called the *square ice shift of finite type*, and we denote it by  $Y^3$ .

Square ice has the following algebraic interpretation and generalization. Consider the set of all  $2^n$  square tiles whose edges are labeled  $1, 2, \dots, n-1$ . Denote the edges as follows: bottom  $a$ , right  $b$ , top  $c$  and left  $d$ . Then the generalized square ice Wang tiles,  $\mathcal{S}_n$ , are the subset consisting of those tiles that satisfy  $a + b = c + d \pmod n$ . The matching rule is that the labels on adjacent edges must match. The corresponding shift of finite type is denoted by  $Y^n$ . Here we consider only the case  $n = 3$ . In this case the algebraic coding can be seen in Figure 3 by labeling the gray edges 1 and the black edges 2.

**PROPOSITION 4.5.** *The square ice shift of finite type  $Y^3$  has a uniform filling set, but it does not have the uniform filling property.*

*Proof.* The fixed point  $z^f \in Y^3$ , defined by  $z^f[\vec{v}] = 1$  for all  $\vec{v}$ , is clearly frozen, so  $Y^3$  does not have the uniform filling property.

One way of seeing that  $Y^3$  has a uniform filling set is to use the fact that  $Y^3$  is a finite to one factor of the three color chessboard  $X^3$  (see [9]). One can show that the factor map in [9] is injective on the filling set  $Z \subseteq X^3$  described above. The result then follows from Lemma 3.4.

An alternative proof uses the lozenge shift of finite type  $Y^L$ , defined in the next section. By Proposition 4.6,  $Y^L$  has a uniform filling set  $Z$ , and since  $Y^L \subseteq Y^3$ ,  $Z \subseteq Y^3$  is a uniform filling set for  $Y^3$ .  $\square$

#### 4.4. Lozenges

We define the *lozenge shift of finite type* by  $Y^L = Y^3 \cap \{1, 2, 3, 4, 5\}$ . In particular,  $Y^L$  is the subshift of square ice that does not use the tile labeled 0. The name comes from the following geometric interpretation. First, we make the *lozenge Wang tiles* by decorating the five nonzero square ice Wang tiles, as shown in Figure 4, by drawing a diagonal line on tiles 2–5.

Given any tiling by these tiles, if we erase the grey lines, we obtain a tiling of  $\mathbb{R}^2$  by the three *lozenge prototiles* shown in Figure 5. Such a tiling is called a *lozenge tiling*. Conversely, given a lozenge tiling such that all the vertices lie in  $\mathbb{Z}^2$ , there is a unique corresponding lozenge Wang tiling and point  $z \in Y^L$ .

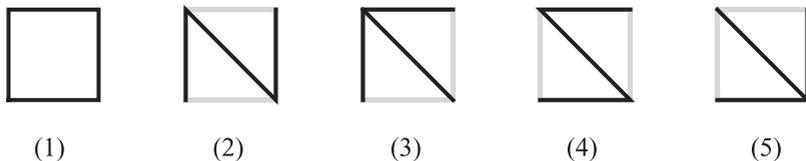


FIGURE 4. The five lozenge Wang tiles.

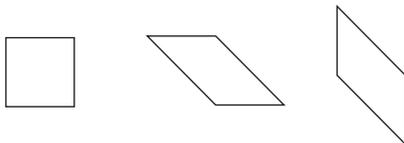


FIGURE 5. The three lozenge prototypes.

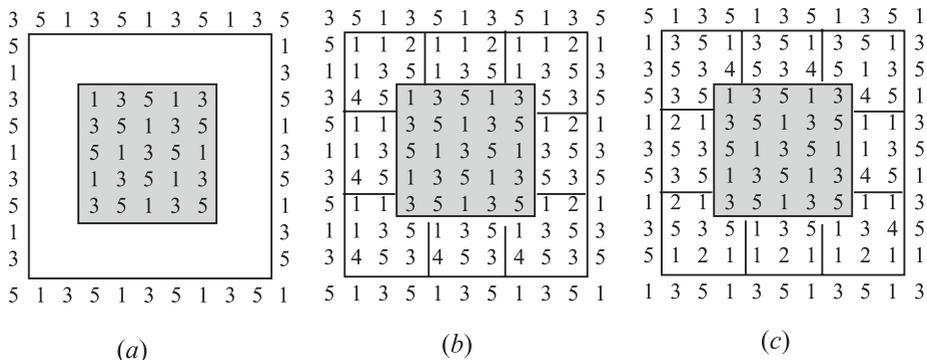


FIGURE 6. (a), (b) These show the case with corner entry 5 before and after the filling. (c) The case with corner entry 1, but only after the filling. Notice that even though the symbols 2 and 4 are never used in  $z$ , they are needed in the fillings.

PROPOSITION 4.6. *The lozenge shift of finite type  $Y^L$  does not have the uniform filling property, but it does have a uniform filling set.*

*Proof.* Since the frozen point  $z^f \in Y^3$  is also in  $Y^L$ , clearly  $Y^L$  does not have the uniform filling property. Also, the filling set for  $Y^3$  obtained in Proposition 4.5 is not a filling set for  $Y$ , because one cannot always avoid using the symbol 0 in the filling collar.

Here we will prove that the set  $Z$  consisting of the orbit of the periodic point  $z'$  defined by  $z'[(i, j)] = 2(|i| + |j| \bmod 3) + 1$  is a filling set for  $Y^L$ .

To begin, we note that we can glue the block  $\mathbf{a} = z'[[\vec{0}, \vec{5}]]$  into any point  $z \in Z$  using a filling collar of width 2. There are two interesting cases (shown in Figure 6) corresponding to whether the symbol at the lower left-hand corner of the hole in  $z$  is a 5 or a 1. The same filling operations, interpreted as lozenge tilings, are shown in Figure 7.

We now argue that  $Z$  is a filling set with filling length 4. Fix  $z, z' \in Z$  and a box  $[\vec{v}, \vec{w}]$ . Denote the configuration  $z[\vec{v}, \vec{w}]$  by  $\mathbf{b}$ . Using no more than two layers of the

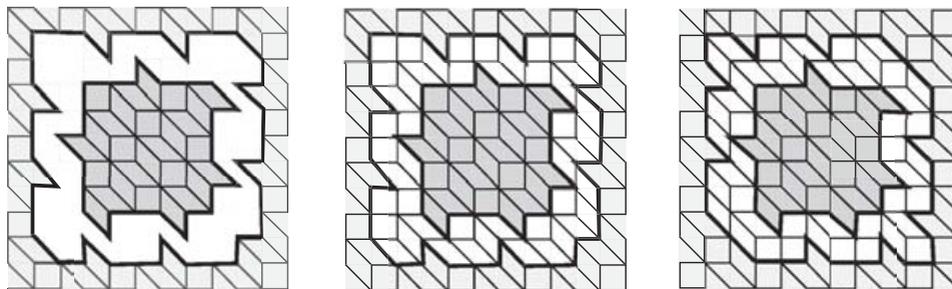


FIGURE 7. The same fillings as those in Figure 6 interpreted geometrically as lozenge tilings. Notice that we have completed any half lozenge prototiles that jut out into the filling collar.

filling collar, we first expand  $\mathbf{b}$  to a new configuration  $\mathbf{c}$  where the shape of  $\mathbf{c}$  is a box of dimensions  $n_0 \times m_0$ , such that the following hold.

- (i)  $n_0 \equiv m_0 \equiv 2 \pmod{3}$ .
- (ii) The symbol appearing in the lower left-hand corner of  $\mathbf{c}$  is 3.

Next we extend either this configuration, or  $z'[\vec{v} - 4, \vec{w} + 4]$ , as necessary to obtain a filling collar of length 2. We call the final configuration being glued in  $\mathbf{d}$ .

Again, the interesting cases are when the symbol in the lower left-hand corner of the hole in  $z$  is a 1 or a 5. We note that  $\mathbf{d}$  can be thought of as horizontally and vertically stacked copies of the block  $\mathbf{a}$ . Notice also that the filling collars in Figures 6(b) and 6(c) have some periodic structure which allow us to stack the filling configuration in those cases vertically and horizontally as many times as necessary to obtain a filling for  $\mathbf{d}$ .  $\square$

#### 4.5. Rudolph tilings

To conclude the examples, we consider the tilings and a corresponding shift of finite type introduced in [6] to provide universal models for free ergodic  $\mathbb{R}^d$  actions. We refer the reader to [6] and [7] for a detailed discussion of the properties of these tilings. In fact, our Theorem 1.1 is a discrete-time version of the main result in [6]. Interestingly, Theorem 1.1 itself for this shift of finite type can be obtained as a corollary of the results in [7], without reference to filling properties. However, the proof that this example has a filling set is interesting in its own right.

For simplicity, we consider only the two-dimensional case here, although the discussion holds, with appropriate modifications, for all  $d \geq 2$ .

Consider the four rectangular prototiles  $a = [0, 1] \times [0, 1]$ ,  $b = [0, 1] \times [0, 1 + \beta]$ ,  $c = [0, 1 + \alpha] \times [0, 1]$  and  $d = [0, 1 + \alpha] \times [0, 1 + \beta]$ , where  $\alpha$  and  $\beta$  are positive irrational numbers. Call the lower left-hand vertex of each tile its *base point*. The *Rudolph tilings* are tilings of  $\mathbb{R}^2$  by these prototiles, subject to the following adjacency rule. For simplicity in describing this rule, we assume that

$$\alpha, \beta < 1/2, \tag{4.1}$$

although this assumption can be eliminated (see [6–8]). The rule, which is illustrated in Figure 8, is as follows.

- (i) If two tiles share a horizontal edge of length greater than  $1/2$ , then the vertical distance between their base points may be 0 or  $\pm\beta$ .

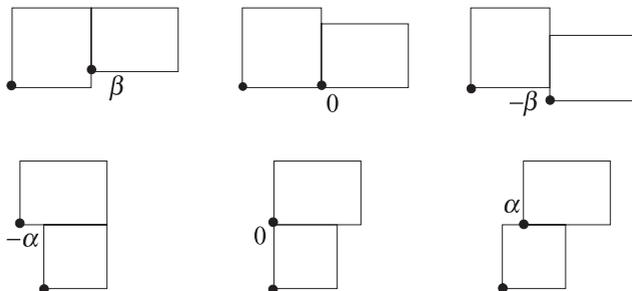


FIGURE 8. The three possible vertical alignments and three possible horizontal alignments of the Rudolph tiles in an allowed tiling.

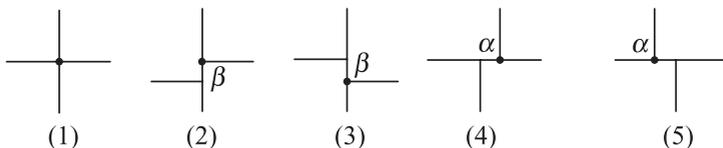


FIGURE 9. The five possible base point configurations.

(ii) If two tiles share a vertical edge of length greater than  $1/2$ , then the horizontal distance between their base points may be  $0$  or  $\pm\alpha$ .

It follows from these adjacency rules and (4.1) that each tile in a Rudolph tiling has well defined vertical (horizontal) ‘neighbors’: those tiles which share a vertical (horizontal) edge of length  $>1/2$ . If we assume that one of the tiles in the tiling has its base point at the origin in  $\mathbb{R}^2$ , then we obtain a correspondence between the tiles in the tiling and the set  $\mathbb{Z}^2$  by passing from a tile to its neighbor.

By (4.1) the tiling rules described above force each base point in a Rudolph tiling to occur in one of the five allowed configurations shown in Figure 9. We label each tile with a symbol  $s_n$ , where  $s \in \{a, b, c, d\}$  is the tile type and  $n \in \{1, 2, 3, 4, 5\}$  is the type of base point configuration as enumerated in Figure 9. Thus we have  $s_n \in \mathcal{A} = \{a, b, c, d\} \times \{1, 2, 3, 4, 5\}$ . We define  $z \in \mathcal{A}^{\mathbb{Z}^2}$  by  $z[(i, j)] = s_n$ , where  $s_n$  is the marking on the tile corresponding to  $(i, j) \in \mathbb{Z}^2$ . The set of all  $z$  thus obtained is a shift of finite type, denoted  $X^R$ , with alphabet  $\mathcal{A}$ .

PROPOSITION 4.7. *The Rudolph tiling shift of finite type  $X^R$  has a uniform filling set, but does not have the uniform filling property.*

*Proof.* Clearly the point  $z^f$  corresponding to the edge-to edge tiling of  $\mathbb{R}^2$  by  $1 \times 1$  squares is frozen. Thus  $X^R$  does not have the uniform filling property.

Now consider the periodic grid tiling (see [6]), as shown in Figure 10(a). We will show that set  $Z$  of all points in  $X^R$  corresponding to translations of this tiling is a filling set with filling distance  $\ell = 3$ .

The proof follows easily from two geometric facts. First, cutting out a rectangular hole from the symbolic point  $z \in Z$  corresponds to cutting out a rectangle with perfectly straight edges from the corresponding tiling in the plane. Second, a block with even dimensions  $2n \times 2m$  corresponds to a union of fundamental domains for the (periodic) grid tiling. If we cut out such a patch, then we can replace it with

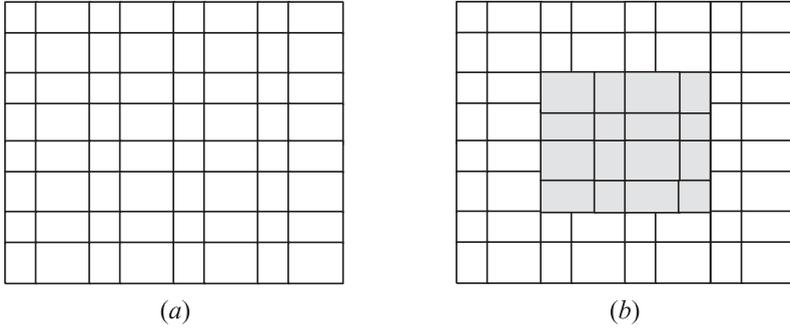


FIGURE 10. (a) A patch of grid tiling. (b) The same grid with a  $4 \times 4$  hole filled perfectly with a different  $4 \times 4$  grid.

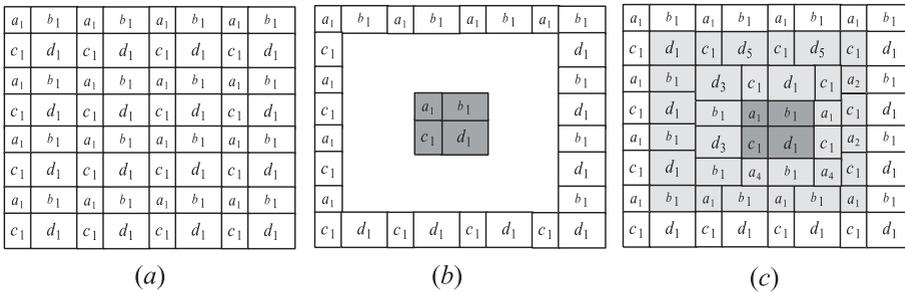


FIGURE 11. (a) A patch of grid tiling with tiles labeled by the symbols  $\mathcal{A}$ . (b) A  $4 \times 4$  hole cut out of the tiling shown in (a) to accommodate a filling collar of width 2 to glue in a new  $2 \times 2$  patch of tiling. (c) The symbolic coding of the new tiling, including the tiling of the filling collar of length 2. This new tiling corresponds to the one shown in Figure 10(b).

any other patch of grid tiling of the same size without violating the matching rules (see Figure 10(b)).

In the tiling picture one can see that the corner configurations for the base points of the tiles just inside and just outside the seam of the glued-in patch no longer correspond to configuration type 1. Thus, in the symbolic representation, this corresponds to cutting out a hole of size  $(2n + 2) \times (2m + 2)$  in the point  $z$ , and using a filling collar of length 2 to fill in a grid tiling patch of size  $2n \times 2m$  (see Figure 11).

If the dimensions of the patch we are gluing in are not even, we can enlarge it by adding at most one additional layer of tiles. We choose the tiles to add so that the patch being glued in becomes the union of fundamental domains, as considered above. Thus all together the filling distance for the shift of finite type  $X^R$  is  $\ell = 3$ . □

### 5. Proof of Theorem 1.1

Let  $A$  be the alphabet and  $s$  the step size of the shift of finite type  $Y$ . Let  $Z$  denote the filling set of  $Y$  and  $\ell$  denote the filling length. Fix  $z \in Z$  for the remainder of the argument.

Recall that to prove the result, it suffices to construct a measurable function  $\phi : X \rightarrow A$  with the property that for  $x \in X$  almost everywhere, we have

$$\text{there exists } y_x \in Y \text{ such that } \phi(T^{\vec{v}}(x)) = y_x[\vec{v}]. \tag{5.1}$$

We will construct such a function  $\phi$  as the limit of a sequence of functions  $\{\phi_i\}$  on levels of Rohlin towers of increasing measure. Our first step is to define inductively an increasing sequence of positive integers  $n_i$ , and a sequence of real numbers  $\epsilon_i$  decreasing to zero with the following properties.

- (i)  $\epsilon_i < 2^{-i}$ .
- (ii)  $n_1 \gg \ell$ .
- (iii)  $2d(\ell + s + n_i)/n_{i+1} < \epsilon_i/10$ .
- (iv) There is a sequence  $\{\tau_i\}$  of Rohlin towers each of shape  $B_{n_i}$  with base set  $E_i$ , and error less than  $\epsilon_i/10$ , such that, for  $x \in E_i$  almost everywhere,

$$\frac{|\{\vec{v} \in B_{n_i} : T^{\vec{v}}x \in \tau_{i-1}\}|}{n_i^d} > 1 - \frac{\epsilon_{i-1}}{5}. \tag{5.2}$$

This is possible by the pointwise ergodic theorem.

### 5.1. Constructing $\phi_1$ and $\phi_2$

We define the first map  $\phi_1$  on the levels of  $T^{[\ell+s, n_1-\ell-s]}(E_1)$  into  $A$ . For all  $\vec{v} \in [\ell + s, n_1 - \ell - s]$ , we set

$$\phi_1(T^{\vec{v}}(E_1)) = z[\vec{v}]. \tag{5.3}$$

We have left a collar of width  $\ell + s$  to help us with the induction step.

Now consider the second tower  $\tau_2$ . For each slice  $T^{[\ell+s, n_2-\ell-s]}x$  of  $\tau_2$ , we call the intersects of the slice with  $\tau_1$  *occurrences* of  $\tau_1$  in  $\tau_2$ . These look like disjoint squares of size  $n_1$ . We want to define a map  $\phi_2$  on the levels of  $T^{[\ell+s, n_2-\ell-s]}(E_2)$  so that it is a refinement of  $\phi_1$  whenever possible. To this end, we want to keep the symbols already assigned to as many as possible of the occurrences of  $\tau_1$  in  $\tau_2$ . Informally, we think of using  $z[B_{n_2}]$  as the ambient configuration appearing on each slice of  $\tau_2$ . We allow this ambient configuration to color the outermost collar of width  $s$  of each occurrence of  $\tau_1$ . We then use the filling property of the point  $z$  and the remaining collar of width  $\ell$  to interpolate between the ambient configuration and the one previously appearing on each copy of  $\tau_1$ . The fact that the outer collar of width  $s$  is assigned the ambient configuration allows us to interpolate each occurrence of  $\tau_1$  independently to  $z[B_{n_2}]$ .

Now we give the details. For  $x \in E_2$  almost everywhere, we are only interested in occurrences of  $\tau_1$  in  $T^{B_{n_2}}x$  which do not intersect the inner collar of width  $\ell + s$  of  $\tau_2$ . These are the occurrences which do not intersect the levels  $T^{\partial^{-(\ell+s)}[B_{n_2}]}x$ . We call such an occurrence a *good copy* of  $\tau_1$ . For the set of  $x \in E_2$  satisfying (5.2), we denote the collection of locations in  $T^{B_{n_2}}x$  which are base points of good copies of  $\tau_1$  by

$$B(x) = \{\vec{b}_1 \in [\ell + s + n_1, n_2 - n_1 - \ell - s] : T^{\vec{b}_1}x \in E_1\}.$$

We now partition  $E_2$  into subsets  $E_2^1, \dots, E_2^{k_2}$  such that the set of indices  $B(x)$  are the same for every  $x \in E_2^t$ ,  $t = 1, \dots, k_2$ . We refer to the relevant set of indices by  $B(t)$ .

Fix such a  $t$ . For all  $x \in E_2^t$  we set  $\phi_2(T^{\vec{v}}x) = z[\vec{v}]$  for the following locations of  $\vec{v}$ .

- (i)  $\vec{v} \in \partial^{-s}[\vec{b} + [0, n_1]]$ ,  $\vec{b} \in B(t)$ .
- (ii)  $\vec{v} \in (\bigcup_{\vec{b} \in B(t)} \vec{b} + [0, n_1])^c \cap [\ell + s, n_2 - \ell - s]$ .

For  $\vec{v} \in \vec{b} + [\ell + s, n_1 - \ell - s]$ ,  $\vec{b} \in B(t)$  we set  $\phi_2(T^{\vec{v}}x) = \phi_1(T^{\vec{v}}x)$ . We note that each good copy of  $\tau_1$  now sees

$$\begin{aligned} &\text{symbols from } z[B_{n_1}] \text{ in positions } \vec{b} + [\ell + s, n_1 - \ell - s] \\ &\text{and from } z[B_{n_2}] \text{ in locations } \vec{b} + \partial^{-s}[0, n_1]. \end{aligned} \tag{5.4}$$

Using (5.4) and the fact that  $Z$  is a uniform filling set, we can now find a point  $z_2^t \in Y$  such that for all  $\vec{b} \in B(t)$

$$\begin{aligned} z_2^t[\vec{b} + [\ell + s, n_1 - \ell - s]] &= z[\ell + s, n_1 - \ell - s] \\ z_2^t[\vec{b} + \partial^{-s}[0, n_1]] &= z[\vec{b} + \partial^{-s}[0, n_1]]. \end{aligned}$$

Now for all  $\vec{v} \in \vec{b} + \partial^\ell[\ell + s, n_1 - \ell - s]$  we set  $\phi_2(T^{\vec{v}}x) = z_2^t[\vec{v}]$ . Repeating the procedure for all  $t = 1, \dots, k_2$ , we define  $\phi_2$  on the levels of  $\bigcup_{t=1}^{k_2} T^{[\ell+s, n_2-\ell-s]}E_2^t$ .

Notice that we have left  $\phi_2$  undefined on the levels  $\partial^{-(\ell+s)}[B_{n_2}]$  of  $\tau_2$ . In addition, our choice of the sequence  $\{n_i\}$  guarantees that by ignoring those occurrences of  $\tau_1$  which were not good copies, we may have changed the partition assignments of points in a set of measure less than

$$\frac{2d(\ell + s + n_1)^{d-1}}{n_2^d} < \frac{\epsilon_1}{5}. \tag{5.5}$$

### 5.2. Constructing $\phi_i$

At stage  $i$  of the construction we define a map  $\phi_i$  which is a refinement of  $\phi_{i-1}$  on as much of the space as possible. At this stage we only worry about occurrences of  $\tau_{i-1}$  in  $\tau_i$ , as opposed to occurrences of all  $\tau_j$ ,  $1 \leq j \leq i - 1$ .

The only difference, then, between the general inductive step and the construction of  $\phi_2$  is that the configurations assigned to slices of  $\tau_{i-1}$  will not necessarily correspond to configurations from  $z[B_{n_i}]$  for any  $z \in Z$ . This is already apparent if one considers  $\tau_2$ . For this reason, in choosing the points  $z_i^t$ , we have to use the analog of (5.4) and invoke Lemma 3.5 to find the point  $z_i^t \in Y$ .

In all other ways the construction is identical. Computations essentially identical to the one given above will yield that (5.2) and our choice of the  $\{n_i\}$  guarantee that if we ignore the bad copies of  $\tau_{i-1}$ , at stage  $i$  of the construction we will have reassigned partition values to a set of measure less than

$$\frac{\epsilon_{i-1}}{5} + \frac{2d(\ell + s + n_{i-1})^{d-1}}{n_i^d} < \frac{\epsilon_{i-1}}{5} + \frac{\epsilon_i}{10} < \epsilon_i. \tag{5.6}$$

### 5.3. Constructing $\phi$

We now claim that there is a set  $G \subseteq X$  of full measure such that for all  $x \in G$  the function

$$\phi(x) = \lim_n \phi_n(x) \tag{5.7}$$

is well defined and satisfies (5.1).

Under our construction the only points for which the limit in (5.7) does not exist are those that fall in  $\partial^{-(\ell+s)}[B_{n_i}]$  for infinitely many  $i$  and those that land in  $\tau_i^c$  for infinitely many  $i$ . By our choice of  $\epsilon_i$  and by expression (5.6), we can use the easy direction of the Borel–Cantelli lemma to conclude that these points form a set of measure 0. Removing them and their orbits gives us an invariant set  $G$  of full measure with the property that (5.7) holds for all  $x \in G$ .

It is clear from the construction that if  $\phi(x)$  is defined, then the point in  $A^{\mathbb{Z}^d}$  defined by  $y_x[\vec{v}] = \phi(T^{\vec{v}}x)$  lies in  $Y$ .

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